Can we understand the unpredictable?
The mathematics of chaos

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Is life predictable or unpredictable?

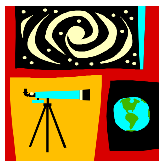
Can we tell what is going to happen

- In the next second?
- In the next hour?
- In the next year?

Does nature have an underlying order and pattern?

First answer …. YES!

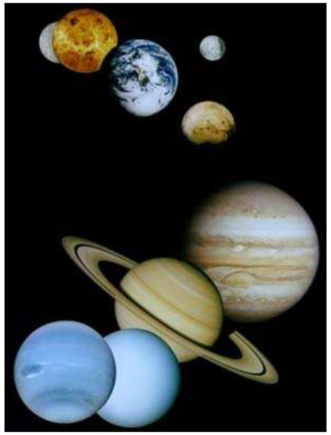
Science is the search for order and pattern in the universe



If we look we can see order and pattern all around us …..

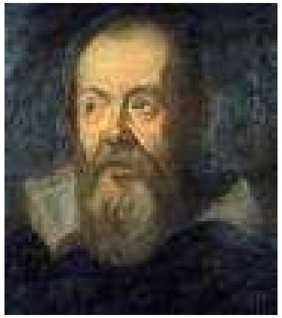
Snow crystals

The animal world



The motion of the planets

One of the first to realise this



Galileo

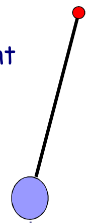
Pisa

1600

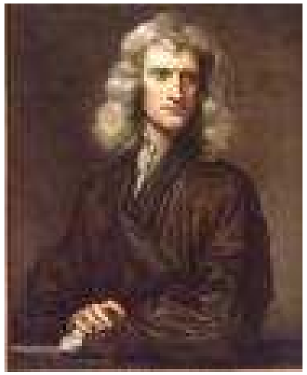
Galileo watched a pendulum swing and realised that it was governed by predictable laws

Swing time of the pendulum was constant

- Regardless of how it was pushed
- Or where it was
- Or when



Mathematically, for small swings the pendulum approximates a harmonic oscillator, and its motion approximates to simple harmonic motion.
Remark: Bánhelyi-Csendes-Garay-Hatvani proved first chaos for a periodically forced and damped pendulum.



Newton

1686

In the Principia Newton showed that this order and pattern could be expressed by using mathematics

$$l\frac{d^2\theta}{dt^2}+k\frac{d\theta}{dt}+g\sin(\theta)=0$$

Pendulum equation

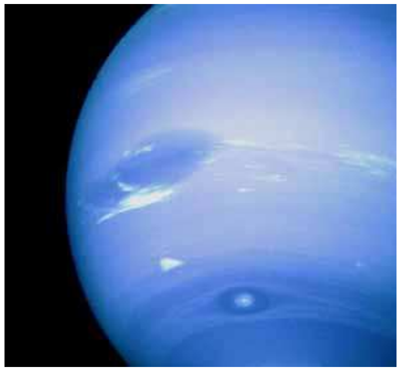
Key idea ….

- Write down the equations describing a physical system
- Solve the equations
- Predict the future

Does this work?

$$\frac{d^2x}{dt^2}= -\frac{GMx}{|x|^3}$$

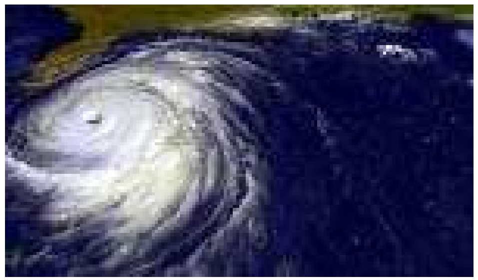
Newton's law of gravitation



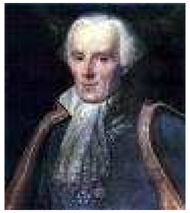
Neptune: discovered by maths

$$u_t+u.\nabla u=-\nabla P+\frac{1}{\mathrm{Re}}\nabla^2 u, \nabla.u=0$$

Navier-Stokes equations



Weather forecasting



Laplace's prediction …

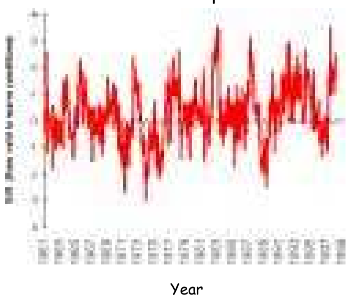
If we were to know the exact position of every particle in the universe then we could predict the future with certainty

No room here for free will!

