# 12. The Antiderivative: Indefinite Integrals

## Definitions, basic rules

If $F'(x)=f(x)$, that is, the function $f(x)$ is the *derivative* of $F(x)$, then the function $F(x)$ is called the *antiderivative* of $f(x)$. Besides $F(x)$, any function $(F(x)+C)$ is also an antiderivative of $f(x)$, where $C$ is an arbitrary constant, since $(C)'=0$. By the *indefinite integral* of $f(x)$ we mean the set of all antiderivatives of $f(x)$, in notation:

$$\int f(x)\,\mathrm{d}x = F(x)+C \quad \Leftrightarrow \quad F'(x)=f(x)$$

where $F(x)$ is one of the infinitely many antiderivatives and $C$ is an indefinite integration constant.

**The rules of indefinite integration** come from the rules of differentiation, most basic are the following:

$$\int (f(x)+g(x))\,\mathrm{d}x = \int f(x)\,\mathrm{d}x + \int g(x)\,\mathrm{d}x \qquad \int cf(x)\,\mathrm{d}x = c\int f(x)\,\mathrm{d}x \qquad (c \text{ constant})$$

i.e., integration can be done term by term and the constant coefficient can be extracted. These two readily imply the rule for a difference and the rule for a linear combination of functions with an arbitrary (but finite) number of terms:

$$\int (c_1 f_1(x)+c_2 f_2(x)+\ldots)\,\mathrm{d}x = c_1\int f_1(x)\,\mathrm{d}x + c_2\int f_2(x)\,\mathrm{d}x+\ldots$$

## Rule for the linear transformation of the independent variable

if $\int f(x)\,\mathrm{d}x = F(x)+C$, then $\int f(ax+b)\,\mathrm{d}x = \dfrac{F(ax+b)}{a}+C$

frequent special cases: $\int f(x+b)\,\mathrm{d}x = F(x+b)+C$ and $\int f(ax)\,\mathrm{d}x = \dfrac{F(ax)}{a}+C$

(when $a$=1 and $b$=0, respectively).

The so called *fundamental integrals* can be obtained by reversing the derivative rules for the elementary functions (see table below). Note that not every elementary function's integral occurs among the fundamental integrals (ln $x$, tan $x$ and cot $x$ are missing). Finding the antiderivatives using the fundamental integrals and the above basic rules is called *basic* or *fundamental (indefinite) integration*.

**Fundamental integrals** *

| Power and exponential functions | Trigonometric functions |
|---|---|
| $\int x^{\alpha}\,dx = \frac{x^{\alpha+1}}{\alpha+1} + C \quad (\alpha \neq -1)$ | $\int \sin x\,dx = -\cos x + C$ |
| $\int \frac{1}{x}\,dx = \ln\lvert x\rvert + C \quad (\text{case } \alpha = -1)$ | $\int \cos x\,dx = \sin x + C$ |
| $\int a^{x}\,dx = \frac{a^{x}}{\ln a} + C \quad (a>0,\, a\neq 1)$ | $\int (\cot^2 x + 1)\,dx = \int \frac{1}{\sin^2 x}\,dx = -\cot x + C$ |
| $\int e^{x}\,dx = e^{x} + C \quad (\text{case } a=e)$ | $\int (\tan^2 x + 1)\,dx = \int \frac{1}{\cos^2 x}\,dx = \tan x + C$ |

* α and $a$ are real constants, $e$=2,71828…

**EXAMPLE:**

$$\int\left(3x^5 - x + \sqrt[3]{2x} + 2 - \frac{4}{x} + 2^x - 6e^{-x} + 5\sin 2x - 8\cos(x-\tfrac{\pi}{5}) - \frac{7}{\sin^2 x} + \frac{9}{\cos^2 2x}\right)dx =$$

$$= 3\frac{x^6}{6} - \frac{x^2}{2} + \sqrt[3]{2}\,\frac{x^{\frac{4}{3}}}{\frac{4}{3}} + 2x - 4\ln\lvert x\rvert + \frac{2^x}{\ln 2} - 6\frac{e^{-x}}{-1} + 5\frac{-\cos 2x}{2} - 8\sin(x-\tfrac{\pi}{5}) - 7(-\cot x) + 9\frac{\tan 2x}{2} + C$$

## Integration by substitution:

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du = F(u) + C = F(g(x)) + C$$

This method is used for a product where a composite function is multiplied by the derivative of its inner function. Here we substitute two things: the inner function g(x) by an auxiliary variable u, and the derivative of the inner function du/dx = g′(x) multiplied by the differential dx, that is:

$$u = g(x)\,, \qquad du = g'(x)\,dx\,.$$

The new integration problem obtained by substitution with the new variable $u$ is hopefully simpler than the original, and if we can find the antiderivative of $f(u)$, denoted by $F(u)$, then by back substitution of $u$ by the inner function $g(x)$, we can get the solution to the original problem.

