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On the delta distribution: Historical remarks, Visualization of delta sequences

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In this paper we present the introduction of the notion delta distribution with special accent on its historical remarks and visualization, by using programme package *GeoGebra* and its dynamic properties.In order to make clear and understandable the properties of delta sequences we prepared 15 Figures, with the graphs of corresponding functions, mostly given in integral forms. The Figures are exported from *GeoGebra*, as eps files, and the properties of the notions connected with the delta sequences are explained geometrically. Under each Figure the hipelink leading to the corresponding animation in *GeoGebra* is given.

1. Introduction

Generalized functions or distributions constitute a space (in mathematical sense- locally convex vector space) adapted for solving partial differential equations. Actually there are many spaces of generalized functions, constituting of distributions, of ultradistributions, of hyperfunctions, of microfunctions,.... All of them serve as frameworks for various methods in the theory of partial differential equations, especially in the microlocal analysis.

Generalized function theory and the microlocal analysis are pretty complicated theories. At least, several years of PhD studies are necessary for understandig these fields. On the other hand, both theories are motivated by the explanations of natural phenomena and the foundation of natural laws and of many consequences which follow from these laws.

A great importance of the generalized function theory and a long period needed for the study require the introduction of basic notions of the theory in modern university courses of analysis on different levels of studies.

2. How delta had appeared?

The most important ”function” which was used, in the beginning in the fields of theoretical physics and today in almost all sciences, is the so called ”delta function”. In the end of 19. century, Heaviside, and later Dirac in the 20-es of the last century used the formal calculus with delta function although they did not have a clear knowledge about it. But they used this calculus with great success. This is evident from the fact that delta function, certainly, is not a function.

In their approach this is a ”function” with the properties

$$f(x) = 0,\ x \neq 0, \quad f(0) = \infty, \quad \int_0^\infty \delta(x)dx = 1.$$

For mathematicians, it is known that such a function does not exist (it must be zero, but then integral may not be equal to one).

Until the thirties of the last century, mathematical foundations were not done. Then in papers of Sobolev and later, around fifties, of Schwartz were given mathematically correct theories in which delta distribution has a complete and clear mathematical sense.

2.1. Applications

In other sciences delta distribution is used for the description of the white noise, the set of all sounds, or white light, the set of all light frequencies, or black holes, or delta shocks (which appear with earthquakes tsunamis...). We can see now-days that in the "up to day" literature of engineers the delta distribution appears as a non-voidable notion in the analysis of concrete models (equations), in the signal transfer, signal boost, in the picture detection and the texture removal of a picture and so on. How to find various school of fish in ocean by the use of sound signals which they produce but which have very low intensity. For example, it can be done by the use of the white noise signal, the stochastic process with the covariance equals $\delta(x-y)$; the extended sound signal is registered and then by the decomposition of the signal (we know the white noise) we conclude what kind of fishes is detected.

2.2. Equations

Spaces of generalized functions are the natural framework for various kinds of partial differential equations Models used for the description of various natural phenomena often (the most often) are formed through differential equations which show the dependence of the change of unknown variables from the known ones. (The measure of the change is the differential of the corresponding function; so the name differential equations.)

Equations often, by their names, express where they are used. Hamilton's equations of classical mechanics, Equation of the radioactive decrease in nuclear physics, wave equation, equation of the heat conduction, Ginzburg-Landau equation is used in modelling superconductivity, Maxwell's equations in the electromagnetism, Einstein's fields equations in general relativity, The Schrödinger equation, the heart of non-relativistic quantum mechanics, Navier Stockes equation in the fluid dynamics, Lotka- Volterra equation in population dynamics, Black-Sholes equation in mathematical finance, Vidal -Wolf advertising model, linear and nonlinear compartment models in pharmacokinetics.

2.3. History

Mathematicians and physicists to whom we are delightful the most for the theory of generalized functions are O. Heviside (1850-1925) who developed the operational calculus in solving differential equations, P. M. A. Dirac (1902-1984) who introduced (~ 1925) Dirac's "bra- cat" calculus in mathematical physics, S. Sobolev (1908-1986) who introduced (~ 1930) the notion of weak derivative in the investigation of weak solutions of hyperbolic systems and we are delightful the most to L. Schwartz (1920-2003) who developed (~ 1950) distribution theory and in general functional analysis in the direction of partial differential equations. He left to the society an excellent monograph which is studied even today within post-graduated studies in the theory of linear and nonlinear PDE (partial differential equations). Their theories had given a strong impulse to the theory of ΨDE (pseudodifferential equations) as well as to the theory of Fourier integral operators which were developed by Calderon, Zygmund, especially Hörmander, then Gelfand, Stein, Boni, and their collaborators.

The approach to the theory of generalized functions, developed by quoted mathematicians is so-called functional analysis approach (by the use of duality).

Another approach to generalized function theory, based on the theory of functions of many complex variables and the cohomology theory was introduced by M. Sato and his pupils T. Kavai and M. Kashiwara. Sato formulated (~ 1960) his hyperfunction theory and the theory of microfunctions (through germs of distributions and hyperfunctions).

In this context it is needed to mention the name of H. Komasu who developed ultradistribution theory and who connected approaches of Schwartz-Hörmander on one side and the approach of Japanese school of Sato, on another side.

