Introduction to nonlinear wave models

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We will consider some simpler one-space-dimensional nonlinear problems and their simple wave solutions. We were dealing mainly with conservation laws which represents basic physical laws – the most important building blocks in science. The prototype is so called the *continuity equation* or he law of mass conservation. We start with it right now.

1. Some physical examples

1.1. Introduction

There is no precise definition of wave, but one can describe it as a signal traveling from one place to another one with clearly visible speed. The signal can be any disturbance, like some kind of maxima or change of some quantity.

In a lot of physical problems a disturbance in a fluid or material can arise. We use its main characteristics to be

density	....	$\rho(x,t)$
flux	....	$q(x,t)$
velocity	....	$u(x,t) = \frac{q(x,t)}{\rho(x,t)} = \frac{\text{flux}}{\text{density}}$.

The relations between them will be handled with some constitutional relation (it is not a physical law, but the relation obtained by a n approximation or an experiment).

Take any space-time interval $[x_1, x_2] \times [t_1, t_2]$. The law of mass conservation (or principle of mass/matter conservation) says that "that the mass of a closed system will remain constant over time". Thus, the change of mass during the above time interval,

$$M(t_2) - M(t_1) := \int_{x_1}^{x_2} \rho(x,t_2)\mathrm{d}x - \int_{x_1}^{x_2} \rho(x,t_1)\mathrm{d}x$$

has to balanced by a difference of inflow and outflow in the end points of the space interval

$$(\Delta M)([x_1,x_2]) := \int_{t_1}^{t_2} q(x_1,t)\mathrm{d}t - \int_{t_1}^{t_2} q(x_2,t)\mathrm{d}t.$$

Note that the single points $\{x_1\}$ and $\{x_2\}$ represents boundary in 1-D case. We have a curve in 2-D case and as surface in 3-D case. The mass conservation law says that

$$M(t_2) - M(t_1) = \Delta Q([x_1,x_2])(\text{ “mass difference equals flow”})$$

for any space-time interval.

So, the real problem above is an integral equation on an arbitrary region. There are not so much of mathematical tools adopted for such problems. That is a reason why we will make the following transition to a partial differential equation. Let us fix (x, t) for a moment and multiply

$$\int_{x_1}^{x_2} \rho(x,t_2) - \rho(x,t_1)\mathrm{d}x = \int_{t_1}^{t_2} q(x_1,t) - q(x_2,t)\mathrm{d}t$$

with $\dfrac{1}{t_2 - t_1}$ and let $t_1, t_2 \to t$. Then

$$\int_{x_1}^{x_2} \lim_{t_2 - t_1 \to 0} \frac{\rho(x,t_2) - \rho(x,t_1)}{t_2 - t_1}\mathrm{d}x = \lim_{t_2 - t_1 \to 0} \frac{\int_{t_1}^{t_2} q(x_1,t) - q(x_2,t)\mathrm{d}t}{t_2 - t_1}$$

that gives

$$\int_{x_1}^{x_2} \frac{\partial \rho}{\partial t}(x,t)\mathrm{d}x. = q(x_1,t) - q(x_2,t).$$

Dividing that equality with $x_2 - x_1$ and letting $x_2 - x_1 \to 0$ we get the following PDE

$$\rho_t + q_x = 0 \tag{1}$$

for every point (x, t).

So, we have single equation with two variables. In fluid dynamics one knows that $q = \rho u$ but one will see some other possibilities bellow. In any case, we have two ways to deal with the above problem:

- We can give a constitutive relation between the variables (ρ and q). That closes the system and we can proceed with its solving.
- We can use another physical law (a conservation of momentum or energy for example) having the same variables. That extends the equation into a system.

We will present both possibilities with some simple but also important models in the next few sections.

1.2. A homogeneous flux models

Homogeneous relation between ρ and q is the simplest one: $q = q(\rho)$. Denote $c(\rho) = q'(\rho)$. The above equation now reads

$$\rho_t + c(\rho)\rho_x = 0 \tag{2}$$

provided q is regular enough.

First, let us note that the characteristics for (2) are given by the following ordinary differential equations

$$\gamma : \frac{\mathrm{d}x}{\mathrm{d}t} = c(\rho).$$

Since we are dealing with a conservation law (right-hand side of the equation equals zero), the curves given by γ are straight lines, i.e. speed of a wave, $c(\rho)$, is constant. That constant is determined by the initial data. So, if characteristics do not cross each other, that is the way to solve the initial data problem.

1.3. Traffic flow

We shell now present one example that is particularly interesting because it is a rare case of quite realistic one-equation model. The most of the models are systems (gas dynamics models are the main examples). In this model, the flow velocity

$$u(\rho) = \frac{q(\rho)}{\rho}$$

is obviously decreasing function with respect to ρ ("lot of cars on the street means low velocity"). It take values from a maximum one, at $\rho = 0$, to zero, as $\rho \to \rho_y$. The later constant denotes the maximal possible car density (when vehicles touches one another). The flux q is therefore a convex function (see Fig. 1) and has a maximal value q_m for some density ρ_m, while $q(0) = q(\rho_y) = 0$.

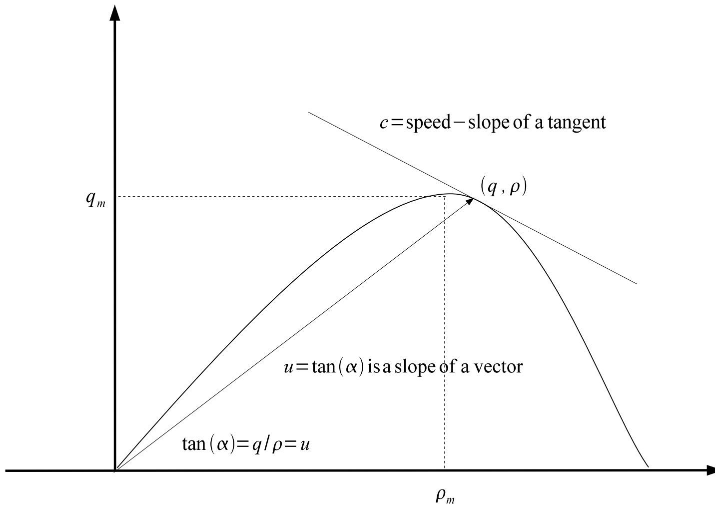

Figure 1: Flux function for traffic flow model

After observations made in Lincoln tunnel, New York (the first one relevant in the field), the experimental data for one right of way are: $\rho_y \approx 225 \frac{\text{vehicles}}{\text{mile}}$, $\rho_m \approx 80 \frac{\text{vehicles}}{\text{hour}}$. (The maximum flow for the above data could be obtained for car speeds $q_m \approx 20 \frac{\text{miles}}{\text{hour}}$).

A rough model for more than one right of way can be obtained by multiplying the above values with their multiplicity.

We supposed that q depends only on ρ, $q := q(\rho)$. Then the speed of waves (the speed of characteristics, too) is given by

$$c(\rho) = q'(\rho) = u(\rho) + \rho u'(\rho).$$

Since $u'(\rho) < 0$, it is less than flow velocity. It means that drivers can see a disturbance ahead.

In this particular case, speed of waves c is speed of cars, and a flow velocity is an average velocity of motion of a road relative to all of cars. Let us note that $c > 0$ for $\rho < \rho_m$ (cars are moving faster than average if density is small) and $c < 0$ for $\rho > \rho_m$ (opposite case: high density of the cars has lower speed than average).

Greenberg's model for the above tunnel, the more realistic than the first one, is determined in the following way: $q(\rho) = a\rho \log \frac{\rho_j}{\rho}$, $a = 17.2 \frac{m}{h}$, $\rho_j = 228 \frac{v}{m}$. $\rho_m = 83 \frac{v}{m}$, $\rho_m = 1430 \frac{v}{h}$. Logarithmic function definitely does not approximate states in a neighborhoods of the point $\rho = 0$ in a good way, but this is practically not interesting case, anyway.

A solution to the above problem is just illustrated in the Fig. 2.

1.4. Sedimentation in a river, chemical reactions

We are continuing with the single equation models with a bit more complicated model. It describes exchange processes between two materials, one us usualy taken to be a fluid while the other one is solid. The main real

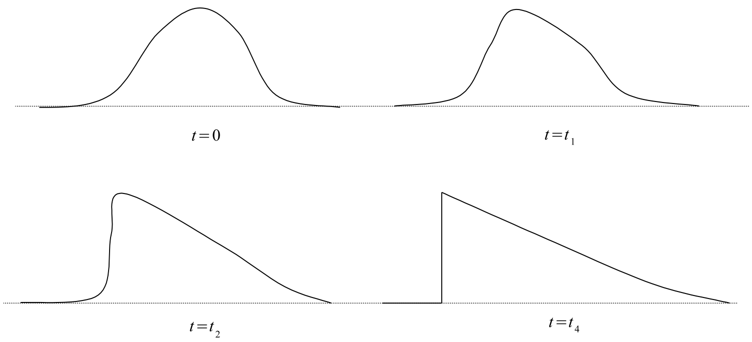

Figure 2: Density of cars

examples are chromatography and a model describing mutual influence of river-bed and fluid in the river, i.e. sedimentation transport, more precisely.

Denote

ρ_1	....	density of a fluid
ρ_0	....	density of a solid material.

