MATHEMATICS BEHIND CHIP

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Department for Mathematics and Informatics
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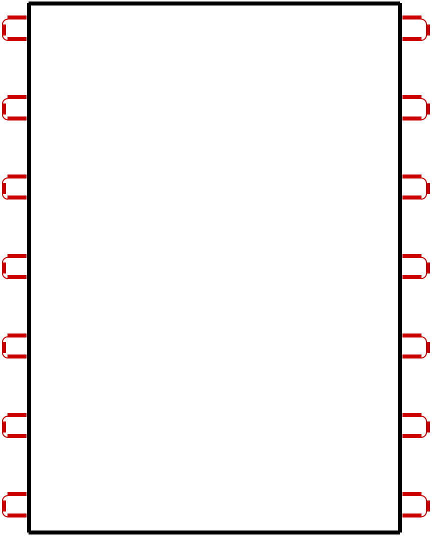
Teaching material

Chip from the outside

Chip from the outside



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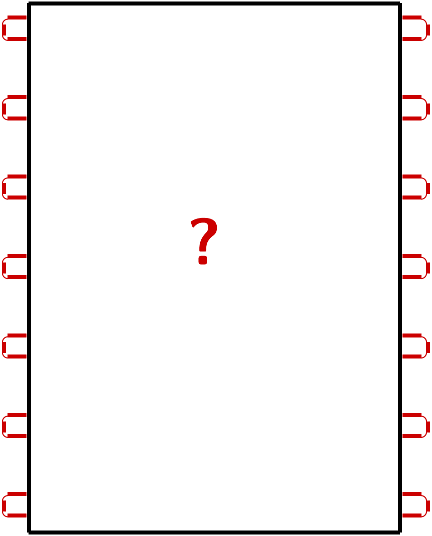

Digital technology

Zeroes and ones

$$\ldots \mathbf{011101010} \ldots \quad \mathbf{?} \rightarrow \mathbf{?} \quad \ldots \mathbf{011000110} \ldots$$

Power set

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$X \subseteq Y \Longleftrightarrow (\forall x)(x \in X \Rightarrow x \in Y)$

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It is also denoted by $C_A(X) = \overline{X}$.

Power set

Properties of set operations

$$(\mathcal{P}(A), \cap, \cup, ^{-}, \emptyset, A, \subseteq)$$

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Diagram

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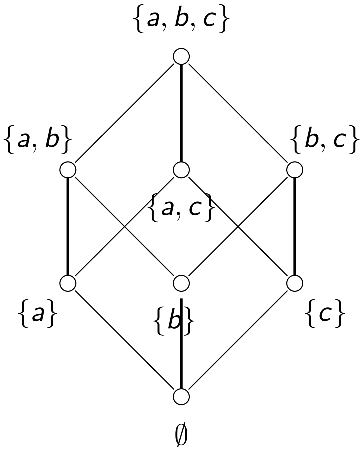

$(\mathcal{P}(\{a, b, c\}), \subseteq)$

Power set

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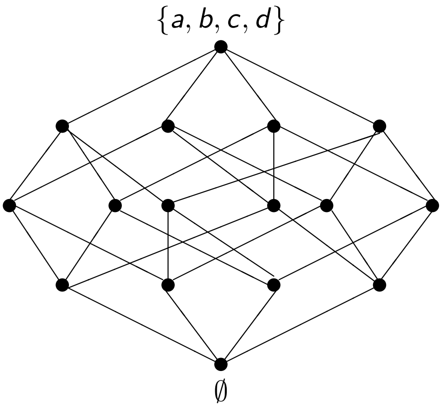

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Boolean algebra

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B - a nonempty set, $\wedge$, $\vee$ - binary operations, $'$ - a unary operation, 0 and 1 - constants, and the following axioms hold:

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b1: $x \wedge y = y \wedge x$
b2: $x \vee y = y \vee x$; (*commutativity laws*)

b3: $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$
b4: $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ (*distributivity laws*)

b5: $x \wedge 1 = x$
b6: $x \vee 0 = x$ (*properties of* 0 *and* 1)

b7: $x \wedge x' = 0$
b8: $x \vee x' = 1$ (*properties of the complement operation*)

b9: $0 \neq 1$.