3. Delta distribution

Functions act on points:

$$x \mapsto f(x), y \mapsto f(y)..., x \mapsto g(x), y \mapsto g(y)...$$

but points also act on functions

$$f \mapsto x(f) = f(x), g \mapsto x(g) = g(x)..., f \mapsto y(f) = f(y), g \mapsto y(g) = g(y)...$$

(This is a good point to understand functional analysis, in general)

So let us write

$$\langle f, x\rangle = f(x) = x(f).$$

In a space with a $\sigma - -$algebra $\mathcal{M}$ Dirac's measure $\delta_{x_0} : \mathcal{M} \to [0, \infty)$ is defined by

$$\delta_{x_0}(A) = 1 \text{ if } x_0 \in A, \ \ \delta_{x_0}(A) = 0 \text{ if } x_0 \notin A.$$

Let K be a space of smooth compactly supported test functions. Dirac's delta function $\delta_{x_0} : K \to \mathbb{C}$

$$\delta_{x_0}(\phi) = \langle\delta(x - x_0), \phi(x)\rangle = \phi(x_0).$$

If one consider the characteristic function of a set A, κ_A and model this function near the boundary points of A to be smooth but very close to it, then one can imagine that Dirac's measure equals Dirac's distributions.

3.1. Basic notions

3.1.1. Spaces of test functions

A norm in a vector space X over the field of complex numbers is a mapping $X \ni x \mapsto \|x\| \in [0, \infty)$ with the properties $(x, y \in X, \lambda \in \mathbb{C})$

1) $\|x\| = 0 \Leftrightarrow x = 0,$

2) $\|\lambda x\| = |\lambda|\|x\|,$

3) $\|x + y\| \leq \|x\| + \|y\|.$

The mapping $d(x, y) = \|x - y\|$ is a metric.

If $(X, \|\cdot\|)$ is complete, it is called a Banach space.

3.1.2. Spaces of test functions

Let $\|\cdot\|^{\sim}_{\mu}, \mu \in \mathbb{N}$, be different norms. Then

$$\|x\|_m = \sup_{\mu \leq m} \|x\|^{\sim}_{\mu} \text{ (or } \|x\|_m = \sum_{\mu=1}^{m} \|x\|^{\sim}_{\mu} \text{ or } \|x\|_m = (\sum_{\mu=1}^{m} \|x\|^{\sim 2}_{\mu})^{\frac{1}{2}})$$

is an increasing sequence of equivalent norms.

Define in $(X, \|\cdot\|_m, m \in \mathbb{N})$ a convergence structure by

$$x_\nu \to x_0, \nu \to \infty, \quad \text{if} \quad \|x_\nu - x_0\|_m \to 0, \nu \to \infty$$

for every $m \in \mathbb{N}$.

Then $(X, \|\cdot\|_m, m \in \mathbb{N})$ is called a Fréchet space if it is complete.

We can have scalar product in a vector space $V : V \times V \ni (x, y) \to (x|y) \in \mathbb{C}$ with the properties

1) $(x|x) \geq 0 \wedge (x|x) = 0 \Rightarrow x = 0$

2) $(\lambda x|y) = \lambda(x|y)$

3) $(x|y + z) = (x|y) + (x|z)$

4) $\overline{(x|y)} = (y|x)$.

Then with $\|x\| = (x|x)^{\frac{1}{2}}$ is defined a norm on V. If $(V, (\cdot|\cdot))$ is complete, it is called a Hilbert space.

Again with the family of scalar products $(x|y)^{\sim}_{\nu}, \nu \in \mathbb{N}$ we have equivalent sequence of norms and we can again assume that this sequence of norms is increasing.

Instead of norms we can have seminorms $p : X \to [0, \infty)$

1) $p(x) = 0 \to x = 0$

2) $p(\lambda x) = |\lambda| p(x)$

3) $p(x + y) \leq p(x) + p(y)$.

We say that an (increasing) sequence of seminorms $(p_m)_{m \in \mathbb{N}}$ defines a Fréchet structure in X if the space X with these seminorms is Hausdorff and complete.

Denote by K a compact set in $\mathbb{R}$ (say a closed interval). Then $C(K) = C^0(K)$- the space of continuous functions in K with the norm

$$\|\phi\|_{0,K} = \sup_{x \in K} |\phi(x)|$$

and $C^m(K)$- the space of functions having continuous derivatives up to the order $m \in \mathbb{N}$ with the norm

$$\|\phi\|_{m,K} = \sup_{x \in K, i \leq m} |\phi^{(i)}(x)|$$

are Banach spaces.

3.1.3. Examples of test spaces

- 1. Fréchet spaces $C^m(\mathbb{R})$. Let $C^m(\mathbb{R})$ be the space of continuous functions in $\mathbb{R}$ with continuous derivatives up to the order $m \in \mathbb{N}$.

  Then for any $K \subset\subset \mathbb{R}$ (K is compact in $\mathbb{R}$)

  $$p_{m,K}(\phi) = \sup_{x \in K, i \leq m} |\phi^{(i)}(x)|$$

  is a seminorm but not a norm in $C^m(\mathbb{R})$.

  Take $(K_\nu)_{\nu \in \mathbb{N}}$ to be increasing sequence of compact sets such that

  $$\bigcup_{\nu=1}^{\infty} K_\nu = \mathbb{R}, \quad K_\nu \subset K_{\nu+1}.$$

  Then $C^m(\mathbb{R})$ with the sequence of seminorms $(p_{m,K_\nu})_{\nu \in \mathbb{N}}$ is a Fréchet space.