Lots of natural and human events seem to be very unpredictable!!

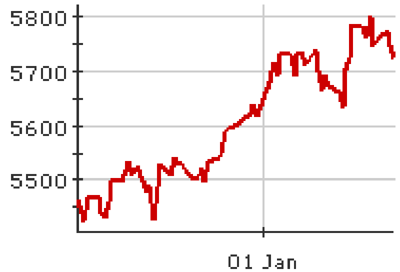
Clouds and the weather (after a week)

Ocean temperature



El Nino

Climate change



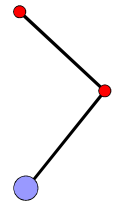
The stock market

Does this complex behaviour arise because nature is really complicated and unexplainable

Or ....

does it arise naturally from Newton's laws???

The Double Pendulum



Motion can be

- Periodic in phase : predictable
- Periodic out of phase : predictable
- Chaotic : unpredictable

Newton's laws apply to the double pendulum!

θ_1 Angle of top part

θ_2 Angle of bottom part

$$\frac{d^2\theta_1}{dt^2} + m\frac{d^2\theta_2}{dt^2}\cos(\theta_2-\theta_1) - m\left(\frac{d\theta_2}{dt}\right)^2\sin(\theta_2-\theta_1) + \sin(\theta_1) = 0$$

$$\frac{d^2\theta_2}{dt^2} + \frac{d^2\theta_1}{dt^2}\cos(\theta_2-\theta_1) - \left(\frac{d\theta_1}{dt}\right)^2\sin(\theta_2-\theta_1) + \sin(\theta_2) = 0$$

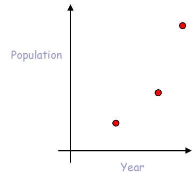
Chaos

Simple rules can lead to complex and unpredictable behaviour

Poincare: discoverer of chaos

Can we predict the population of a town?

$$x_n$$



Population of the town in the year n

Can we relate this years population: x_n

To next years population: x_{n+1}

Malthus ....

$$\boxed{x_{n+1} = ax_n}$$

Birthrate/Deathrate

- a = 1 ... population stays constant
- a > 1 ... population increases
- a < 1 ... population decreases

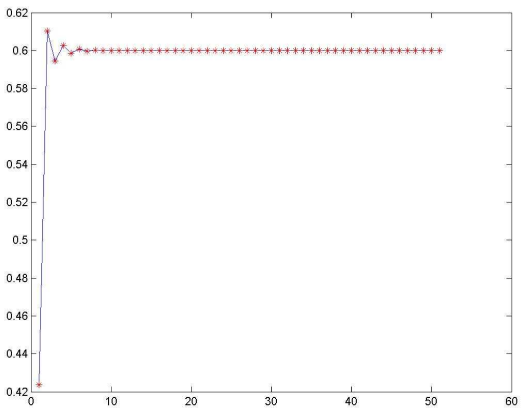
Problem ... population runs out of resources