Then, the density is given by

$$\rho = \rho_0 + \rho_1$$

and flux by $q = u\rho_1$, where u is a fluid speed. Conservation of mass law is given by

$$(\rho_1 + \rho_0)_t + u\rho_{1x} = 0,$$

with the supposition that fluid speed is a constant.

Reaction between these two materials are given by

$$\frac{\partial \rho_0}{\partial t} = k_1(A_1 - \rho_0)\rho_1 - k_2\rho_0(A_2 - \rho_1),$$

where k_1 and k_2 are coefficients depending on a reaction speed, and A_1, A_2 are constants depending on material specifications (both solid and fluid ones).

Let us take a special case, so called quasi-equilibrium, when changes of solid material density due chemical reactions are neglected, i.e.

$$\frac{\partial \rho_0}{\partial t} = 0.$$

We shall also suppose that space-time position is negligible, i.e.

$$\rho_0 = r(\rho_1).$$

Then we have the following system

$$\rho_{1t}(1 + r'(\rho_1)) + u\rho_{1x} = 0$$

i.e.

$$\rho_{1t} + \frac{u}{1 + r'(\rho_1)}\rho_{1x} = 0.$$

In some models, one can take

$$r(\rho_1) = \frac{k_1 A_1 \rho_1}{k_2 B + (k_1 - k_2)\rho_1}.$$

Equation which describes waves in this case follows from law of mass conservation. In general, flux is given by $q = \rho u$, where $u \neq \mathrm{const}$, so we need one more equation (for speed u).

1.5. Shallow water equations

That is our main example for systems of conservation laws. Later on we shall use it for demonstration of the main wave type solutions: shocks and rarefaction waves. The first step is its construction from the physical model. Let us fix some notation first.

ρ .... height of water level – its depth ($\approx$ density)
u .... speed of water flow

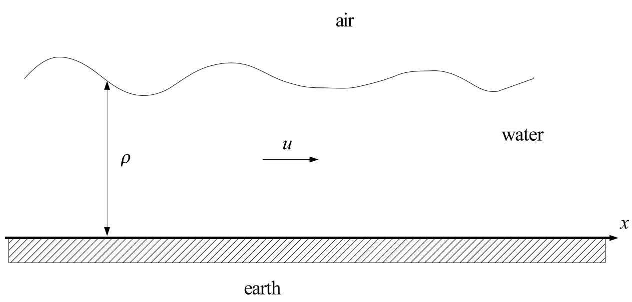

Figure 3: Shallow water

This model is used for description of river flow when depth is not so big (in the later case one can safely take that the depth equals infinity). It can be also used for flood, sea near beach, channel flow, avalanche,...

Basic assumption in this model is that a fluid is incompressible and homogenous (forming of "waves", moving of a water visible on its surface, is possible). Bottom of a river is not necessary flat, but for a flat one equations are homogenous – flux is independent of space-time coordinates. That eases finding global solutions to a system.

Mass conservation law gives

$$\rho_t + (\rho u)_x = 0. \tag{3}$$

In order to solve the above equation we shall introduce new partial differential equation involving the speed u and Newton's second law:

$$(mu)^{\cdot} = f \text{ ("force = impuls change per time").}$$

Take a space interval $[x_1, x_2]$ during a time interval $[t_1, t_2]$. Then

$$\int_{x_1}^{x_2} \rho(x,t_2)u(x,t_2)\mathrm{dx} - \int_{x_1}^{x_2} \rho(x,t_1)u(x,t_1)\mathrm{dx}$$
$$= \int_{t_1}^{t_2} \big(\rho(x_1,t)u^2(x_1,t) - \rho(x_2,t)u^2(x_2,t)\big)\mathrm{dt}$$
$$+ \int_{t_1}^{t_2} \big(p(x_1,t) - p(x_2,t)\big)\mathrm{dt}$$

"impuls change per time = kinetic energy + force due to preassure"

Contraction of a time-space interval: $t_1, t_2 \longrightarrow t$ and $x_1, x_2 \longrightarrow x$ for some pint (x,t), gives the following PDE

$$(\rho u)_t + (\rho u^2)_x + p_x = 0. \tag{4}$$

The pressure in the above equation is the hydraulic pressure. One gets (we shall assume that density of water equals 1)

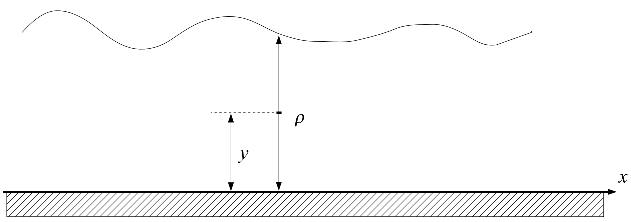

Figure 4: Hydraulic pressure

$$\pi(y) = g(\rho - y) \text{ . . . . hydraulic preassure,}$$

where g is the universal gravitational constant (see Fig. 4), and

$$p = \int_0^\rho \pi(y)\mathrm{dy} = \int_0^\rho g(\rho - y)\mathrm{dy} = g\frac{\rho^2}{2}.$$

Substituting this relation into (3) and (4) gives

$$\begin{aligned} &\rho_t + (\rho u)_x = 0 \\ &(\rho u)_t + \left(\rho u^2 + g\frac{\rho^2}{2}\right)_x = 0. \end{aligned} \tag{5}$$

Let us differentiate the second equation in the above system assuming enough regularity of solutions:

$$\rho_t u + \rho u_t + 2\rho u u_x + \rho_x u^2 + g\rho\rho_x = 0.$$

Then substitute ρ_t from the first equation in the modified second equation. After that procedure we get

$$u_t + uu_x + g\rho\rho_x = 0,$$

and finally the system becomes

$$\begin{aligned} &\rho_t + (\rho u)_x = 0 \\ &u_t + \left(\frac{u^2}{2} + g\rho\right)_x = 0. \end{aligned} \tag{6}$$

If solutions are not necessarily differentiable, one substitute $\omega = \rho u$ (ω is a flux) into system (5) so we get a different one

$$\begin{aligned} &\rho_t + \omega_x = 0 \\ &\omega_t + \left(\frac{\omega^2}{\rho} + g\frac{\rho^2}{2}\right)_x = 0. \end{aligned} \tag{7}$$

In subsequent sections one will see that systems (6) and (7) are not equivalent in practice (concerning weak solutions) due to the use of differentiation.

1.6. Gas dynamics (viscous)

Gas dynamics is the most influential area on the mathematical theory of conservation laws. Each important theorem, definition, observation or procedure is usually checked in some model of gas dynamics. We shell briefly describe some possibilities. One will see that the above shallow water system is the same as a special case of isentropic gas dynamic model described here. The main point of the isentropic model (=entropy (physical) is constant) is that the energy equation is missing and the energy plays the role of a factor which determines the proper solution ("energy cannot increase in a closed system"). We will explain that in the part about choosing admissible solutions.

1.6.1. Isentropic gas dynamics

We shall use the following notation:

ρ	. . . .	gas density
u	. . . .	gas velocity (gas molecule speed)
σ	. . . .	stress (force/area)

As before, we have the following system of conservation laws

$$\rho_t + (\rho u)_x = 0$$
$$(\rho u)_t + (\rho u^2)_x - \sigma_x = 0.$$

The following relation holds in general:

$$\sigma = -p + \nu u_x,$$

where p is a pressure of a gas without moving, and ν is a viscosity ($\ll 1$) (see Fig. 5).

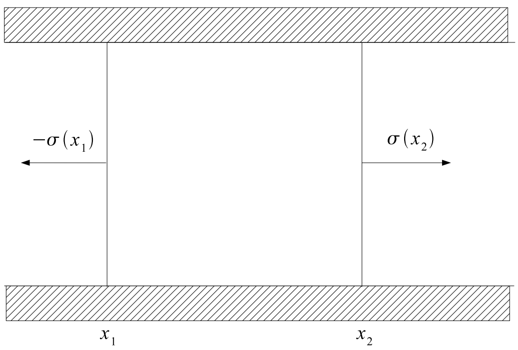

Figure 5: Pressure in share gas

Thus,

$$\rho_t + (\rho u)_x = 0$$
$$(\rho u)_t + (\rho u^2)_x + p_x = \nu u_{xx}$$

holds for viscous fluids. For gases it holds $\nu \to 0$, so one can often take $\nu \equiv 0$.

In more than one space dimensions we have well known Navier-Stokes equation

$$\begin{aligned}
&\rho_t + \operatorname{div}(\rho\vec{u}) = 0\\
&(\rho u)_t + \vec{u}\cdot\operatorname{grad}(\rho\vec{u}) + (\rho\vec{u})\cdot\operatorname{div}\vec{u} + \operatorname{grad}p = \nu\triangle\vec{u}\\
&(\text{or} = 0 \text{ for inviscid fluids}).
\end{aligned}$$

If density can be taken to be a constant, the above system reduces to

$$\vec{u}_t + \vec{u}\cdot\nabla\vec{u} = -\nabla p,$$

where $\nabla\vec{u}$ is the tensor derivative of the vector u. Or $\vec{u}\cdot\nabla\vec{u}$ should be understood as $\nabla(\|\vec{u}\|^2/2) + (\nabla\times\vec{u})\times\vec{u}$.

Let us take thermodynamical effects in gases now in order to close the system. Let $p = p(\rho, S)$, where the new independent variable S stands for entropy.

In order to close the system we need an extra equation or constitutive relation. For adiabatic case one can take

$$S_t + uS_x = 0,$$

for example.