- 2. Banach spaces $C_0^k(K)$ Let $\phi \in C(\mathbb{R})$. The support of a function ϕ is defined as
$$\text{supp}\phi = \overline{\{x; \phi(x) \neq 0\}}.$$
We define $C_0^k(K)$ to be the subspace of $C^k(\mathbb{R})$ containing functions ϕ which are supported by K. Then $\|\phi\|_{k,K} = \sup_{x\in\mathbb{R}} |\phi^{(i)}(x)|$ is a norm and $C^k(K)$ is a Banach space.
- 3. Fréchet space $C_0^\infty(K)$ If $k = \infty$, then $\phi \in C_0^\infty(\mathbb{R})$ means that ϕ has all derivatives continuous on $\mathbb{R}$ and $\text{supp}\phi \subset K$ for some compact set K.
Let $K \subset\subset \mathbb{R}$ be fixed. Define in $C_0^\infty(K)$
$$\|\phi\|_\nu = \sup_{x\in K, i\leq\nu} |\phi^{(i)}(x)|.$$
Then $(\|\cdot\|_\nu)_{\nu\in\mathbb{N}}$ is a sequence of norms and $C_0^\infty(K)$ is a Fréchet space with these norms.
- 4. Fréchet space $\mathcal{S}(\mathbb{R})$ Let $\phi \in C^\infty(\mathbb{R})$. Define a sequence of norms
$$\|\phi\|_k = \sup_{x\in\mathbb{R}, i\leq k} (1 + |x|^2)^{k/2} |\phi^{(i)}(x)|,\ k \in \mathbb{N}.$$
Then
$$\mathcal{S}(\mathbb{R}) = \{\phi \in C^\infty(\mathbb{R}); \|\phi\|_k < \infty\}$$
equipped with this sequence of norms is a Fréchet space. It is called the Schwartz space of rapidly decreasing functions.

Let us note that the definition of $\mathcal{D}(\mathbb{R})$ is not given in this article; for the calculus with delta distributions in this article the Schwartz space $\mathcal{S}(\mathbb{R})$ is sufficient.

3.1.4. Dual spaces

Let X be a test space (which means we have a norm or a sequence of norms in X).

Then X' is the space of linear continuous mappings $f : X \to \mathbb{C}$, i.e.
$$f(\alpha\phi + \beta\psi) = \alpha f(\phi) + \beta f(\psi)$$
and
$$\phi_\nu \to \phi \text{ in any norm implies} f(\phi_\nu) \to f(\phi)$$
For $(X, ||\cdot||)$, $f \in X'$ iff
$$\exists M > 0 \ :\ |f(x)| \leq M||x||, \quad x \in X \tag{$*$}$$
$$\begin{aligned} ||f||_{X'} &= \inf\{M :\ \text{such that } (*) \text{ holds}\} \\ &= sup_{x\neq 0} \frac{|f(x)|}{||x||_X} = sup_{||x||\leq 1}|f(x)| = sup_{||x||=1}|f(x)| \end{aligned}$$
For $(X, (||\cdot||_\nu)$, $f \in X'$ iff
$$\exists M > 0\ \exists \nu_0 \in \mathbb{N} \ :\ |f(x)| \leq M||x||_{\nu_0}$$
The same holds if we have seminorms.

Let
$$C_{\text{fin}} = \{f \in C(\mathbb{R}) : \lim_{|x|\to\infty} f(x) = 0\}$$
and
$$||f|| = \sup_{x\in\mathbb{R}} |f(x)|.$$
Then this is a Banach space and its dual $(C_{\text{fin}})' = \mathcal{M}$ is the space of Radon measures.

$\mathcal{S}' = L(\mathcal{S}, \mathbb{C})$ is the space of tempered distributions.

3.1.5. Examples of measures and distributions

Let f be a function on $\mathbb{R}$ so that $\int_{\mathbb{R}} |f(x)|\, dx < \infty$. Then

$$C_{\text{fin}} \ni \phi \mapsto \int_{\mathbb{R}} f(x)\phi(x)\, dx \quad : \quad \phi \mapsto \tilde{f}(\phi) =< f, \phi >$$

is continuous and linear.

So, $\tilde{f} \in \mathcal{M}$ and $\tilde{f} \in \mathcal{S}'$. Note, $\mathcal{M} \subsetneq \mathcal{S}'$.

If $f(x) = P_n(x)$ is a polynomial of degree n, then

$$\tilde{f}(\varphi) = \int_{\mathbb{R}} f(x)\varphi(x)dx, \qquad \varphi \in \mathcal{S}$$

is an element of $\mathcal{S}'$ but not of $\mathcal{M}$.

Any function f bounded by a polynomial defines $\tilde{f} \in \mathcal{S}'$ by

$$\tilde{f}(\varphi) = \langle \tilde{f}, \varphi \rangle = \int_{\mathbb{R}} f(x)\varphi(x)dx.$$

3.1.6. The Dirac distribution

But there exist measures and tempered distributions which are not defined by ”ordinary” functions.

For example,

$$\varphi \mapsto \varphi(0) = \langle \delta, \phi \rangle, \quad \varphi \in C_{\text{fin}} \quad \text{is in } \mathcal{M}$$
$$\varphi \mapsto \varphi(0) = \langle \delta, \phi \rangle, \quad \varphi \in \mathcal{S} \quad \text{is in } \mathcal{S}'$$

3.1.7. Differentiation in $\mathcal{S}'$

Differentiation in $\mathcal{S}'$ is defined by

$$\langle f', \varphi \rangle = -\langle f, \varphi' \rangle.$$

If f, f' are continuous and bounded by polynomials, then

$$f \mapsto \tilde{f} \in \mathcal{S}' \qquad f' \mapsto \tilde{f}' \in \mathcal{S}'$$

$$\langle \tilde{f}', \varphi \rangle = -\langle \tilde{f}, \varphi' \rangle = -\int_{-\infty}^{\infty} f(x)\varphi'(x)dx$$

$$\langle \tilde{f}', \varphi \rangle = \int_{-\infty}^{\infty} f'(x)\varphi(x)dx.$$

Since $\int_{-\infty}^{\infty} f'(x)\varphi(x)dx = -\int_{-\infty}^{\infty} f(x)\varphi'(x)dx$, we have $\tilde{(f')} = (\tilde{f})'$.