**Example:**

$$\int \frac{x}{\sqrt[3]{1-x^2}}\,dx = \int (-\tfrac{1}{2})(1-x^2)^{-\frac{1}{3}}(-2x)\,dx = \int (-\tfrac{1}{2})u^{-\frac{1}{3}}du = -\frac{1}{2}\frac{u^{\frac{2}{3}}}{\frac{2}{3}} + C = -\frac{3}{4}(1-x^2)^{\frac{2}{3}} + C$$

where, after a simple modification we applied the substitution $u=1-x^2$, $du=-2x\,dx$.

## Integration by parts (partial integration):

$$\int f'(x)\,g(x)\,dx = f(x)g(x) - \int f(x)\,g'(x)\,dx$$

This method is used for such a product, where we know the antiderivative of one factor $f'(x)$ (it is $f(x)$), and the derivative of the other factor $g(x)$ makes the new problem on the right side of the rule simpler than the original left side. The method cannot always be used successfully; typical examples when it works are: $x^\alpha \log_a x$, $xa^x$, $xe^{\alpha x}$, $x\sin\alpha x$, $x\cos\alpha x$, etc. When integrating $f'(x)$, occasionally it is useful to choose a nonzero integration constant for its antiderivative $f(x)$ to make possible the simplification of the function to be integrated in the right side of the rule.

**Ex. 1:**

$$\int 2x\ln(x+1)\,dx = (x^2-1)\ln(x+1) - \int (x^2-1)\frac{1}{x+1}dx = (x^2-1)\ln(x+1) - \int (x-1)\,dx = (x^2-1)\ln(x+1) - \frac{x^2}{2} + x + C$$

Here, for $f'(x)=2x$ we chose the antiderivative as $f(x)=x^2-1$ instead of $x^2$, which choice allowed us a simplification in the present case.

**Ex. 2: two-step** $\int x^2e^{-x}\,dx = -e^{-x}x^2 - \int (-e^{-x})\cdot 2x\,dx = -e^{-x}x^2 + \int e^{-x}\cdot 2x\,dx =$

$$= -e^{-x}x^2 - e^{-x}\cdot 2x - \int (-e^{-x})\cdot 2\,dx = -e^{-x}x^2 - e^{-x}\cdot 2x - 2e^{-x} + C = -e^{-x}(x^2+2x+2) + C$$

where in both partial integration steps we used the choice $f'(x)=e^{-x}$.

**Ex. 3: two-step implicit** e.g. $e^x \sin x$, $e^x \cos x$ (presented in the lecture)

## PROBLEM 1 FUNDAMENTAL INTEGRATION

Find the indefinite integral of the given function using fundamental integration. When necessary, first modify / simplify the function using identities to transform it into a linear combination of fundamental integrals!

*a)* $2x^2-9x+12$ *b)* $\dfrac{\sqrt[4]{\sqrt[3]{\sqrt{x}}}}{x}$ *c)* $\dfrac{\pi-3x+2x^2-x^3}{x^2}$

d) $\dfrac{\sqrt{2x}-3x+e^2-x^{-\frac{1}{2}}}{\sqrt{x}}$ e) $(x^2-2^x)(x^2+2^x)$ f) $\dfrac{3-4\sin^3 x}{\sin^2 x}$

g) $\cos\frac{x}{2}-\dfrac{1}{x-1}$ h) $\dfrac{\pi-3e^x+e^{3x}-3^x}{e^{2x}}$ i) $\dfrac{3}{2x-1}-5e^{-2x+1}$

j) $\dfrac{-2}{(x-1)^2}+(x+10)^9$ k) $2005^{-x}+x^{-2005}$ l) $\tan^2 x$

m) $\dfrac{\sin(x-2006)}{2005}$ n) $\dfrac{\cos^2(\pi x)-e^{x+\pi}}{e^x\cos^2(\pi x)}$ o) $\dfrac{e^{2x}-e^{-2x}}{2}$

p) $x(x+1)^8$ q) $\dfrac{1}{\sin^2 2x}$ r) $\dfrac{\cos 2x}{\sin x+\cos x}$

s) $\left(\sqrt[3]{x}+\dfrac{1}{\sqrt[3]{x}}\right)^3$ t) $\dfrac{x^3+1}{x+1}$

## PROBLEM 2 INTEGRATION BY SUBSTITUTION

Integrate the function using the method of substitution. (Some modification might be necessary!)