For an isotropic inviscid gas one takes

$$S \equiv \text{const}, \text{ and } \nu \equiv 0.$$

Now, the inviscid case is modeled by

$$\begin{aligned}
&\rho_t + (\rho u)_x = 0\\
&(\rho u)_t + (\rho u^2)_x + (p(\rho))_x = 0\\
&p(\rho) = \kappa\rho^\gamma,\ 1 < \gamma < 3,\ \gamma = 1 + 2/n,
\end{aligned}$$

where κ stands for universal gas constant, and n is a number of atoms in gas molecule. The last relation in the above system is constitutive relation named "ideal gas relation". It capture the well known fact that pressure increase with density (and vice versa), but neglects the another well known fact that temperature (read as internal energy) also increases with pressure (the important cooling mechanism used in the ordinary life is based on that property). In order to catch that, we need the third equation instead of the constitutive relation $p(\rho) = \kappa\rho^\gamma$.

Let us remark that for constant density, $\rho = \rho_0 \in \mathbb{R}$, there is no changes in pressure and speed of the gas – no gas movements. Note that for $\gamma = 2$ (and $\kappa = g$ to be precise) we have the above shallow water system. Let us introduce new dependent variable,

1.6.2. Euler system of gas dynamics

As we have mentioned above, the pressure cannot be independent on a temperature i.e. energy in a real situations. Therefore we will substitute the constitutive relation with another conservation law – energy conservation. Let us use the following notation in the sequel:

e	. . . .	internal energy
m	. . . .	momentum
E	. . . .	energy
S	. . . .	entropy

The third equation is now

$$\left(\rho e + \frac{1}{2}\rho u^2\right)_t + \left((\rho e + \frac{1}{2}\rho u^2 + p)u\right)_x = 0.$$

Again we have to use a constitutive equation $p = p(\rho, e)$. We shall present also important quantity now. It is used to exclude all non-physical weak solution to the above equation. Let $S = S(\rho, e)$ be an *entropy density* in the sense of thermodynamics: It is a solution to

$$\rho^2 S_\rho + pS_e = 0,\ S_e > 0.$$

(The inverse $T = 1/S_e$ is so-called absolute temperature.) For any decreasing real function h the *convex* function

$$\eta := \rho h \circ S$$

is called *mathematical entropy*. The function $Q := u\eta$ is called *entropy flux* and every *smooth* solution (ρ, u, e) satisfies

$$\eta_t + Q_x = 0.$$

We shall return to that notion later on. Let us just say that a weak solution u is *admissible* (*entropic*) if

$$\eta_t + Q_x \leq 0$$

in generalised sense (distributional inequality and it will be described bellow). When we interpret the above inequality in the physical sense it means that the real entropy cannot decrease with time (and it is constant for strong (classical) solutions).

We can immediately transform Euler system into the *canonical* (or evolutionary) form (using so called*Lagrange coordinates*):

$$\begin{aligned} &\rho_t + m_x = 0 \\ &m_t + (m^2/\rho + p)_x = 0 \\ &E_t + ((E + p)u)_x = 0, \end{aligned}$$

where $E = \rho e + \frac{1}{2}\rho u^2$. The main problem with that form of the system is that there is no good way to describe a vacuum state which cannot be avoided for some initial data.

2. Solving conservation laws

2.1. Single 1-D equation

2.1.1. Rankin-Hugoniot conditions

Let $u \in C^1(\mathbb{R} \times [0, \infty))$ be a solution to the following partial differential equation

$$\begin{aligned} &u_t + (f(u))_x = 0 \\ &u(x, 0) = u_0(x). \end{aligned} \tag{8}$$

Take $\varphi \in C_0^1(\mathbb{R} \times [0, \infty))$, i.e. smooth function such that its support intersected by $\mathbb{R} \times [0, \infty)$ is compact.

Then

$$\begin{aligned} 0 &= \int_0^\infty \int_{-\infty}^\infty (u_t(x, t) + (f(u))_x \varphi(x, t) \mathrm{dtdx} \\ &= -\int_0^\infty \int_{-\infty}^\infty f(u)\varphi_x \mathrm{dtdx} + \int_{-\infty}^\infty u(x, t)\varphi(x, t)\mathrm{dx}\Big|_{t=0}^{t=\infty} \\ &\quad - \int_0^\infty \int_{-\infty}^\infty u\varphi_t \mathrm{dxdt} \\ &= -\int_0^\infty \int_{-\infty}^\infty (u\varphi_t + f(u)\varphi_x)\mathrm{dxdt} - \int_{-\infty}^\infty u_0(x)\varphi(x, 0)\mathrm{dx}. \end{aligned}$$

The above calculation inspired the following definition of weak solution for (8).

Definition 1 $u \in L^\infty(\mathbb{R} \times (0,\infty))$ *(u is bounded function up to a set of Lebesgue measure zero) is called* weak solution *of (8) if*

$$\int_0^\infty \int_{-\infty}^\infty (u\varphi_t + f(u)\varphi_x)\mathrm{dxdt} + \int_{-\infty}^\infty u_0(x)\varphi(x,0)\mathrm{dx} = 0,$$

for every $\varphi \in C_0^1(\mathbb{R} \times [0,\infty))$

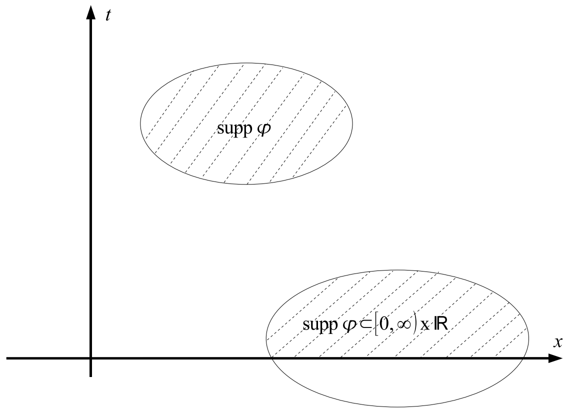

Figure 6: Supports of test functions in halfplane

Remark 2

1. *All classical solutions are also weak.*
2. *If u is a weak solution, then u is also a distributive solution.*
3. *If $u \in C^1(\mathbb{R} \times [0,\infty))$ is a weak solution, then it is a classical, too.*

If we do not say differently, "solution" will mean weak solution from now on.

In a few steps we shall find necessary conditions for existence of piecewise differentiable weak solution to some conservation law.

Theorem 3 *Necessary and sufficient condition that*

$$u(x,t) = \begin{cases} u_l(x,t), & x < \gamma(t),\ t \geq 0 \\ u_d(x,t), & x > \gamma(t),\ t \geq 0, \end{cases}$$

where u_l and u_d are C^1 solutions on their domains, be a weak solution to (8) is

$$\dot{\gamma} = \frac{f(u_d) - f(u_l)}{u_d - u_l} =: \frac{[f(u)]_\gamma}{[u]_\gamma}. \tag{9}$$

Proof. The proof will be given in few steps.

1. Let

$$u(x,t) = \begin{cases} u_l(x,t), & x < \gamma(t),\ t \geq 0 \\ u_d(x,t), & x > \gamma(t),\ t \geq 0, \end{cases}$$

where u_l and u_d are defined above, be a weak solution to (8). Then

$$\int_0^\infty \int_{-\infty}^\infty (u\varphi_t + f(u)\varphi_x)\mathrm{dxdt} + \int_{-\infty}^\infty u(x,0)\varphi(x,0)\mathrm{dx} = 0,$$

for every $\varphi \in (\mathbb{R} \times [0, \infty))$.

Also $(u_l)_t + f(u_l)_x = 0$ for $x < \gamma(t)$ and $t > 0$ as well as $(u_d)_t + f(u_d)_x = 0$ for $x > \gamma(t)$ and $t > 0$.

That is consequence of the fact that

$$\begin{aligned} 0 &= \int\int u_l\varphi_t + f(u_l)\varphi_x \mathrm{dxdt} \\ &= -\int\int (u_l)_t\varphi + (f(u_l))_x\varphi \mathrm{dxdt}, \end{aligned}$$

for every φ, supp$\varphi \subset \{(x,t):\ x < \gamma(t),\ t > 0\}$ and C^1-function u_l. And since φ is arbitrary, we have

$$(u_l)_t + (f(u_l))_x = 0.$$

The same arguments hold for u_d, too.