3.1.8. The Heaviside function

The Heaviside function

$$H(x) = \begin{cases} 1, & x > 0 \\ 0, & x \leq 0 \end{cases}$$

defines a distribution.

$$H(x) \mapsto \tilde{H}(x)$$

$$\langle \tilde{H}, \varphi \rangle = \tilde{H}(\varphi) = \int_{-\infty}^{\infty} H(x)\varphi(x)dx = \int_{0}^{\infty} \varphi(x)dx.$$

We have $\tilde{H}' = \delta$:

$$\langle \tilde{H}', \varphi \rangle = -\langle \tilde{H}, \varphi' \rangle = -\int_{0}^{\infty} \varphi'(x)dx = \varphi(0) = \langle \delta, \varphi \rangle.$$

3.1.9. Convolution

For $\varphi, \psi \in L^1_{\text{loc}}(\mathbb{R})$ let

$$(\varphi * \psi)(x) = \int_{\mathbb{R}} \varphi(x-t)\psi(t)dt = \int_{\mathbb{R}} \varphi(t)\psi(x-t)dt$$

If φ, ψ are functions equal to zero on $(-\infty, 0)$ then

$$(\varphi * \psi)(x) = \int_{0}^{x} \varphi(t)\psi(x-t)dt$$

4. On the visualization of the delta sequences

4.1. The function given by integral

In this part we shall draw the graph of the function given by the integral

$$F(t) = \int_{0}^{t} f(x)dx, \qquad t > 0, \tag{1}$$

for given function f. Let us remark that in relation (1) the function F depends on variable t, but in order to make visualization of definite integral we have to take care about the variable x, too. This problem can be solved by using dynamic programme packages, as for example, *Mathematica, GeoGebra,* and so on. By using *sliders* in mentioned programme packages one can consider parameters besides the variables, and the parameters can be changed *almost continuously*. In our work we shall use the programme package *GeoGebra*. Since the variable x, in relation (1), can not be fixed, we consider t, as a parameter.

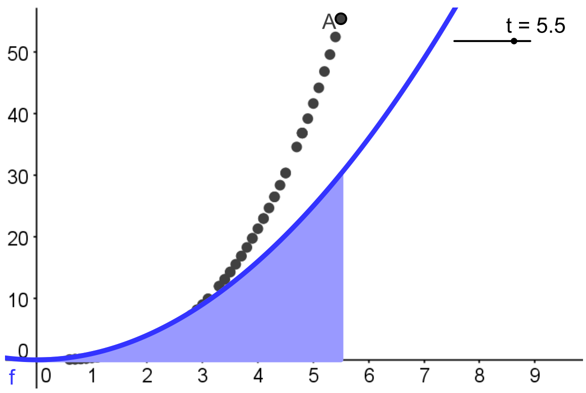

Figure 1: Function defined by integral

This means that the slider has to be introduced, first, to be denoted by t, and then the value of the function $F(t) = \int_0^t f(x)dx$ can be determined for each value of t. Since this definite integral geometrically represent area, for each t, it can be visualized, too. The graph of the function F can be obtained by using the trace of the point $(t, P(t))$, where P represent the corresponding area under the graph of f.

On Figure 1 we consider the function $f(x) = x^2$, $x > 0$, and the area (under the parabola) given by

$$F(t) = \int_0^t x^2 dx = \frac{1}{3}t^3, \quad \text{for fixed} \quad t = 4.2,$$

which is colored. By changing t, (Figure 2) using the corresponding slider, one obtains different values for area. If we introduce the point $A(t, P(t))$, and include the trace, then we shall get the points on the graph of the function F.

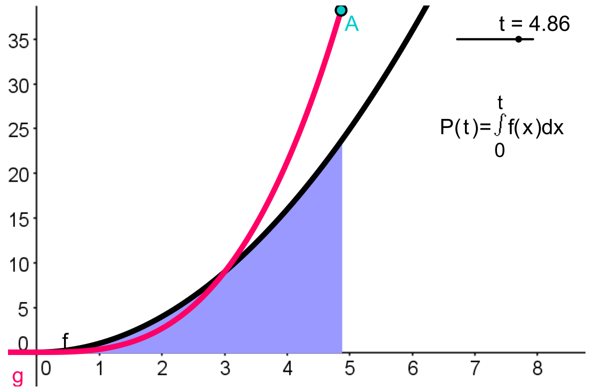

Figure 2: Function defined by integral2

It is interesting to consider the visualization of the logarithmic function given in form of definite integral

$$F(t) = \int_1^t \frac{dx}{x}, \qquad t > 0. \tag{2}$$

On Figure 3 it is drawn the graph of the function

$$f(x) = \begin{cases} 0 & x < 1 \\ \frac{1}{x} & x \geq 1 \end{cases},$$

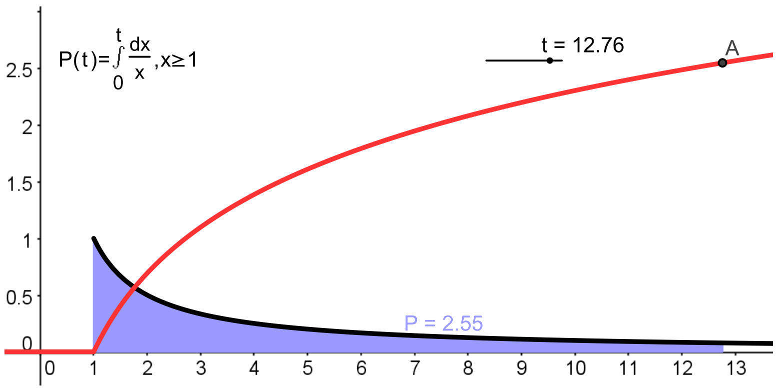

Figure 3: Logarithmic function, $x > 1$

and the slider $t \in [1, 50]$ is introduced. It is shown that the point $A(t, f(t))$ with the trace belongs to the curve $F(t) = \ln t$, $t \in [1, \infty)$. The graphs of these piecewise defined functions can be drawn by using command *if*.

On Figure 4 we consider the function given by integral (2), for $0 < t \leq 1$. It is interesting to remark that the in this case the function F is negative, but the corresponding area is drawn as the positive one.