Improved model proposed by May

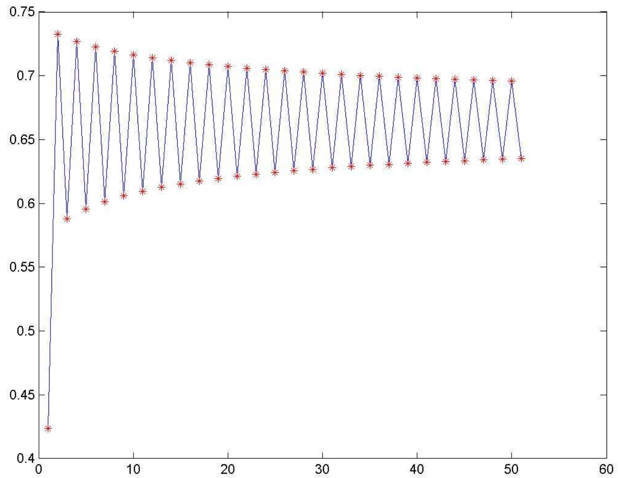
$$\boxed{x_{n+1} = ax_n(M - x_n)}$$

Maximum population

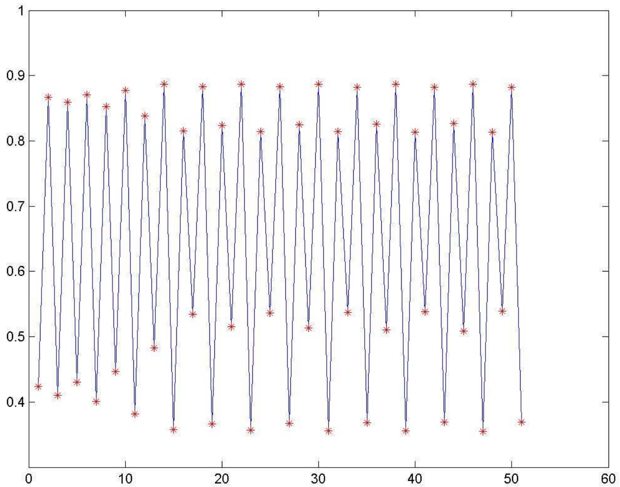
What does this predict?



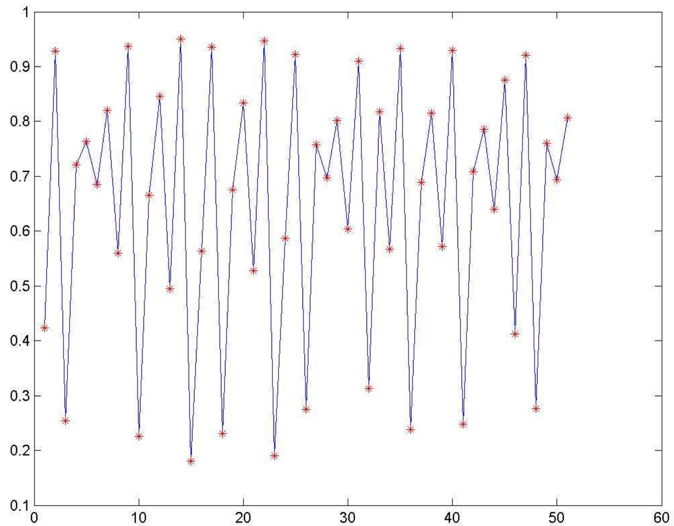
a = 2 Single point



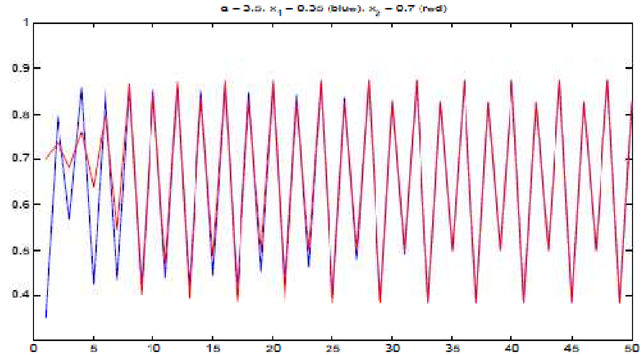
a = 3 Two points

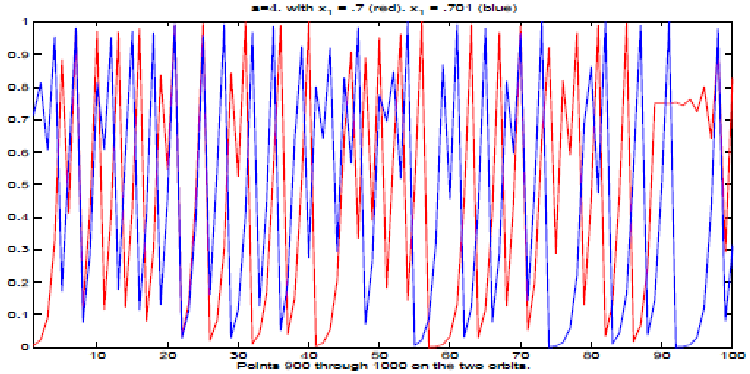


a = 3.55 8 points



a = 4 Chaos





The Problem

A set X and a function $f : X \to X$ are given.