2.

$$\begin{aligned} &\int_0^\infty \int_{-\infty}^\infty (u\varphi_t + f(u)\varphi_x)\mathrm{dxdt} + \int_{-\infty}^\infty u_0(x)\varphi(x,0)\mathrm{dx} \\ =&\int_0^\infty \int_{-\infty}^{\gamma(t)} (u_l\varphi_t + f(u_l)\varphi_x)\mathrm{dxdt} + \int_0^\infty \int_{\gamma(t)}^\infty (u_d\varphi_t + f(u_d)\varphi_x)\mathrm{dxdt} \\ &+ \int_{-\infty}^\infty u_0(x)\varphi(x,0)\mathrm{dx}. \end{aligned}$$

3. Let us calculate the first integral from above. It holds

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{\gamma(t)} u_l\varphi \mathrm{dx} \\ =&\dot\gamma(t)u_l(\gamma(t),t)\varphi(\gamma(t),t) + \int_{-\infty}^{\gamma(t)} ((u_l)_t\varphi + u_l\varphi_t)\mathrm{dx}. \end{aligned}$$

That implies

$$\begin{aligned} &\int_0^\infty \int_{-\infty}^{\gamma(t)} u_l\varphi_t \mathrm{dxdt} = -\int_0^\infty \int_{-\infty}^{\gamma(t)} (u_l)_t\varphi \mathrm{dxdt} \\ &-\int_0^\infty \dot\gamma(t)u_l(\gamma(t),t)\varphi(\gamma(t),t)\mathrm{dt} + \int_0^\infty \frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{\gamma(t)} u_l\varphi \mathrm{dxdt}. \end{aligned}$$

On the other hand,

$$\begin{aligned} &\int_0^\infty \int_{-\infty}^{\gamma(t)} f(u_l)\varphi_x \mathrm{dxdt} = -\int_0^\infty \int_{-\infty}^{\gamma(t)} f(u_l)_x\varphi \mathrm{dxdt} \\ &+ \int_0^\infty f(u_l(\gamma(t),t))\varphi(\gamma(t),t))\mathrm{dt} \end{aligned}$$

Adding these terms and using the fact that u_l is a solution of PDE on the left-hand side of the curve $(\gamma(t), t)$, one gets the following

$$\int_0^\infty (f(u_l) - \dot\gamma u_l)\varphi \mathrm{dt} + \int_0^\infty \frac{d}{\mathrm{dt}}\int_{-\infty}^{\gamma(t)} u_l\varphi \mathrm{dxdt}$$

as a value of that integral.

4. Analogously, concerning the right-hand side, one can see that the second integral equals

$$-\int_0^\infty (f(u_d) - \dot{\gamma} u_d)\varphi \mathrm{dt} + \int_0^\infty \frac{d}{\mathrm{dt}} \int_{\gamma(t)}^\infty u_d \varphi \mathrm{dxdt}.$$

5. After adding all the above integrals one gets

$$\begin{aligned} 0 = &\int_0^\infty (f(u_l) - f(u_d) - (u_l - u_d)\dot{\gamma})\varphi \mathrm{dt} \\ &+ \int_0^\infty \frac{\mathrm{d}}{\mathrm{dt}} \int_{-\infty}^\infty u\varphi \mathrm{dxdt} + \int_{-\infty}^\infty u_0(x)\varphi(x,0)\mathrm{dx}, \\ &\text{and} \\ &\int_{-\infty}^\infty u(x,t)\varphi(x,t)\mathrm{dx}\Big|_{t=0}^{t=\infty} = -\int_{-\infty}^\infty u_0(x)\varphi(x,0)\mathrm{dx}. \end{aligned}$$

That is true if

$$\dot{\gamma} = \frac{f(u_d) - f(u_l)}{u_d - u_l} =: \frac{[f(u)]_\gamma}{[u]_\gamma}.$$

Obviously the above condition is sufficient. The proof is complete.

Condition (9) is called *Rankine-Hugoniot* (RH) condition.

Example 4 *Consider the following Riemann problem*

$$\begin{aligned} &u_t + \left(\frac{u^2}{2}\right)_x = 0 \\ &u_0 = \begin{cases} u_l \in \mathbb{R}, & x < 0 \\ u_d \in \mathbb{R}, & x > 0. \end{cases} \end{aligned} \tag{10}$$

Since u_l and u_d are constants, there exist two trivial solutions of (10) out of the discontinuity curve, and RH-condition gives

$$\dot{\gamma}(t) = \frac{u_d^2 - u_l^2}{2(u_d - u_l)} = \frac{u_d + u_l}{2},$$

i.e. $\dot{\gamma}(t) = ct$, $c = \frac{u_l + u_d}{2}$ and (see Fig. 7)

$$u(x,t) = \begin{cases} u_l, & x < ct \\ u_d, & x > ct, \end{cases} \tag{11}$$

If $u_l < u_d$, then except the above solution there exist also the following solutions (Fig. 8):

$$u(x,t) = \begin{cases} u_l, & x < u_l t \\ \frac{x}{t}, & u_l t \leq x \leq u_d t \\ u_d, & x > u_d t \end{cases} \tag{12}$$

or, (Fig. 9))

$$u(x,t) = \begin{cases} u_l, & x < u_l t \\ \frac{x}{f}, & u_l t \leq x \leq at \\ a, & at \leq x \leq \frac{a+u_d}{2} t \\ u_d, & x \geq \frac{a+u_d}{2} t, \end{cases} \tag{13}$$

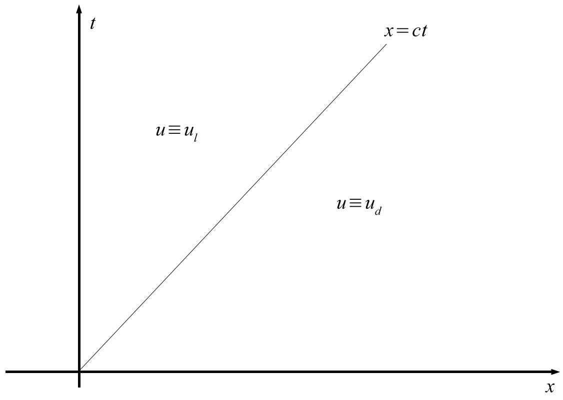

Figure 7: Shock wave

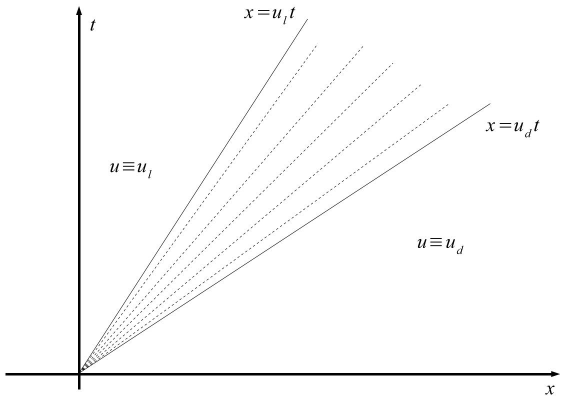

Figure 8: Rarefaction wave

for some $a \in (u_l, u_d)$.

One can see that there is no uniqueness of solution in the case $u_l < u_d$. *That problem (finding admissible or so called "entropy" solutions) will be approached later on.*

Example 5 *Let us multiply partial differential equation(10) by* u *and transfer it into divergence form*

$$\begin{aligned} &u_t + uu_x = 0 \quad / \cdot u \\ &uu_t + u^2 u_x = 0 \\ &\left(\frac{1}{2}u^2\right)_t + \left(\frac{1}{3}u^3\right)_x = 0. \end{aligned}$$

After nonlinear change of variables $\frac{1}{2}u^2 \mapsto v$, *one gets the following conservation law*

$$\begin{aligned} &v_t + (\frac{2\sqrt{2}}{3}v^{3/2})_x = 0 \\ &v\Big|_{t=0} = \begin{cases} v_l = \frac{1}{2}u_l^2, & x < 0 \\ v_d = \frac{1}{2}u_d^2, & x > 0. \end{cases} \end{aligned}$$

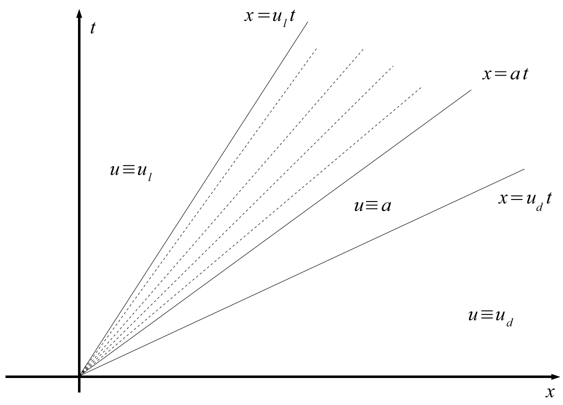

Figure 9: Non-entropic weak solution

RH-conditions give the following speed of shock wave c and the discontinuity line is $\gamma = ct$:

$$\dot{\gamma}(t) = \frac{[\frac{3}{2}v^{3/2}]}{[v]} = \frac{\frac{2\sqrt{2}}{3}\frac{1}{2}(u_d^2)^{3/2} - \frac{2\sqrt{2}}{3}\frac{1}{2}(u_l^2)^{3/2}}{\frac{1}{2}(u_d^2 - u_l^2)}$$
$$= \frac{\frac{1}{3}(u_d^3 - u_l^3)}{\frac{1}{2}(u_d^2 - u_l^2)} \neq \frac{u_l + u_d}{2} \text{ in general.}$$

(For example, for $u_l = 1$, $u_d = 0$ one has $\frac{\frac{1}{3}}{\frac{1}{2}} \neq \frac{1}{2}$.)

This was an "unpleasant" example, because simple but nonlinear transformations of variables do not preserve solutions.

Because of that a precise interpretation of a physical model is of the crucial importance.

2.1.2. Rarefaction waves

Solution of equation (8) of the form $u(x,t) = \tilde{u}(\frac{x}{t})$ is called *selfsimilar solution*. Now we shall try to find such a solution of (8) in a simple way, just by substituting a function of this form into the equation. After the differentiation we have

$$-\frac{x}{t^2}\tilde{u}'\Big(\frac{x}{t}\Big) + f'\Big(\tilde{u}\Big(\frac{x}{t}\Big)\Big)\frac{1}{t}\tilde{u}'\Big(\frac{x}{t}\Big) = 0$$

after multiplication of the equation with t and the substitution $\frac{x}{t} \mapsto y$ one gets the ODE

$$\tilde{u}'(y)(f'(\tilde{u}(y)) - y) = 0$$

After neglecting constant, so called trivial solutions ($\tilde{u}' \neq 0$), one can see that solution is given by the implicit relation

$$f'(\tilde{u}) = y, \text{ ie. } \tilde{u}(y) = f'^{-1}(y),$$

if f' is bijection (locally).