By clicking on the corresponding links one get directly the visualizations in GeoGebra. The dynamic properties of this package can be used to show the visualization of resented process. Namely, by hanging the values of t, (using slider t) one can follow the trace of the point $A(t, f(t))$.

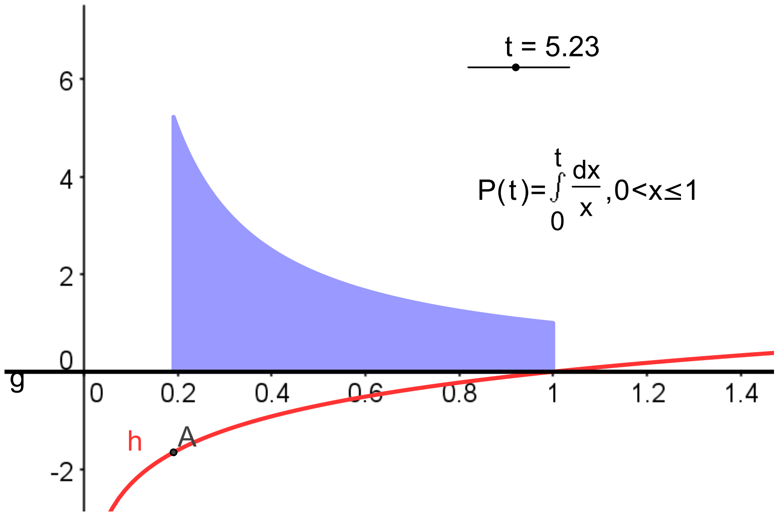

Figure 4: Logarithmic function, $0 < x < 1$

On Figure 5 it can be seen that the graph of exponential function $f(x) = e^x$, $x \in R$ is obtained as the inverse curve of the graph of logarithmic function. Namely, each point A' is the inverse one of the point A, which belongs to the graph of logarithmic function.

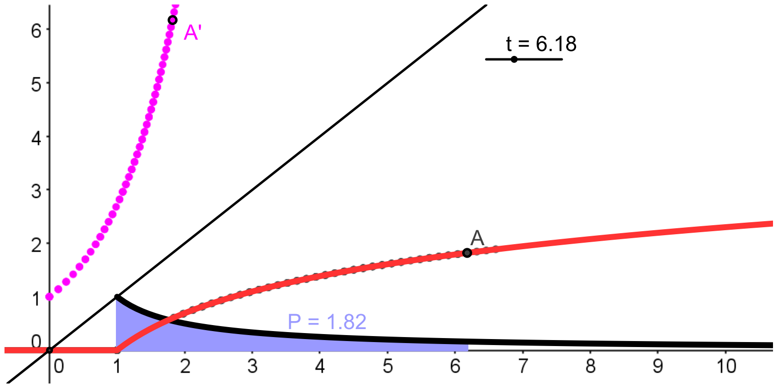

Figure 5: Exponential function, $x > 1$

5. On the visualization of the convolution of the functions

Let us consider the convolution of two functions locally integrable functions f and g given by

$$(f * g)(t) = (g * f)(t) = \int_0^t f(t-x)g(x)dt. \tag{3}$$

As it is told before the functions f and g are equal zero for $x \in (-\infty, 0)$.

The function $f * g$ is defined by definite integral, representing area under the graph of the function $f(t-x)g(x)$.

For example, if we take the the functions $f(x) = e^x$, $g(x) = x^2$, then their convolution can be considered as:

$$P(t) = \int_0^t e^{t-x}x^2 dx = e^t \int_0^t e^{-x}x^2 dx = e^t \left(2 - 2te^{-t} - t^2e^{-t} - 2e^{-t}\right) = 2e^t - 2t - t^2 - 2, \quad t > 0.$$

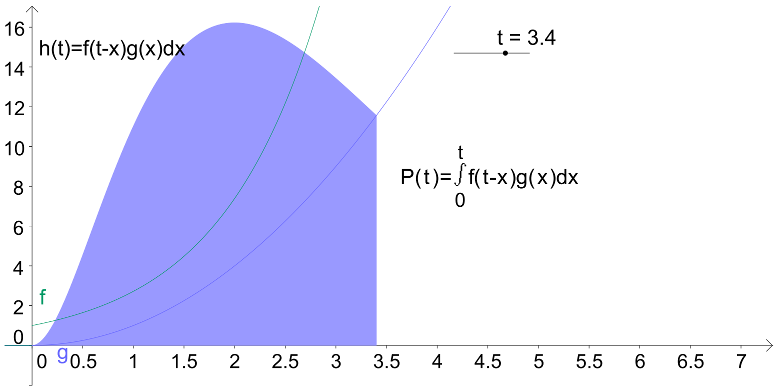

Figure 6: The Convolution of two functions

On Figure (6) we consider the graphs of function f, g, and $f * g$, and the value of P, for fixed $t = 3.4$. Let us remark that in this case, the area representing convolution P is the area under the graph of function

$$h(t) = f(t-x)g(x) = e^t(e^{-x}x^2).$$

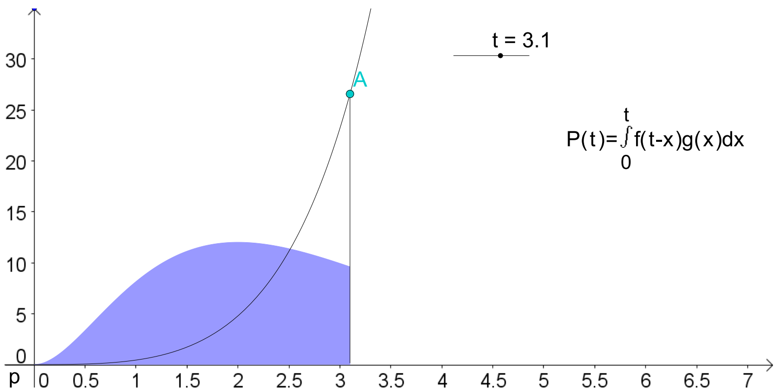

Figure 7: The Convolution of two functions

On Figure (7), the graph of function $r(t) = 2e^t - 2t - t^2 - 2$, and the trace of the point $(t, P(t))$, are drawn. Of course, the point A belongs to the graph of the r.