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	$'$
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Example with positive integers

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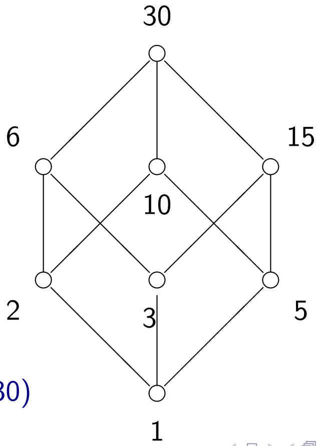

$(D(30), gcd, lcm, 30/x, 1, 30)$

Language of Boolean algebras

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- $1 \wedge x'$
- $((u' \vee v)' \wedge (u' \wedge v')) \vee v$

Examples

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- $1 \wedge x'$
- $((u' \vee v)' \wedge (u' \wedge v')) \vee v$
- $(x_1 \wedge x_2' \wedge x_4') \vee (x_2' \wedge x_3 \wedge x_5) \vee (x_1 \wedge x_6')$

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To each subset of A there corresponds a characteristic function and vice versa.

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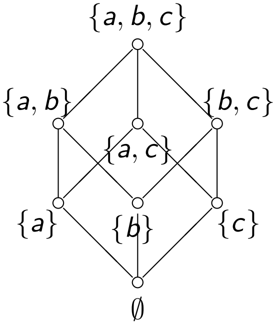
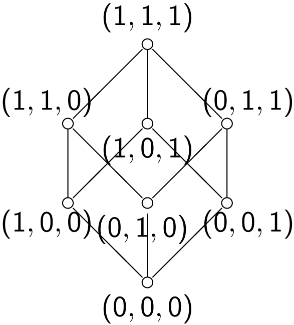
Boolean algebra of characteristic functions

Three-element set



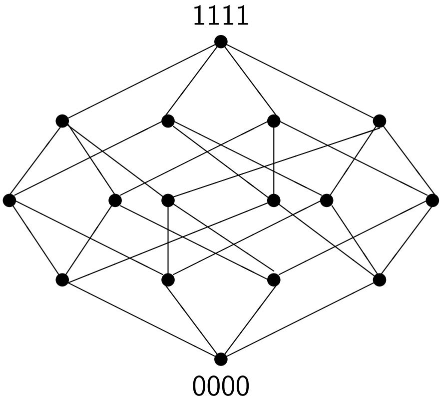
$\mathcal{P}(\{a, b, c\})$



$\{\mathcal{K}_X \mid X \subseteq \{a, b, c\}\}$

Boolean algebra of characteristic functions

Four-element set



$$\{\mathcal{K}_X \mid X \subseteq \{a, b, c, d\}\}$$

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Number of elements

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$n = 2$	$\begin{matrix} 0 & 0 & & 1 & 0 \\ 0 & 1 & & 1 & 1 \end{matrix}$	$2^2 = 4$
$n = 3$	$\begin{matrix} 0 & 0 & 0 & & 1 & 0 & 0 \\ 0 & 0 & 1 & & 1 & 0 & 1 \\ 0 & 1 & 0 & & 1 & 1 & 0 \\ 0 & 1 & 1 & & 1 & 1 & 1 \end{matrix}$	$2^3 = 8$

Boolean algebra of characteristic functions

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0	0	0	0	1	1	0	0
0	0	0	1	1	1	0	1
0	0	1	0	1	1	1	0
0	0	1	1	1	1	1	1
0	1	0	0	1	1	0	0
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$2^4 = 16$

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Single words are **code words**.

Counting with code words is performed coordinatewise, using two-element Boolean-algebra operations:

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$$\varphi : \{0,1\}^3 \to \{0,1\}$$

x	y	z	$\varphi(x,y,z)$
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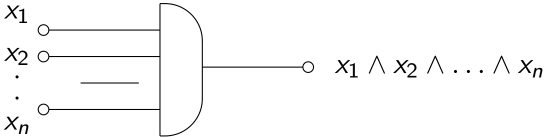
$$f(x,y,z) = (x' \wedge y' \wedge z') \vee (x' \wedge y \wedge z) \vee (x \wedge y' \wedge z).$$

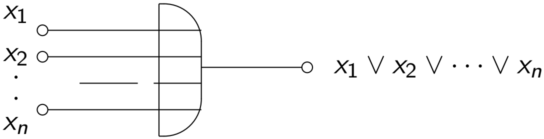
Logical circuits

Logical circuits

AND-gate:



OR-gate:



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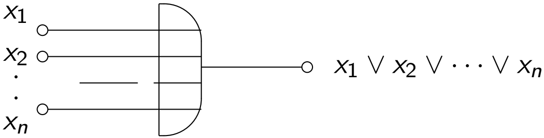
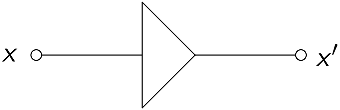


Inverter or **NOT-gate**:



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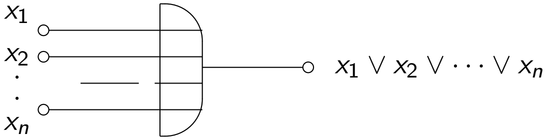

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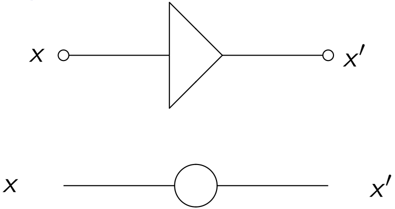

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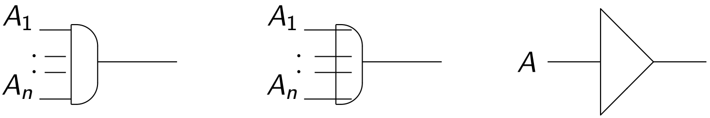

Boolean functions

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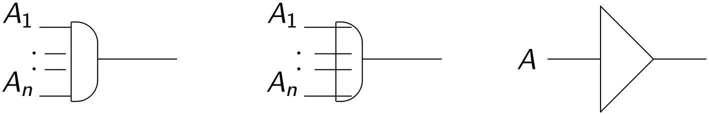

- Logical circuits are obtained only by a finite number of application of the previous two rules.

Example

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$$(x \vee (y' \wedge z))'$$

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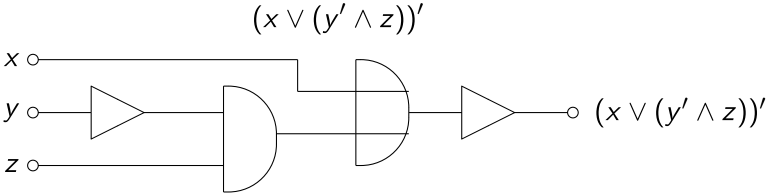

Half adder and adder

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Addition of numbers represented in binary system;
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$$x \oplus y = (x \vee y) \wedge (x \wedge y)' \quad \text{and} \quad r = x \wedge y.$$

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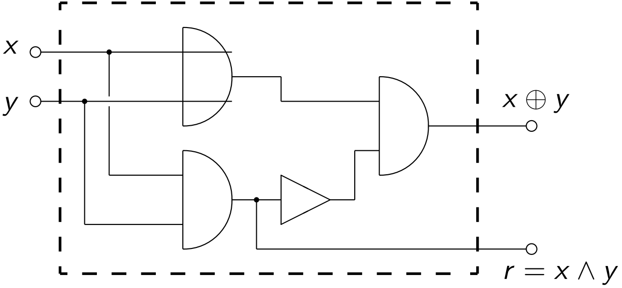

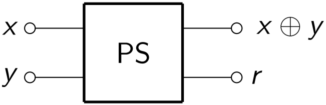

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x	y	z	$x \oplus y \oplus z$	R
0	0	0	0	0
0	0	1	1	0
0	1	0	1	0
0	1	1	0	1
1	0	0	1	0
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$R = r \vee (z \wedge (x \oplus y))$; r - reminder, transfer.

Adder

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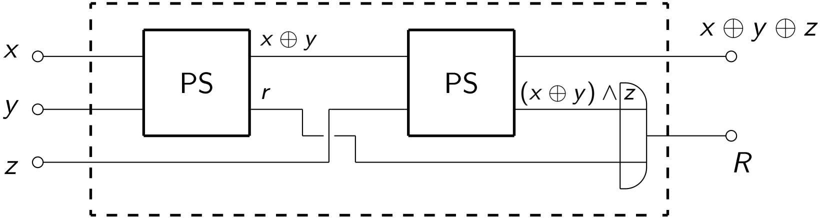
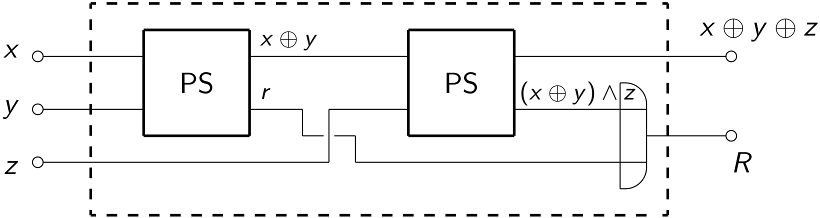
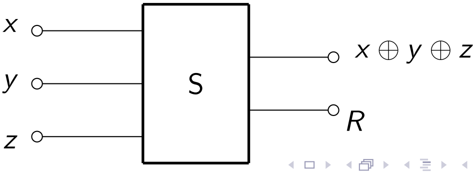