For $x \in X$ define the orbit starting from x:

$\{x, f(x), f(f(x)), f(f(f(x))), \ldots\} = \{x, f(x), f^2(x), f^3(x), \ldots\}$

Notation: $f^k = f \circ f \circ f \circ \ldots \circ f$ (k-times)

$f^0(x) = x$, $f^{k+1}(x) = f(f^k(x))$ for $k \in \{0, 1, 2, \ldots\}$

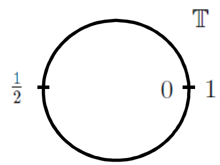
Determine the long time behaviour of the sequences $(f^k(x))_{k=0}^{\infty}$

$$f^k(x) \to ? \text{ as } k \to \infty$$

How does the answer depend on $x \in X$ and on $f : X \to X$?

A Model Chaotic System

$\mathbb{T} = \{x \in [0,1] : 0 \text{ and } 1 \text{ are identified}\}$



$t : \mathbb{T} \to \mathbb{T}$, $t(x) = 2x \pmod 1$

Dyadic (binary) expansion of $x \in \mathbb{T}$:

$x = 0.a_0a_1a_2a_3\ldots \qquad a_i \in \{0,1\}$

$t(x) = 0.a_1a_2a_3\ldots$

t is a left shift on the sequences of two symbols 0 and 1:

$t : (a_0a_1a_2a_3\ldots) \mapsto (a_1a_2a_3\ldots)$

For any 0-1 sequence $(s_n)_0^{\infty}$ the point $x = 0.s_0s_1s_2\ldots$

has the property $\quad t^k(x) \in [0, \frac{1}{2}]$ if $s_k = 0$

$\qquad t^k(x) \in [\frac{1}{2}, 1]$ if $s_k = 1$

Periodic points:

For the repeating sequence $x = 0.\overline{a_0 a_1 \ldots a_{n-1}}$
we have $t^n(x) = x$, that is, x is n-periodic

For given $x \in \mathbb{T}$ and $\epsilon > 0$ there is a periodic point $y \in \mathbb{T}$ with $|x - y| < \epsilon$

$x = 0.a_0 a_1 a_2 \ldots a_n a_{n+1} \ldots$
$y = 0.\overline{a_0 a_1 a_2 \ldots a_n}$
choose n so that $\frac{1}{2^n} < \epsilon$

A dense orbit

$x_* = 0.01\ 00\,01\,10\,11\ 000\,001\,010\,011\,100\,101\,110\,111 \ldots$
Given $y \in \mathbb{T}$ and $\epsilon > 0$ there is k so that
$t^k(x_*)$ and y are ϵ-close

Sensitive dependence on initial conditions

$x = 0.a_0 a_1 a_2 \ldots a_n a_{n+1} \ldots \in \mathbb{T}$ given
Choose $y = 0.a_0 a_1 a_2 \ldots a_n b_1 b_2 b_3 \ldots$
so that $b_1 = 1 - a_{n+1}$, $b_j = a_{n+j}$ for $j \geq 2$
Distance of x and y is $< \frac{1}{2^n}$
Distance between
$t^n(x) = 0.a_{n+1} a_{n+2} a_{n+3} \ldots$ and $t^n(y) = 0.b_1 b_2 b_3 \ldots$
is $\frac{1}{2}$

This property is called sensitive dependence on initial conditions (butterfly effect)

The dynamics is unpredictible:
a small change in the initial point results a large change after a certain number of iterations

A map $f : X \to X$ is called chaotic if

(i) the periodic points of f are dense in X;
(ii) there is a sensitive dependence on initial conditions;
(iii) for all open sets U, V in X there are k and $x \in U$ with $f^k(x) \in V$

The existence of a dense orbit implies (iii).

An order in the dynamics

Define $t^{-1}(H) = \{x \in \mathbb{T} : t(x) \in H\}$, $H \subset \mathbb{T}$

$t^{-1}(H) - \frac{1}{2}H \cup \left(\frac{1}{2} + \frac{1}{2}H\right)$

Define $|H|$ as the measure (length) of H

It is the usual length for a disjoint union of intervals, not defined for all subsets of $\mathbb{T}$.

We have $|H| = |t^{-1}(H)|$

$|\cdot|$ is an invariant measure for t

Birkhoff's ergodicity theorem implies

For $0 \le a < b \le 1$, and for almost all $x \in \mathbb{T}$

$$\lim_{n\to\infty} \frac{1}{n} \#\{k \in \{0, 1, \ldots, n-1\} : t^k(x) \in [a, b]\} = b - a$$

For almost all $x \in \mathbb{T}$, the sequence

$$\{x, t(x), t^2(x), \ldots\}$$

is uniformly distributed in $\mathbb{T}$.