One can interpret the initial data in the following way:

$$u(x,0) = \begin{cases} u_l, & x < 0 \\ u_d, & x > 0 \end{cases} \Longrightarrow \tilde{u}(+\infty) = u_d,\ \tilde{u}(-\infty) = u_l. \tag{14}$$

If $f'' > 0$ (f is convex), then f' is an increasing function and solution $\tilde{u}$ to the equation satisfying (14) exists if $u_l < u_d$. Such solution is called *centered rarefaction wave* (the initial data has a singularity at zero).

2.2. Linear hyperbolic systems

We shall look at linear systems before we start with systems of conservation laws. Homogeneous linear scalar Cauchy problem with constant coefficients

$$\begin{aligned} &u_t + \lambda u_x = 0 \\ &u(x,0) = \bar{u}(x),\ \lambda \in C(\mathbb{R}),\ \bar{u} \in C^1([0,\infty) \times \mathbb{R}) \end{aligned} \tag{15}$$

has a simple solution in a traveling wave form

$$u(x,t) = \bar{u}(x - \lambda t). \tag{16}$$

If $\bar{u} \in L^1_{loc}$, then the above function (16) is a weak solution to (15), what one can show easily.

Let a homogeneous system with constant coefficients

$$\begin{aligned} &u_t + A u_x = 0 \\ &u(x,0) = \bar{u}(x) \end{aligned} \tag{17}$$

be given, where A is $n \times n$ hyperbolic matrix with real characteristic values $\lambda_1 < \ldots < \lambda_n$ and left-hand sided l_i (resp. right-hand sided r_i), $i = 1, \ldots, n$, eigenvectors. They are chosen in a way that $l_i r_i = \delta_{ij}$, $i, j = 1, \ldots, n$. Denote by $u_i := l_i u$ coordinates of the vector $u \in \mathbb{R}^n$ with respect to the base $\{r_1, \ldots, r_n\}$. Multiplying (17) from the left-hand side with l_i one gets

$$\begin{aligned} &(u_i)_t + \lambda_i (u_i)_x = (l_i u)_t + \lambda_i (l_i u)_x = l_i u_t + l_i A u_x = 0 \\ &u_i(x,0) = l_i \bar{u}(x) =: \bar{u}_1(x). \end{aligned}$$

So, (17) decouples into n scalar Cauchy problems, which can be solved like (15), one by one. Using (16) one can see that

$$u(x,t) = \sum_{i=1}^{n} \bar{u}_i(x - \lambda_i t) r_i \tag{18}$$

is solution to (17) because

$$u_t(x,t) = \sum_{i=1}^{n} -\lambda_i (l_i \bar{u}_x(x - \lambda_i t)) r_i = -A u_x(x,t).$$

Thus, initial profile $\bar{u}$ decouples into a sum of n waves with speeds $\lambda_1, \ldots, \lambda_n$.

As a special case, take Riemann problem

$$\bar{u}(x) = \begin{cases} u_l, & x < 0 \\ u_d, & x > 0. \end{cases}$$

Let us write down a solution to (18) using

$$u_d - u_l = \sum_{j=1}^{n} c_j r_j$$

and define the intermediate states by

$$w_i := u^l + \sum_{j \le i} c_j r_j,\ i = 0, \ldots, n,$$

such that $w_i - w_{i-1}$ is $(i-n)$-th characteristic vector of A. Solution is of the form (Fig. 10)

$$u(x,t) = \begin{cases} w_0 = u_l, & \frac{x}{t} < \lambda_1 \\ \ldots, & \\ w_i, & \lambda_i < \frac{x}{t} < \lambda_{i+1} \\ \ldots, & \\ w_n = u_d, & \frac{x}{t} > \lambda_n. \end{cases} \tag{19}$$

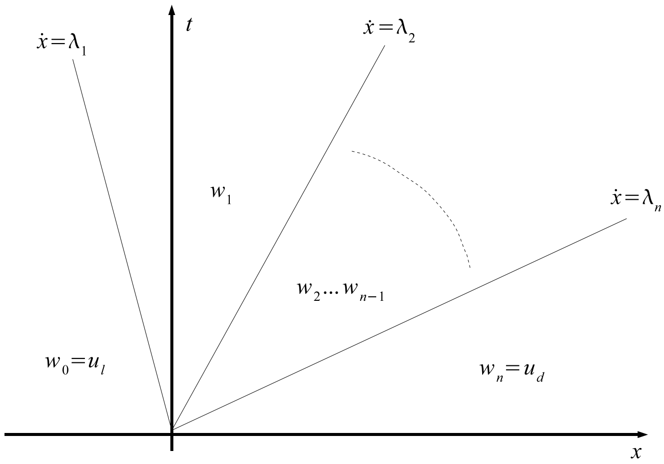

Figure 10: Waves and linear system

2.3. Systems of conservation laws – shallow water example

Let us use the procedure for finding a shock wave for a single equation described above. Note that we are able to find an appropriate speed for any pair of initial data there. But now

First, we shall solve the Riemann problem for (5), scaling variables so that the gravitational constant g equals 1. That will be our toy model and one will see lot of important notions and procedures there. Let us start with the system written in canonical form when $m = \rho u$ is the momentum:

$$\begin{aligned} &\rho_t + m_x = 0 \\ &m_t + \left(\frac{m^2}{\rho} + \rho^2\right)_x = 0. \end{aligned} \tag{20}$$

The system written in quasilinear form reads

$$\partial_t \begin{bmatrix} \rho \\ m \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -\frac{m^2}{\rho^2} + 2\rho & 2\frac{m}{\rho} \end{bmatrix} \partial_x \begin{bmatrix} \rho \\ m \end{bmatrix} = 0.$$

In the matrix notation, the above equation read as

$$U_t + AU_x = 0. \tag{21}$$

The eigenvalues of A equals

$$\lambda_1 = \frac{m}{\rho} - \sqrt{2\rho} < \lambda_2 = \frac{m}{\rho} + \sqrt{2\rho}, \text{for } \rho > 0,$$

so the above system is strictly hyperbolic in the physical domain of positive density. But, one will see later that a vacuum state ($\rho = 0$) will also be needed for a general solution to Riemann problem and the system should be written in the original form with ρ and u as dependent variables. The eigenvector are taken to be

$$r_1 = \left(1, -\frac{2\rho^3 - m^2}{\rho^2\sqrt{\rho} + m\rho}\right), \text{and } r_2 = \left(1, \frac{2\rho^3 - m^2}{\rho^2\sqrt{\rho} - m\rho}\right).$$

2.3.1. Shock and rarefaction waves

We shall use the same procedure for finding a shock wave solution as we have done with a single equation. Let us fix the initial data

$$(\rho, m) = \begin{cases} (\rho_l, m_l), & x < 0 \\ (\rho_r, m_r), & x > 0 \end{cases} \tag{22}$$

First, we look for a shock wave solution

$$(\rho, m) = \begin{cases} (\rho_l, m_l), & x < ct \\ (\rho_r, m_r), & x > ct \end{cases}$$

to (20,22) like it was done for a single equation above. Contrary to that case, each equation in the system will determine its own speed of the wave. So, a shock wave cannot exists for each initial data as it was for a single equation – one has to determine the set of possible initial data such that it exists. We will show that in our model bellow.

The RH condition implies from the first equation

$$c = \frac{m_r - m_l}{\rho_r - \rho_l}.$$

From the second one we have

$$c = \frac{m_r^2/\rho_r + \rho_r^2 - m_l^2/\rho_l + \rho_l^2}{m_r - m_l}.$$

These speeds should be the same, so we get the condition which initial data has to satisfy:

$$\frac{m_r - m_l}{\rho_r - \rho_l} = \frac{m_r^2/\rho_r + \rho_r^2 - m_l^2/\rho_l + \rho_l^2}{m_r - m_l}.$$

That is, if the left-handed side is fixed, all the possible points (ρ_r, m_r) lies on the curve

$$m_r = \frac{m_l\rho_r - (\rho_r - \rho_l)\sqrt{\rho_l\rho_r(\rho_l + \rho_r)}}{\rho_l}, \text{ with } c = \frac{m_l}{\rho_l} - \sqrt{\frac{(\rho_l + \rho_r)\rho_r}{\rho_l}}, \tag{23}$$

or

$$m_r = \frac{m_l\rho_r + (\rho_r - \rho_l)\sqrt{\rho_l\rho_r(\rho_l + \rho_r)}}{\rho_l}, \text{ with } c = \frac{m_l}{\rho_l} + \sqrt{\frac{(\rho_l + \rho_r)\rho_r}{\rho_l}}. \tag{24}$$

These sets (curves) are presented at the figure 11: For a fixed left-handed state, if (ρ, m) lies on these curves, it can be connected with (ρ_l, m_l) by a shock wave. The name of that set is *Hugoniot locus*.