a1) $\tan x$ a2) $\cot x$ b) $x\sin x^2$ c) $x^2e^{-x^3}$

d) $\dfrac{\sqrt[3]{\sqrt{x}+1}}{\sqrt{x}}$ e) $\dfrac{(\sqrt[3]{x}+1)^{2005}}{\sqrt[3]{x^2}}$ f) $\dfrac{\ln x}{x}$ g) $\dfrac{1}{x\ln x}$

h) $\dfrac{\sin(\ln x)}{x}$ i) $\dfrac{\cos x}{\sin^2 x}$ j) $\dfrac{\sqrt{\cot x-2}}{\sin^2 x}$ k) $\dfrac{1}{e^{\tan x}\cos^2 x}$

l) $\dfrac{x^2+1}{3-6x-2x^3}$ m) $x\cdot\sqrt{x^2+1}$ n) $e^x\sin e^x$ o) $\dfrac{e^{2\ln|x|}}{x}$

p) $\dfrac{\tan\pi x}{\cos^2(\pi x)}$ q) $\dfrac{\cos x-\sin x}{\cos x+\sin x}$ r) $\dfrac{x+\sin 2x}{x^2-\cos 2x}$ s) $\dfrac{x}{\sin^2(x^2+1)}$

t) $\dfrac{2^x}{2^x+\sqrt{2}}$ u) $\dfrac{2\sin x}{2-\cos x}$ v) $\tan^3 x+\tan x$

w) $\dfrac{3\cdot\sqrt{x}}{2(\sqrt{x^3}-1)^2}$ x) $(4x-2x^3)\sqrt[3]{x^2-2}$ y1) $\sin^2 x$

y2) $\cos^2 x$ z1) $\sin^3 x$ z2) $\cos^3 x$

## PROBLEM 3 INTEGRATION BY PARTS

*a)* $\lg x$ *b)* $x^{2005}\log_2 x$ *c)* $\sqrt{x}\ln x$ *d)* $\frac{\ln x}{x}$

*e)* $x\cdot 2^x$ *f)* $(2x-1)e^{-2x}$ *g)* $\frac{\ln(\sin^2 x)}{\cos^2 x}$ *h)* $x\sin\frac{x}{2}$

*i)* $(x+1)\cos 2x$ *j)* $\sin^2 x$ *k)* $\cos^2\frac{x}{2}$ *l)* $\ln^2 x$

*m)* $(x^{-1}\ln x)^2$ *n)* $x^2\cos 2x$ *o)* $x^2 2^x$

*p)* $(x^2+2)\sin x$ *q)* $e^{-x}\sin x$ *r)* $2^x\cos x$ *s)* $4x\cos^2 x$

*t)* $\frac{\ln(\ln x)}{x}$ *u)* $\cos(\ln x)$

## SOLUTIONS

### Fundamental integration

*a)* $\frac{2}{3}x^3-\frac{9}{2}x^2+12x+C$ *b)* $24\cdot\sqrt[24]{x}+C$ *c)* $-\frac{\pi}{x}-3\ln|x|+2x-\frac{1}{2}x^2+C$ *d)* $\sqrt{2}x-2x^{\frac{3}{2}}+2e^2x^{\frac{1}{2}}-\ln|x|+C$ *e)* $\frac{1}{5}x^5-\frac{1}{\ln 4}4^x+C$ *f)* $-3\cot x+4\cos x+C$ *g)* $2\sin\frac{x}{2}-\ln|x-1|+C$ *h)* $-\frac{\pi}{2}e^{-2x}+3e^{-x}+e^x+\frac{1}{2-\ln 3}(\frac{3}{e^2})^x+C$ *i)* $\frac{3}{2}\ln|2x-1|+\frac{5}{2}e^{-2x+1}+C$ *j)* $2(x-1)^{-1}+0{,}1(x+10)^{10}+C$ *k)* $-\frac{1}{\ln 2005}2005^{-x}-\frac{1}{2004}x^{-2004}+C$ *l)* $\tan x-x+C$ *m)* $-\frac{1}{2005}\cos(x-2006)+C$ *n)* $-e^{-x}-\frac{e^{\pi}}{\pi}\tan(\pi x)+C$ *o)* $\frac{1}{4}(e^{2x}+e^{-2x})+C$ *p)* $\frac{1}{10}(x+1)^{10}-\frac{1}{9}(x+1)^9+C$ *q)* $-\frac{1}{2}\cot 2x+C$ *r)* $\sin x+\cos x+C$ *s)* $\frac{1}{2}x^2+\frac{9}{4}x^{\frac{4}{3}}+\frac{9}{2}x^{\frac{2}{3}}+\ln|x|+C$ *t)* $\frac{1}{3}x^3-\frac{1}{2}x^2+x+C$