Here almost all means:

for all $x \in \mathbb{T} \setminus S$ with $S \subset \mathbb{T}$ having the property:

for any $\epsilon > 0$ S can be covered by a sequence of open intervals of total length $< \epsilon$

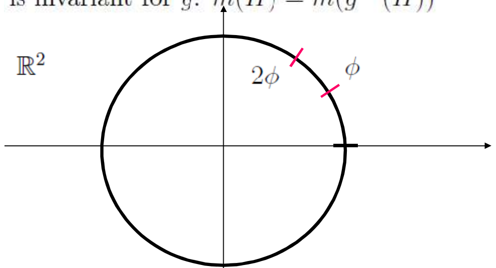
Consequence:

$S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$

$g : S^1 \to S^1 \qquad g(\phi) = 2\phi$

The measure $m(H) = \frac{\text{arclength}(H)}{2\pi}$

is invariant for g: $m(H) = m(g^{-1}(H))$



AIM: to get analogous result for $Q(x) = 4x(1-x)$, $x \in [0, 1]$

Project S^1 onto $[-1, 1]$:

$h : S^1 \to [-1, 1] \qquad h(\phi) = \cos\phi$

Then we have

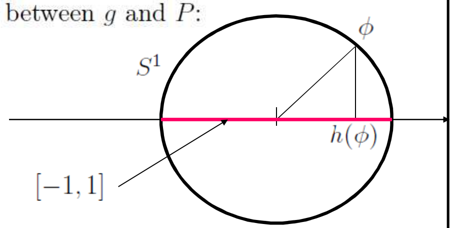
$h \circ g(\phi) = h(2\phi) = \cos(2\phi) = 2(\cos\phi)^2 - 1 = 2[h(\phi)]^2 - 1$

This motivates

$P : [-1, 1] \to [-1, 1] \qquad P(x) = 2x^2 - 1$

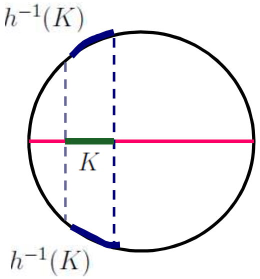
There is a semi-conjugacy between g and P:

$P \circ h = h \circ g$



Define a measure μ on $[-1,1]$ by

$\mu(K) := m(h^{-1}(K)) = \frac{\text{arclength}(h^{-1}(K))}{2\pi}$



It follows that

$\mu(K) = m(h^{-1}(K)) = m(g^{-1} \circ h^{-1}(K))$

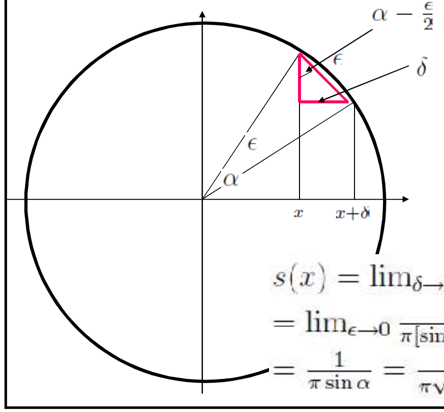
$= m(h^{-1} \circ P^{-1}(K)) = \mu(P^{-1}(K))$

Thus, P preserves μ.

Look for a density function of μ, that is:

$s : (-1,1) \to \mathbb{R}$ so that for $-1 \leq a < b \leq 1$

$$\mu([a,b]) = \int_a^b s(x)\,dx$$



$\delta = 2\sin\frac{\epsilon}{2}\sin(\alpha - \frac{\epsilon}{2})$

$= \sin\epsilon\sin\alpha - 2(\sin\frac{\epsilon}{2})^2\cos\alpha$

$s(x) = \lim_{\delta\to 0}\frac{[x,x+\delta]}{\delta} = \lim_{\delta\to 0}\frac{2\epsilon/(2\pi)}{\delta}$

$= \lim_{\epsilon\to 0}\frac{\epsilon}{\pi[\sin\epsilon\sin\alpha - 2(\sin\frac{\epsilon}{2})^2\cos\alpha]}$