Let us now try to find a rarefaction wave solution to the system. Again, we shall start as in the case of a single equation: Substitute $(\rho, m) = (\rho, m)(x/t)$ into the system. The initial data reduces to $(\rho, m)(-\infty) = (\rho_l, m_l)$, and $(\rho, m)(\infty) = (\rho_r, m_r)$ and the equations become

$$\begin{aligned} 0 &= \rho_t + m_x = -\frac{x}{t^2}\rho' + \frac{1}{t}m' \\ 0 &= m_t \left(\frac{m^2}{\rho} + \rho^2\right)_x = -\frac{x}{t^2}m' + \frac{1}{t}\left(\frac{2mm' - m^2\rho'}{\rho^2} + 2\rho\rho'\right). \end{aligned}$$

After multiplication by t and change of variables $x/t \to y$ we have the following system of ODEs

$$\begin{aligned} &-y\rho' + m' = 0 \\ &\left(-\frac{m^2}{\rho^2} + 2\rho\right)\rho' + \left(2\frac{m}{\rho^2} - y\right)m' = 0. \end{aligned}$$

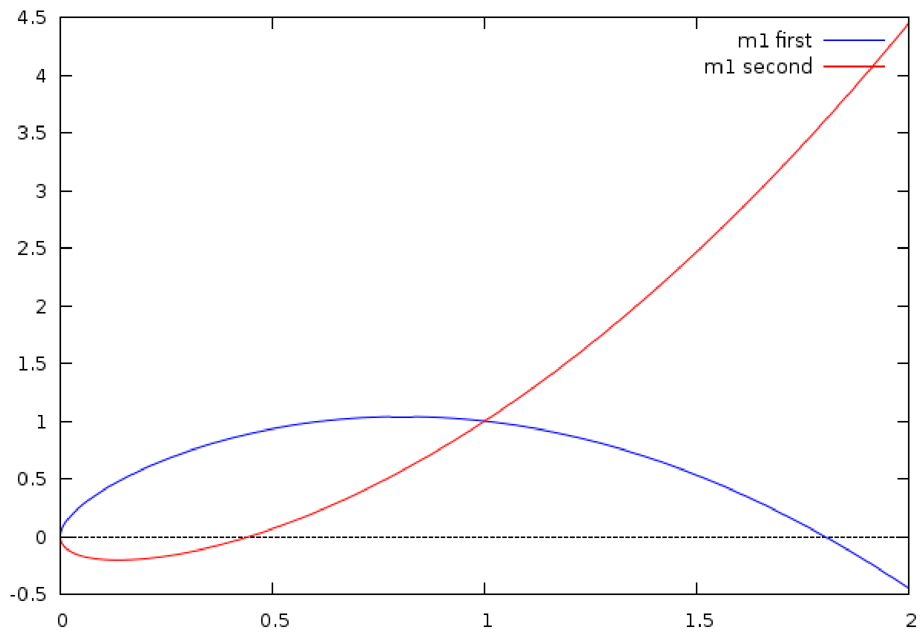

Figure 11: Hugoniot locus

A trivial solution to the above system is when ρ and m are constants, but that was the case covered by the above search for shock waves. So, only one possibility left – the determinant of above system has to be zero,

$$\begin{vmatrix} -y & 1 \\ \left(-\frac{m^2}{\rho^2}+2\rho\right) & \left(2\frac{m}{\rho^2}-y\right) \end{vmatrix} = 0.$$

But the above relation is equivalent to the fact that y is an eigenvalue of the matrix A (see (21) while (ρ', m') is an eigenvector of A, that is (ρ, m) is an integral curve of eigenvector of the matrix A. It takes values from (ρ_l, m_l) to (ρ_r, m_r), since we have to connect these constant states from the left and the right-hand side. The argument of a solution (ρ, m) should go from $x/t = \lambda_i(\rho_l, m_l)$ to some $x/t = \lambda_i(\rho_r, m_r)$ (see Figure 12).

In order to have a well defined function, $\lambda_i(\rho_l, m_l)$ have to be less than $\lambda_i(\rho_r, m_r)$ while we move from left to right-hand state. Thus, λ_i have to increase along the integral curve of eigenvectors, i.e.

$$\nabla\lambda_i \cdot r_i > 0,\ i = 1, 2.$$

Using that we have to multiply the previous choice for r_1 by -1, so take

$$r_1 = \left(-1, \frac{2\rho^3 - m^2}{\rho^2\sqrt{\rho} + m\rho}\right), \text{and } r_2 = \left(1, \frac{2\rho^3 - m^2}{\rho^2\sqrt{\rho} - m\rho}\right)$$

in the sequel.

So, we have two types or rarefaction wave solution, one for each eigenvalue. The first one connects the states (ρ_l, m_l) and $(\rho, m)(s_1)$, for some $s_1 > 0$, where (ρ, m) solves

$$\begin{aligned} \frac{\mathrm{d}\rho}{\mathrm{d}s} &= -1,\ \rho(0) = \rho_l \\ \frac{\mathrm{d}m}{\mathrm{d}s} &= \frac{2\rho^3 - m^2}{\rho^2\sqrt{\rho} + m\rho},\ m(0) = m_l. \end{aligned}$$

The above system is autonomous, so we can reduce it on a single ODE and the initial data:

$$\frac{\mathrm{d}m}{\mathrm{d}\rho} = -\frac{2\rho^3 - m^2}{\rho^2\sqrt{\rho} + m\rho},\ m(\rho_l) = m_l,\ \rho < \rho_l.$$

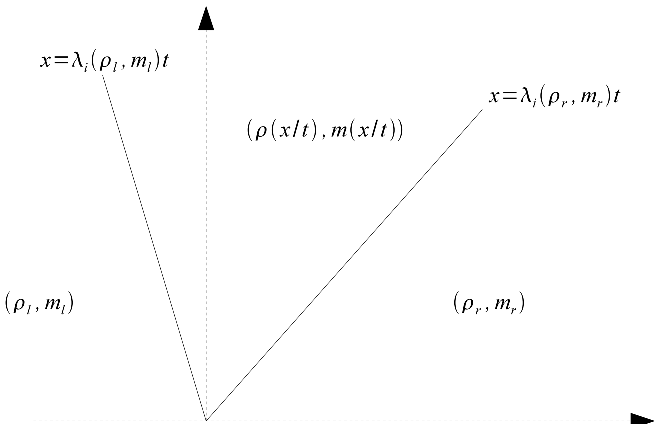

Figure 12: Rarefaction wave

Solution to that initial data problem is given by

$$m = \left(\frac{m_l}{\rho_l} + 2\sqrt{2\rho_l} - 2\sqrt{2\rho}\right)\rho,\ \rho < \rho_l$$

and called *1-rarefaction curve* (R_1).

In the same way we get the *2-rarefaction curve* (R_2),

$$m = \left(\frac{m_l}{\rho_l} - 2\sqrt{2\rho_l} + 2\sqrt{2\rho}\right)\rho,\ \rho > \rho_l.$$

Both curves are shown in the Figure 13.

2.3.2. Admissible waves

One can find a fairly complete list of admissibility criteria in the appendix bellow. Here we shell use just one of them, Lax entropy condition for shocks. It will ensure weak solution uniqueness for Riemann problem (20, 22).

Definition 6 *Let U be a shock wave solution with a speed c to Riemann problem for system (20, 22). The shock wave is* admissible *(or* entropic*) if*

$$\lambda_2(U_l) > \lambda_1(U_l) \geq c \geq \lambda_1(U_r),\ \lambda_2(U_r) > c \text{ (so called 1-shock), or}$$
$$\lambda_2(U_l) \geq c \geq \lambda_2(U_r) > \lambda_1(U_r),\ c > \lambda_1(U_l) \text{ (so called 2-shock).}$$

(It is easy to extend that definition for any strictly hyperbolic conservation law.)

Let us now check which part of Hugoniot locus is admissible. We shell do that in details for one possible case, while all other can be done in a quite similar way.

1. Suppose that $\rho_r > \rho_l$ and take m from (23).

   Then

$$\lambda_1(\rho_l, m_l) = \frac{m_l}{\rho_l} - \sqrt{2\rho_l} > \frac{m_l}{\rho_l} - \sqrt{\frac{(\rho_l+\rho_r)\rho_r}{\rho_l}} = c.$$

   Also,

$$c = \frac{m_l}{\rho_l} - \sqrt{\frac{(\rho_l+\rho_r)\rho_r}{\rho_l}} > \lambda_1(\rho_r, m_r) = \frac{m_r}{\rho_r} - \sqrt{2\rho_r} = \frac{m_l}{\rho_l} - (\rho_l+\rho_r)\sqrt{\frac{1}{\rho_l} + \frac{1}{\rho_r}} + \sqrt{2\rho_r},$$