### Integration by substitution

*a1)* $-\ln|\cos x|+C$ *a2)* $\ln|\sin x|+C$ *b)* $-\frac{1}{2}\cos x^2+C$ *c)* $-\frac{1}{3}e^{-x^3}+C$ *d)* $\frac{3}{2}(\sqrt{x}+1)^{\frac{4}{3}}+C$ *e)* $\frac{3}{2006}(\sqrt[3]{x}+1)^{2006}+C$ *f)* $\frac{1}{2}\ln^2 x+C$ *g)* $\ln|\ln x|+C$ *h)* $-\cos(\ln x)+C$ *i)* $-(\sin x)^{-1}+C$ *j)* $-\frac{2}{3}(\cot x-2)^{\frac{3}{2}}+C$ *k)* $-e^{-\operatorname{tg} x}+C$ *l)* $-\frac{1}{6}\ln\left|3-6x-2x^3\right|+C$ *m)* $\frac{1}{3}(x^2+1)^{\frac{3}{2}}+C$ *n)* $-\cos e^x+C$ *o)* $\frac{1}{2}e^{2\ln|x|}+C=\frac{1}{2}x^2+C$ *p)* $\frac{1}{2\pi}\tan^2\pi x+C$ *q)* $\ln|\sin x+\cos x|+C$ *r)* $\frac{1}{2}\ln\left|x^2-\cos 2x\right|+C$ *s)* $-\frac{1}{2}\operatorname{ctg}(x^2+1)+C$ *t)* $\frac{1}{\ln 2}\ln(2^x+\sqrt{2})+C$ *u)* $2\ln(2-\cos x)+C$ *v)* $\frac{1}{2}\tan^2 x+C$ *w)* $-(x^{\frac{3}{2}}-1)^{-1}+C$ *x)*

$-\frac{3}{7}(x^2-2)^{\frac{7}{3}}+C$ *y1)* $\frac{1}{2}x-\frac{1}{4}\sin 2x+C$ *y2)* $\frac{1}{2}x+\frac{1}{4}\sin 2x+C$ *z1)* $\frac{1}{3}\cos^3 x-\cos x+C$ *z2)* $-\frac{1}{3}\sin^3 x+\sin x+C$

- ## Integration by parts

*a)* $x(\lg x-\frac{1}{\ln 10})+C=x\lg(x/e)+C$ *b)* $\frac{1}{2006}x^{2006}(\log_2 x-\frac{1}{2006\ln 2})+C$ *c)* $\frac{2}{3}x^{\frac{3}{2}}(\ln x-\frac{2}{3})+C$ *d)* $\frac{1}{2}\ln^2 x+C$ *e)* $\frac{1}{\ln 2}2^x(x-\frac{1}{\ln 2})+C$ *f)* $-xe^{-2x}+C$ *g)* $\tan x\cdot\ln(\sin^2 x)-2x+C$ *h)* $-2x\cos\frac{x}{2}+4\sin\frac{x}{2}+C$ *i)* $\frac{1}{2}(x+1)\sin 2x+\frac{1}{4}\cos 2x+C$ *j)* $\frac{1}{2}(x-\sin x\cos x)+C$ *k)* $\frac{x}{2}+\sin\frac{x}{2}\cos\frac{x}{2}+C$ *l)* $x(\ln^2 x-2\ln x+2)+C$ *m)* $-\frac{1}{x}(\ln^2 x+2\ln x+2)+C$ *n)* $(\frac{x^2}{2}-\frac{1}{4})\sin 2x+\frac{x}{2}\cos 2x+C$ *o)* $2^x(\frac{1}{\ln 2}x^2-\frac{2}{\ln^2 2}x+\frac{2}{\ln^3 2})+C$ *p)* $2x\sin x-x^2\cos x+C$ *q)* $-\frac{1}{2}e^{-x}(\sin x+\cos x)+C$ *r)* $\frac{1}{1+\ln^2 2}2^x(\sin x+(\ln 2)\cos x)+C$ *s)* $x^2+x\sin 2x+\frac{1}{2}\cos 2x+C$ *t)* $(\ln x)\cdot(\ln(\ln x)-1)+C$ *u)* $\frac{1}{2}x(\sin(\ln x)+\cos(\ln x))+C$