$= \frac{1}{\pi\sin\alpha} = \frac{1}{\pi\sqrt{1-x^2}}$

P can be transformed to Q by another conjugacy:

$$\begin{tikzcd} {[-1,1]} \arrow[r, "P"] \arrow[d, "j"'] & {[-1,1]} \arrow[d, "j"] \\ {[0,1]} \arrow[r, "Q"] & {[0,1]} \end{tikzcd}$$

$P(x) = 2x^2 - 1$

$j(x) = \frac{1}{2}(1-x)$

$Q(x) = 4x(1-x)$

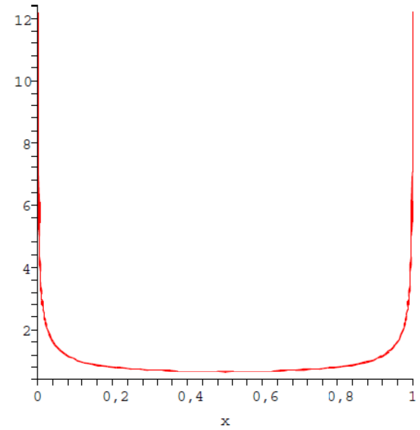
$j \circ P = Q \circ j$

Define a measure on $[0,1]$ by $\nu(K) = \mu(j^{-1}(K))$

ν is an invariant measure for Q

In particular

$\nu([a,b]) = \int_a^b \frac{1}{\pi\sqrt{y(1-y)}}\,dy$



For $Q(x) = 4x(1-x)$, $0 \le x \le 1$, we have:

$Q : [0,1] \to [0,1]$ *is chaotic.*

For $0 \le a < b \le 1$, *and for almost all* $x \in [0,1]$

$$\lim_{n\to\infty} \frac{1}{n} \#\{k \in \{0,1,\dots,n-1\} : Q^k(x) \in [a,b]\} = \int_a^b \frac{1}{\pi\sqrt{y(1-y)}}\, dy$$

Results for

$$Q_\lambda(x) = \lambda x(1-x), \quad 0 \le x \le 1, \quad 0 < \lambda \le 4$$

$\omega_\lambda(x)$ is the set of accummulation points of $(Q_\lambda^n(x))_{n=0}^\infty$

How does $\omega_\lambda(x)$ depend on x ?

For each $\lambda \in (0,4]$ *there is a unique set* $\Omega_\lambda \subset [0,1]$ *such that* $\omega_\lambda(x) = \Omega_\lambda$ *for almost all* $x \in [0,1]$.

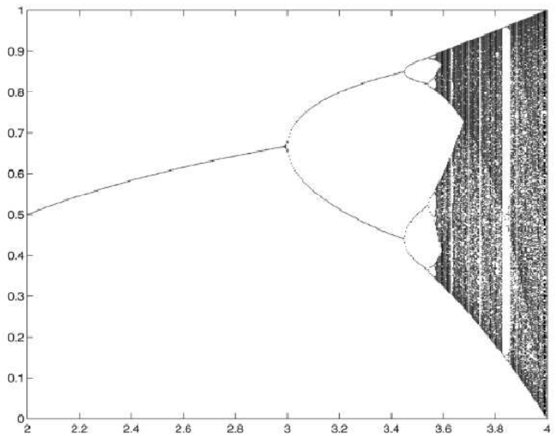
Ω_λ *can have the following structures:*

(i) a periodic cycle;

(ii) an attracting Cantor set of zero Lebesgue measure;

(iii) a finite union of intervals.

All cases do appear.



The structure in parameter space:

$\mathcal{P} := \{\lambda : \Omega_\lambda$ *is a periodic cycle*$\}$ *is dense in* $(0,1]$.
It consists of countable infinitely many nontrivial intervals.

$\mathcal{C} := \{\lambda : \Omega_\lambda$ *is a Cantor set*$\}$ *is a set of Lebesgue measure zero.*

$\mathcal{I} := \{\lambda : \Omega_\lambda$ *is a union of intervals*$\}$ *is a set of positive Lebesgue measure.*

Small perturbation of $\lambda = 4$ can lead to completely different asymptotics:

There is a sequence $(\lambda_n)_0^\infty$ with $\lambda_n \to 4$ so that Q_{λ_n} has an attracting n-cycle containing $1/2$ and only one point in $(\frac{1}{2}, 1]$.