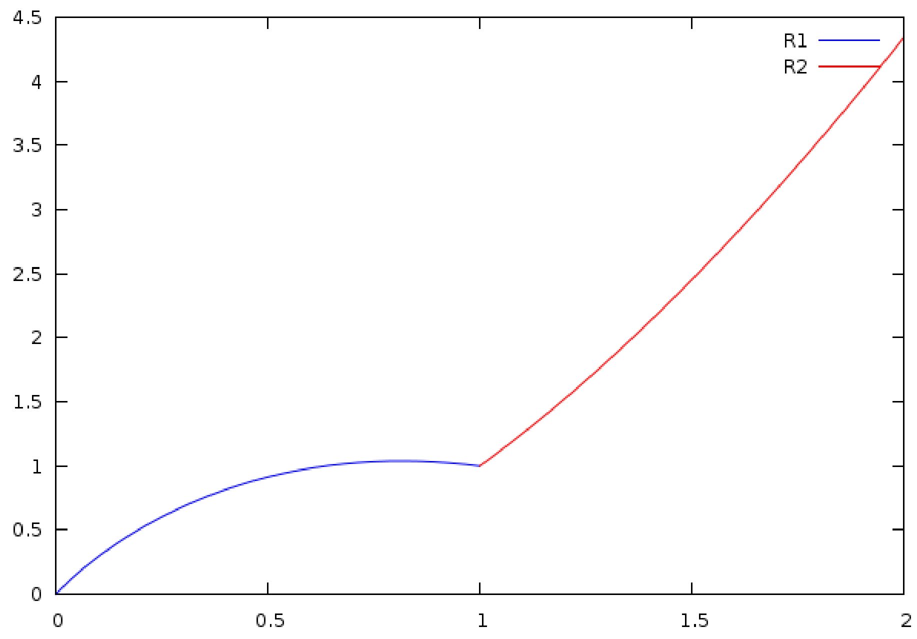

Figure 13: Rarefaction curves

after we substitute m_r from (23). The above inequality will be true if and only if

$$\sqrt{2\rho_r} > \rho_l\sqrt{\frac{\rho_l+\rho_r}{\rho_l\rho_t}} = \sqrt{\frac{\rho_l+\rho_r}{\rho_t}}\rho_l.$$

But this is obviously true, so the curve (23) is in fact *1-shock curve* (S_1) for $\rho_r > \rho_l$.
It cannot represent *2-shock curve* (S_2) because

$$c = \frac{m_l}{\rho_l} - \sqrt{\frac{(\rho_l+\rho_r)\rho_r}{\rho_l}} \geq \lambda_2(\rho_r, m_r) = \frac{m_r}{\rho_r} + \sqrt{2\rho_l} = \frac{m_l}{\rho_l} + \sqrt{2\rho_r} - (\rho_r-\rho_l)\sqrt{\frac{1}{\rho_l}+\frac{1}{\rho_l}},$$

where we have used (23) in the last equality. The above relation is true if and only if

$$0 \geq \sqrt{2\rho_r} + \rho_0\sqrt{\frac{1}{\rho_l}+\frac{1}{\rho_l}},$$

which is obviously not true.

2. Suppose that $\rho_r < \rho_l$ and take m from (23).
   Using the same type of calculations as before, one will get it does not represent neither 1-shock not 2-shock curve.

3. Suppose that $\rho_r > \rho_l$ and take m from (24).
   Like in the previous case, one will see that this part of curve contains points of non-admissible shocks only.

4. Suppose that $\rho_r < \rho_l$ and take m from (24).
   One can now check that
   $$\lambda_2(\rho_l, m_l) > c > \lambda_2(\rho_r, m_r),$$
   so we have 2-shock curve, now.

The rarefaction waves are always admissible if exists, and finally here it is an illustration of admissible waves (Figure 14).

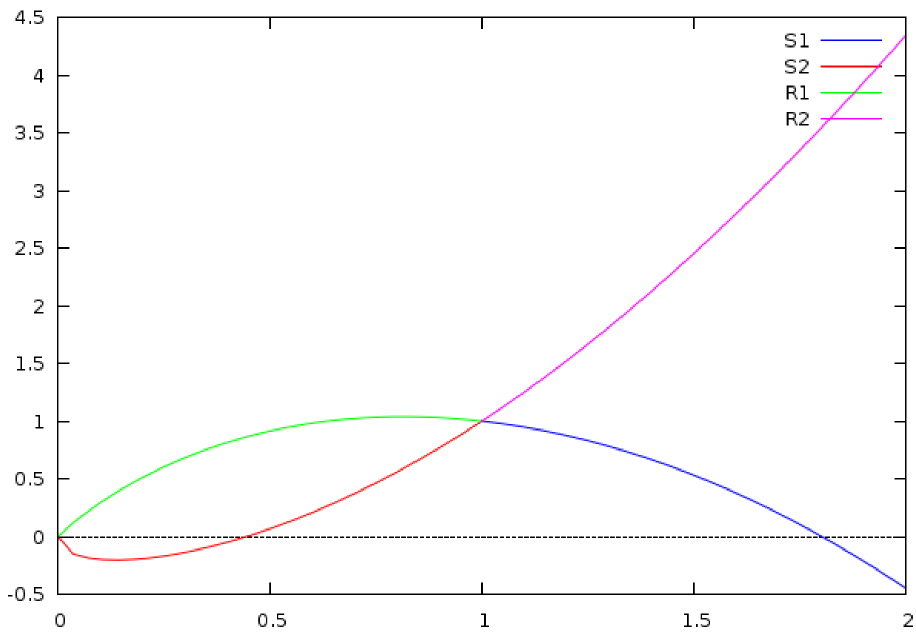

Figure 14: Admissible waves

2.4. Riemann problem

Now we are in position to use the admissible waved obtained above for construction of a solution to an arbitrary Riemann problem.

- Denote the set bellow R_2 and above S_1 by I.

  Take any point $(\rho_r, m_r) \in I$. The solution to (20,22) will consists from an 1-shock connecting the start-state (ρ_l, m_l) with an inter-state (ρ_s, m_s) followed by a 2-rarefaction conecting the state (ρ_s, m_s) with the end-state (ρ_r, m_r).

  That follows from the fact that the system

$$m_s = \frac{m_l\rho_s - (\rho_s - \rho_l)\sqrt{\rho_l\rho_s(\rho_l + \rho_s)}}{\rho_l} \quad \text{(1-shock)}$$

$$m_r = \left(\frac{m_s}{\rho_s} - 2\sqrt{2\rho_s} + 2\sqrt{2\rho_r}\right)\rho_r \quad \text{(2-rarefaction)}$$

  has a soltion (ρ_s, m_s) such that $r_s > r_l$ and $r_r > r_s$.

- Denote the set bellow S_1 and bellow S_2 by II.

  If $(\rho_r, m_r) \in I$, then the system

$$m_s = \frac{m_l\rho_s - (\rho_s - \rho_l)\sqrt{\rho_l\rho_s(\rho_l + \rho_s)}}{\rho_l} \quad \text{(1-shock)}$$

$$m_r = \frac{m_s\rho_r + (\rho_r - \rho_s)\sqrt{\rho_s\rho_r(\rho_s + \rho_r)}}{\rho_s} \quad \text{(2-shock)}$$

  has a unique solution (ρ_s, m_s) such that $r_s > r_l$ and $r_r < r_s$, so the solution to Riemann problem is an 1-shock followed by a 2-shock.

- Denote the set bellow R_1 and above S_2 by III.

  Again, the system

$$m_s = \left(\frac{m_l}{\rho_l} + 2\sqrt{2\rho_l} - 2\sqrt{2\rho_s}\right)\rho_s \quad \text{(1-rarefaction)}$$
$$m_r = \frac{m_s\rho_r + (\rho_r - \rho_s)\sqrt{\rho_s\rho_r(\rho_s + \rho_r)}}{\rho_s} \quad \text{(2-shock)}$$

  has a unique solution, so the solution to Riamann problem consists from 1-rarefaction followed by 2-shock.

- Denote the set above R_1 and above R_2 by IV.

  Here, the situation is not so simple at the first glance. The system

$$m_s = \left(\frac{m_l}{\rho_l} + 2\sqrt{2\rho_l} - 2\sqrt{2\rho_s}\right)\rho_s \quad \text{(1-rarefaction)}$$
$$m_r = \left(\frac{m_s}{\rho_s} - 2\sqrt{2\rho_s} + 2\sqrt{2\rho_r}\right)\rho_r \quad \text{(2-rarefaction)}$$

  doen not always have a solution with $\rho_s > 0$: The only solution is $(\rho_s, m_s) = (0, 0)$. So, the solution to the Riemann problem looks a bit different. We have a 1-rarefaction connecting (ρ_l, m_l) and the vacuum state $\rho = 0$ ($m = \rho u$ and $m = 0$ must hold). Note that for the shallow water system written in original variables (5) (and $g = 1$ of course) vacuum state always solve that system whatever u is. Then connect the vacuum state with (ρ_r, m_r) by a 2-rarefaction (note that each rarefaction curve passes trough $(0, 0)$). So, the solution to Riemann problem in that case is 1-rarefaction and 2-rarefaction connected by vacuum state between them.

3. Introduction into numerical methods

We will restrict our presentation at the very basic level, just to let the readers to get a feeling what can be done in that very important area for nonlinear wave theories.

3.1. Conservative schemes

As one could see, weak solutions of conservation law systems are not unique in general and that produces a lot of numerical problems. But the situation for nonlinear problems could be even worse, as one can see in the following example.

Example 7 *Take Burgers' equation*

$$u_t + uu_x = 0,$$

with the initial data

$$U_j^0 = \begin{cases} 1, & j < 0 \\ 0, & j \geq 0. \end{cases}$$

One can take simple scheme for the above equation under the hypothesis $U_j^n \geq 0$, for every j, n:

$$U_j^{n+1} = U_j^n - \frac{k}{h} U_j^n (U_j^n - U_{j-1}^n).$$

That gives $U_j^1 = U_j^0$ for every j. Thus, $U_j^n = U_j^0$ for every j, n, and approximate solution converges to $u(x, t) = u_0(x)$, which is not even solution to the given equation.