Adder

$$r = x \wedge y \quad R = r \vee (z \wedge (x \oplus y))$$





Example

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Construction of a logical circuit for the addition of three two-digit binary numbers.

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\begin{array}{cccc}
 & & a_1 & a_0 \\
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 & \oplus & c_1 & c_0 \\
\hline
d_3 & d_2 & d_1 & d_0
\end{array}
$$

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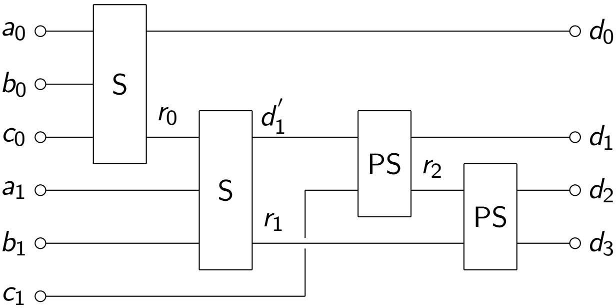
Algorithm for the addition:

$$\begin{array}{rcll} a_0 \oplus b_0 \oplus c_0 & = & d_0 & \qquad \text{(transfer } r_0\text{)} \\ r_0 \oplus a_1 \oplus b_1 & = & d_1' & \qquad \text{(transfer } r_1\text{)} \\ d_1' \oplus c_1 & = & d_1 & \qquad \text{(transfer } r_2\text{)} \\ r_1 \oplus r_2 & = & d_2 & \qquad \text{(transfer } d_3\text{)} \end{array}$$



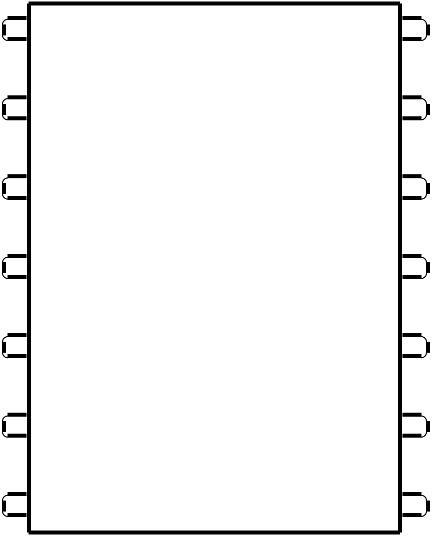
a_0
b_0
c_0
a_1
b_1
c_1
S
r_0
S
d_1'
r_1
PS
r_2
PS
d_0
d_1
d_2
d_3

Back to beginning

Chip from the inside

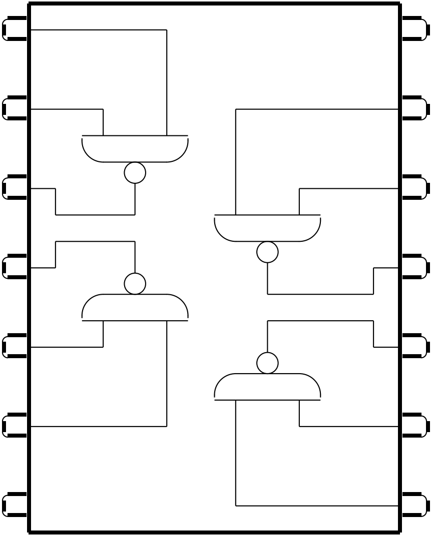
Back to beginning

Chip from the inside



Back to beginning

Chip from the inside



Field GF(2)

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G*alois* **F***ield (E. Galois, French mathematician, 1811-1832.)*

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When applied, field operations are performed faster then Boolean algebra operations.