5.1. On the visualization of Delta sequences

Delta sequences are functional sequences $(\delta_n(x))_{n\in N}$ satisfying the following:

1. $\lim\limits_{n\to\infty} \delta_n(x) = \delta(x)$;
2. $\lim\limits_{n\to\infty} \int_{-\infty}^{\infty} \delta_n(x)g(t-x)dx = g(x)$, for local integrable function g.

In this paper we shall consider the following delta sequences:

$$\delta_n(x) = \begin{cases} n^2 - nx, & x \in [0, \frac{1}{n}], \quad 0, \quad x > \frac{1}{n} \\ n - n^2x, & x \in [-\frac{1}{n}, 0], \quad 0, \quad x < -\frac{1}{n} \end{cases}, \tag{4}$$

$$\delta_n(x) = \frac{\sin((n+1/2)x)}{2.4\pi sin(x/2)}; \quad x \in [-\frac{2\pi}{2n+1}, \frac{2\pi}{2n+1}]. \tag{5}$$

$$\delta_n(x) = \frac{1}{2}ne^{-n|x|}; \tag{6}$$

$$\delta_n(x) = \frac{1}{\pi}\frac{n}{e^{nx} + e^{-nx}}, n \in \mathbb{N}. \tag{7}$$

It can be proved that all previous functional sequences satisfy the following conditions:

1. $\lim\limits_{n\to\infty} \delta_n(0) = \infty$;
2. $\int_{-\infty}^{\infty} \delta_n(x)dx = 1$, for each n.

Next, we shall show this properties visually by using package *GeoGebra*.

On Figure 8 we present the example (4), and we denote by h the term of corresponding delta sequence, for fixed n. It can be seen that $P = 1$, and $poly1 = 1$, means that

$$\int_{-1/n}^{1/n} h(x)dx = 1, \quad \text{for each} \quad n.$$

By increasing n the $y-$coordinate of vertex A, of triangle ABC is becoming greater and is tending to infinity, but the $x-$coordinate of vertices B and C are tending zero.

Let us remark that the functions of two or more variables, in package *GeoGebra*, are considered as the function with only one variable, because the other variables are treated as parameters and they are presented with the sliders. Therefore, in our visualization process, we shall write the same notions as it is done in *GeoGebra*, in order to make it easier.

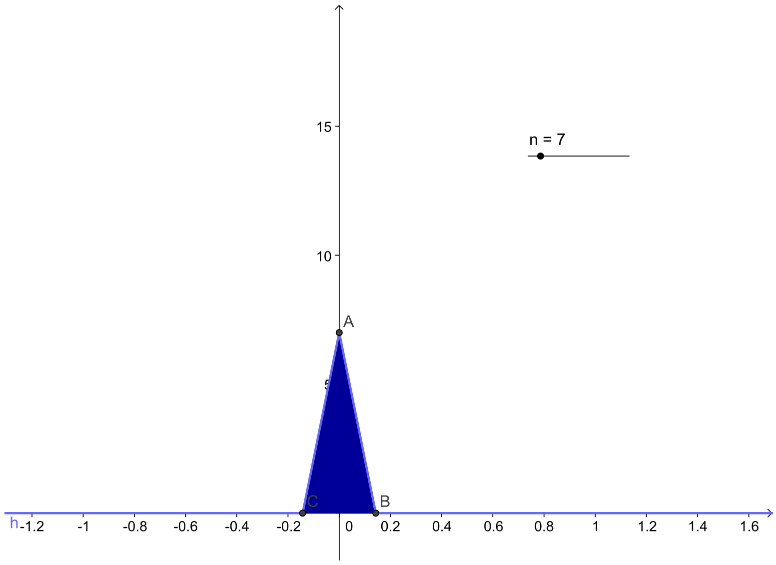

Figure 8: The delta sequence given by (4)

On Figure 9 it is visually shown, for the example (4), that

$$\int_{-1/n}^{1/n} h(x)g(t-x)dx \to g(t), \tag{8}$$

when n is increasing, for $g(x) = x^2$.

Namely, first the graph of the function $f(x) = h(x)g(t-x)$, for fixed parameters n and t, is drawn, and then the integral, representing convolution (8), denoted by P is drawn, coloring the area under the graph of h. Further, the point $D(t,P)$, for fixed n and t is drawn.

For example, for $n = 2$ we can seen the points ordered by A, with the trace included, are very closed to the graph of g. It can be followed, by using sliders, that the point are more closer to the same graph, with increasing n.

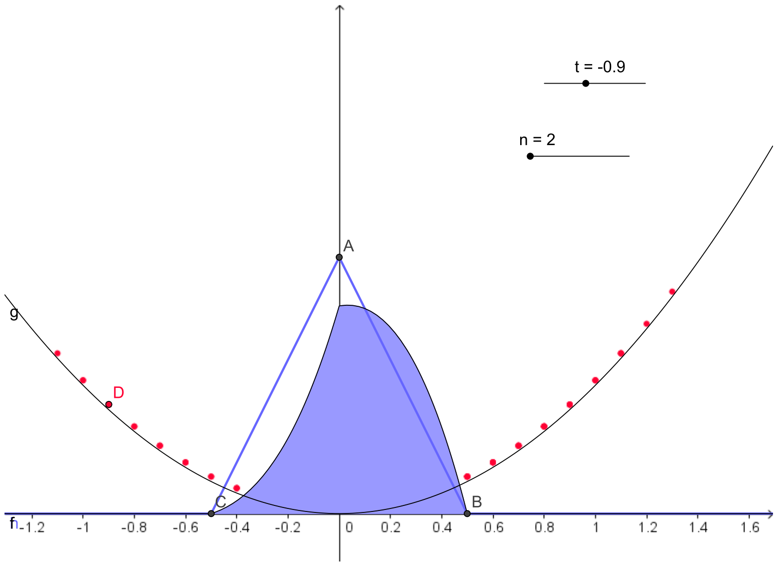

Figure 9: The delta sequence given by (4)

Let us remark that, in all previous examples, one can change the function g and make the same dynamical process.

On Figure 10 it is visually shown, for the example (5), that

$$\int_{-\infty}^{\infty} h(x)dx = 1, \quad \text{for each} \quad n,$$

and the point $A(t, P_1 + P_2)$, with the trace, for $n = 1$..

In this case the functions δ_n are not defined for $x = 0$ and therefore we had to consider separately P_1 and P_2

$$P_1 = \int_{-\frac{2\pi}{2n+1}}^{-.0001} h(x)g(t-x)dx, \quad \text{and} \quad P_1 = \int_{.0001}^{\frac{2\pi}{2n+1}} h(x)g(t-x)dx,$$

for $n = 1$ Figure 10, and for $n = 6$, Figure 11.

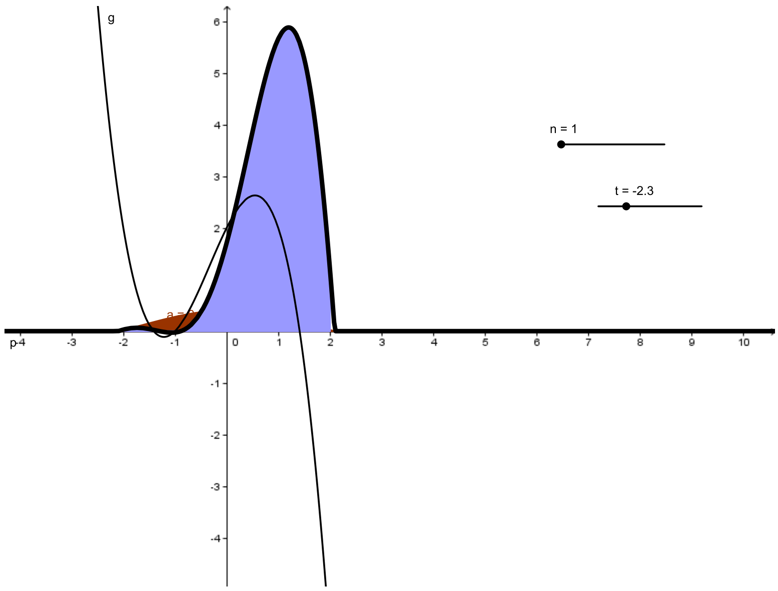

Figure 10: The delta sequence given by (5), for $n = 1$

On Figure 12 we applied the explained procedure for delta sequence given by

$$\delta_n(x) = \frac{sin(nx)}{3.69x},$$

and for the function

$$g(x) = \frac{x^2 - 1}{x^2 + 1}.$$

We determine the graph of the the function, which can be considered as the approximate one,for the function g, for $n = 4$, by using *Spreadsheet*, and *fitting curve* property of *GeGebra* package.

Namely, for $n = 4$ we denoted the points, formed the *list1* and using *fitting curve* we determined the ”green curve”, which can represent the approximation of the graph of function g.

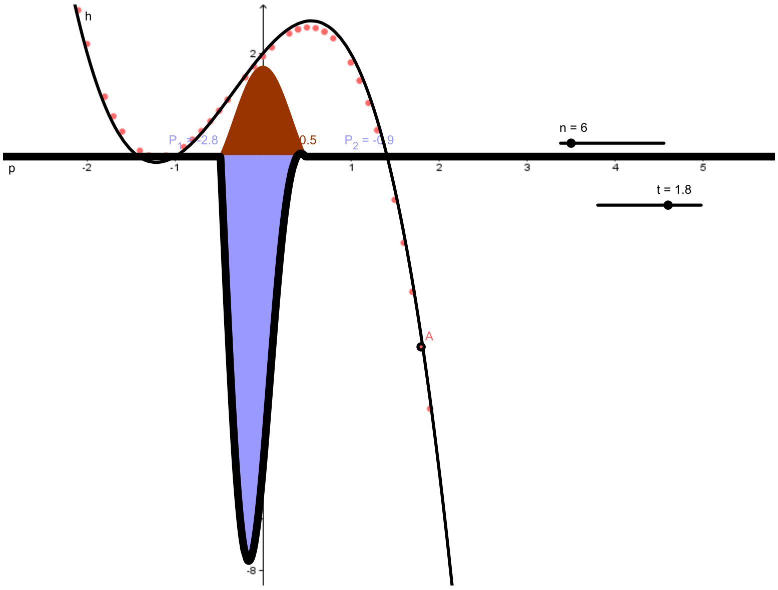

Figure 11: The delta sequence given by (5), for $n = 6$

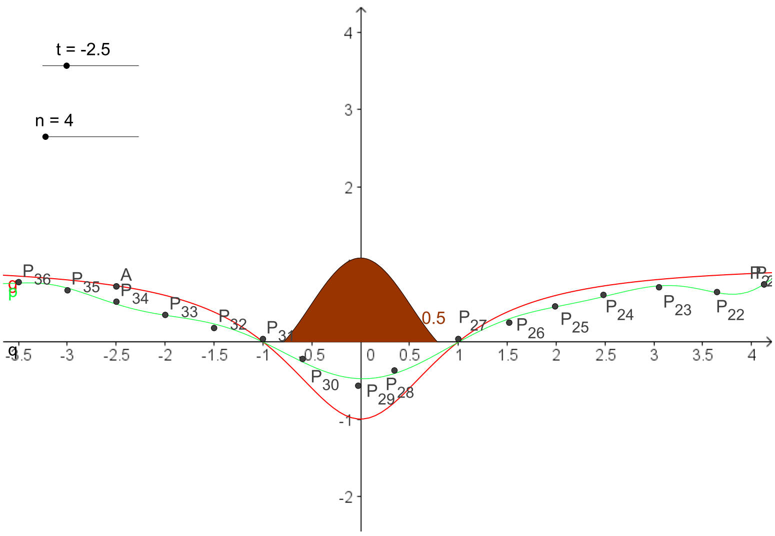

Figure 12: The approximations obtain by using delta sequence

On Figure 13 we consider the delta sequence given by

$$\delta_n(x) = \frac{1}{2} n e^{-n|x|},$$

and the function

$$g(x) = \frac{x^2 - 1}{x^2 + 1},$$

and we obtain the corresponding points for $n = 1$ Figure 13, and $n = 5$ Figure 14

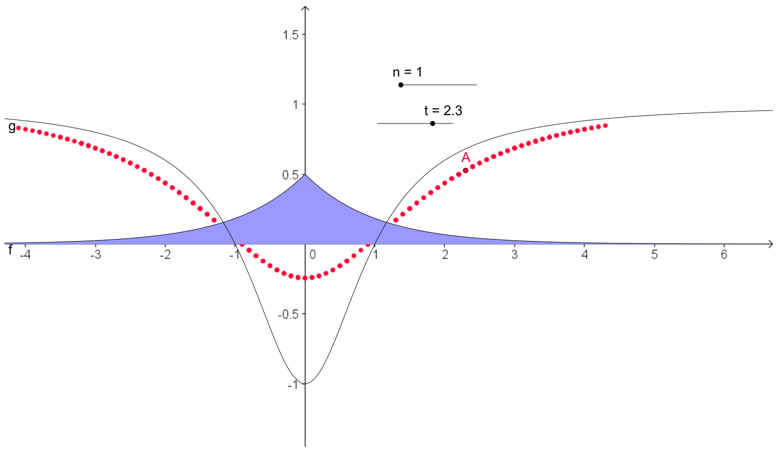

Figure 13: $n = 1$

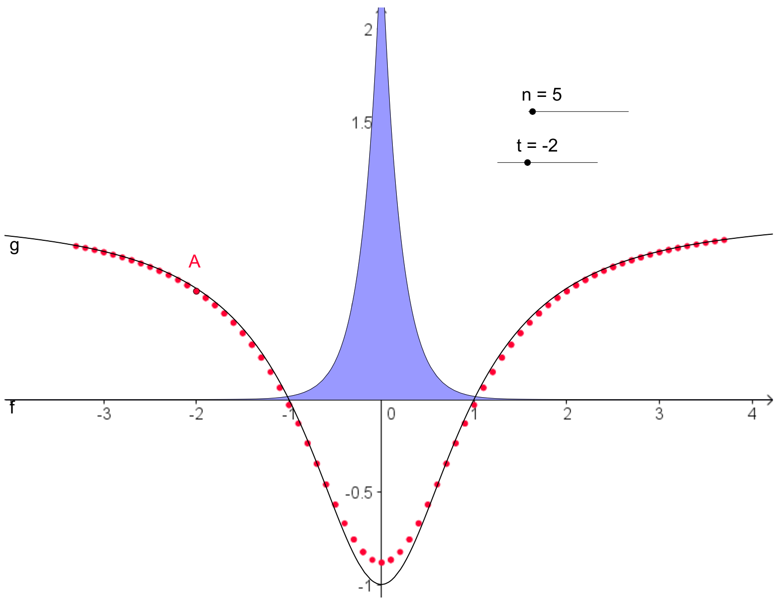

Figure 14: $n = 4$

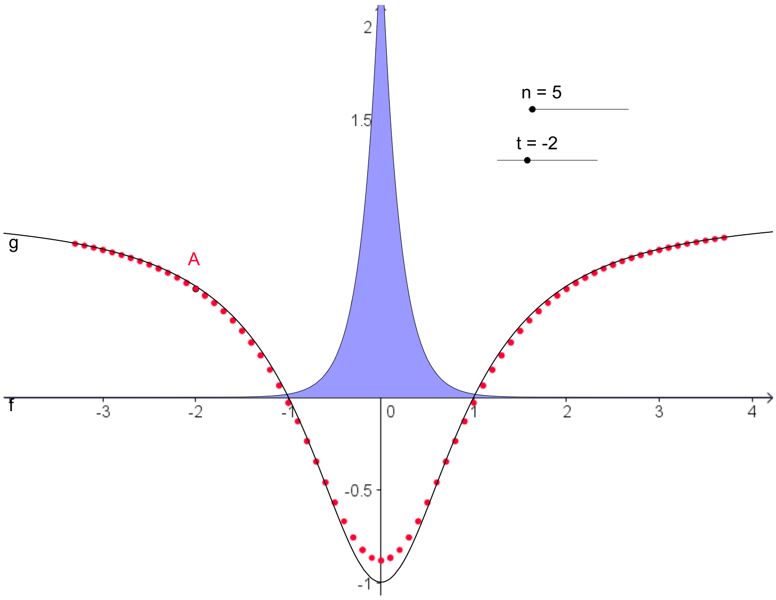

Figure 15: The delta sequence given by (7)

On Figure 15 we consider the delta sequence given by

$$\delta_n(x) = \frac{1}{\pi}\frac{n}{e^{nx} + e^{-nx}}, n \in \mathbb{N}.$$

and the function $g(x) = e^x$, and draw the corresponding graph for $n = 1$ and $n = 10$.

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