Because of this one can use more appropriate procedure. One good class are so called *conservative procedures (schemes).*

Definition 8 *Numerical procedure is* conservative *if it can be written in the following form*

$$U_j^{n+1} = U_j^n - \frac{k}{h}[F(U_{j-p}^n, U_{j-p+1}^n, ..., U_{j+q}^n) - F(U_{j-p-1}^n, U_{j-p}^n, ..., U_{j+q-1}^n)]. \tag{25}$$

Function F is called numerical flux function.

In the simplest case, for $p = 0$ and $q = 1$, relation (25) is

$$U_j^{n+1} = U_j^n - \frac{k}{h}[F(U_j^n, U_{j+1}^n) - F(U_{j-1}^n, U_j^n)]. \tag{26}$$

Let U_j^n be an average value of u in $[x_{j-1/2}, x_{j+1/2}]$ defined by

$$\overline{u}_j^n = \frac{1}{h}\int_{x_{j-1/2}}^{x_{j+1/2}} u(x, t_n)\mathrm{dx}.$$

Since weak solution $u(x,t)$ satisfies the integral form of conservation law, we have

$$\int_{x_{j-1/2}}^{x_{j+1/2}} u(x, t_{n+1})\mathrm{dx} = \int_{x_{j-1/2}}^{x_{j+1/2}} u(x, t_n)\mathrm{dx}$$
$$- [\int_{t_n}^{t_{n+1}} f(u(x_{j+1/2}, t))\mathrm{dt} - \int_{t_n}^{t_{n+1}} f(u(x_{j-1/2}, t))\mathrm{dt}].$$

Dividing it by h gives

$$\overline{u}_j^{n+1} = \overline{u}_j^n - \frac{1}{h}[\int_{t_n}^{t_{n+1}} f(u(x_{j+1/2}, t))\mathrm{dt} - \int_{t_n}^{t_{n+1}} f(u(x_{j-1/2}, t))\mathrm{dt}].$$

One can see that

$$F(U_j, U_{j+1}) \sim \frac{1}{k}\int_{t_n}^{t_{n+1}} f(u(x_{j+1/2}, t))\mathrm{dt}.$$

For the simplicity of notation we shall use

$$F(U^n; j) = F(U_{j-p}^n, U_{j-p+1}^n, ..., U_{j+q}^n),$$

so (25) can be written in the form

$$U_j^{n+1} = U_j^n - \frac{k}{h}[F(U^n; j) - F(U^n, j-1)]. \tag{27}$$

Definition 9 *Numerical procedure (26) is* consistent *with an original conservation law if for $u(x,t) \equiv \overline{u}$ it holds*

$$F(\overline{u}, \overline{u}) = f(\overline{u}),$$

for every $\overline{u} \in \mathbb{R}$.

For the consistency one finds that F should be Lipschitz continuous with respect to all its variables.

In general, if F is a function of more than two variables, consistency condition reads

$$F(\overline{u}, \overline{u}, ..., \overline{u}) = f(\overline{u}),$$

and for Lipschitz condition there has to exist a constant K such that

$$|F(U_{j-p}, ..., U_{j+q}) - f(\overline{u})| \leq K \max_{-p \leq i \leq q} |U_{j+i} - \overline{u}|,$$

holds true for all U_{j+i} close enough to $\overline{u}$.

The following theorem is of the crucial importance for numerical solving of conservation law systems.

Theorem 10 (Lax-Wendorff) *Let a sequence of schemes indexed by $l = 1, 2, \dots$ with parameters $k_l, h_l \to 0$, as $l \to \infty$. Let $U_l(x,t)$ be a numeric approximation obtained by a consistent and conservative procedure at l-th scheme. Suppose that $U_l \to u$, as $l \to \infty$. Then, a function $u(x,t)$ is a weak solution to conservation law system.*

In order to prove that a weak solution $u(x,t)$, obtained by a conservative procedure, satisfy entropy condition, it is enough to prove that it satisfies so called *discrete entropy condition* (see [8])

$$\eta(U_j^{n+1}) \le \eta(U_j^n) - \frac{k}{h}[\Psi(U^n; j) - \Psi(U^n; j-1)], \tag{28}$$

where Ψ is appropriate *numerical entropy flux* consistent with a entropy flux ψ in the same sense as F with f is.

3.2. Godunov method

The basic idea of this procedure is the following: Numerical solution U^n is used for defining piecewise constant function $\widetilde{u}^n(x,t_n)$ which equals U_j^n in a cell $x_{j-1/2} < x < x_{j+1/2}$. Given function is not a constant in $t_n \le t < t_{n+1}$. Because of that we use $\widetilde{u}^n(x,t_n)$ as an initial data for conservation law, which we analytically solve in order to get $\widetilde{u}^n(x,t)$ for $t_n \le t \le t_{n+1}$. After that we define the approximate solution U^{n+1} at time t_{n+1} as a mean value of the exact solution at time t_{n+1},

$$U_j^{n+1} = \frac{1}{h}\int\limits_{x_{j-1/2}}^{x_{j+1/2}} \widetilde{u}^n(x,t_{n+1})\mathrm{dx}. \tag{29}$$

So, we have values for a piecewise constant function $\widetilde{u}^{n+1}(x,t_{n+1})$ and procedure continues. One can easily obtain (29) from integral form of the conservation law. Namely, since $\widetilde{u}$ is a weak solution of the conservation law, there holds

$$\begin{aligned}\int\limits_{x_{j-1/2}}^{x_{j+1/2}} \widetilde{u}^n(x,t_{n+1})\mathrm{dx} = \int_{x_{j-1/2}}^{x_{j+1/2}} \widetilde{u}^n(x,t_n)\mathrm{dx} + \int_{t_n}^{t_{n+1}} f(\widetilde{u}^n(x_{j-1/2},t))\mathrm{dt}\\ - \int\limits_{t_n}^{t_{n+1}} f(\widetilde{u}^n(x_{j+1/2},t))\mathrm{dt}.\end{aligned} \tag{30}$$

After division of the above expression by h, one uses (29) and the fact that $\widetilde{u}^n(x,t_n) \equiv U_j^n$ in the interval $(x_{j-1/2}, x_{j+1/2})$ to transform (30) to

$$U_j^{n+1} = U_j^n - \frac{k}{h}[F(U_j^n, U_{j+1}^n) - F(U_{j-1}^n, U_j^n)].$$

Here, the numerical flux function F is given by

$$F(U_j^n, U_{j+1}^n) = \frac{1}{k}\int_{t_n}^{t_{n+1}} f(\widetilde{u}^n(x_{j+1/2},t))\mathrm{dt}. \tag{31}$$

That proves that Godunov procedure is conservative (it can be written in the form (26)). Additionally, calculation of integral 31 is very simple, since $\widetilde{u}^n$ in constant in (t_n, t_{n+1}) at the point $x_{j+1/2}$. That follows from the fact that a solution to Riemann problem is a constant along a characteristic curve

$$(x - x_{j+1/2})/t = \text{const}.$$

Since $\widetilde{u}^n$ depends only on U_j^n i U_{j+1}^n along the line $x = x_{j+1/2}$, $\widetilde{u}^n$ can be denoted by $u^*(U_j^n, U_{j+1}^n)$. Then numerical flux (31) becomes

$$F(U_j^n, U_{j+1}^n) = f(u^*(U_j^n, U_{j+1}^n)), \tag{32}$$

and Godunov procedure is now given by

$$U_j^{n+1} = U_j^n - \frac{k}{h}[f(u^*(U_j^n, U_{j+1}^n)) - f(u^*(U_{j-1}^n, U_j^n))].$$

Obviously, (32) is consistent with f because

$$U_j^n = U_{j+1}^n \equiv \overline{u}$$

implies

$$u^*(U_j^n, U_{j+1}^n) = \overline{u}.$$

Lipschitz continuity follows from smoothness of f.

But constancy of $\widetilde{u}^n$ in interval (t_n, t_{n+1}) at the point $x_{j+1/2}$ depends on a length of the interval. If a time interval is to long, then interaction of waves obtained by solving the closest Riemann problems may occur. Since speeds of these waves are bounded by characteristic values of the matrix $f'(u)$ and since sequential discontinuity points (origins of appropriate Riemann problems) are separated by h, $\widetilde{u}^n(x_{j+1/2}, t)$ is constant in the interval $[t_n, t_{n+1}]$ for k small enough. So, in order to avoid interactions, one introduces the condition

$$\left| \frac{k}{h} \lambda_p(U_j^n) \right| \leq 1, \tag{33}$$

for every λ_p and U_j^n.

Definition 11 *The number*

$$\mathrm{CFL} = \max_{j,p} \left| \frac{k}{h} \lambda_p(U_j^n) \right|$$

is called Courant number *or CFL (Courant-Friedrichs-Levy) for short. The condition*

$$\mathrm{CFL} \leq 1$$

is called CFL condition.

4. Appendix – beyond an example of a conservation law system

Here we put some important mathematical points missed in the above presentation. They are more mathematically or technically demanded for non-specialist students, but needed when one wants to really use the abode theory in practice both in applied or theoretical meaning.

4.1. Quasilinear hyperbolic systems of balance laws

We consider the following system of balance laws

$$\partial_t H(U(x,t),x,t) + \mathrm{div} G(U(x,t),x,t) = \Pi(U(x,t),x,t), \tag{34}$$

where $x \in \mathbb{R}^m$ and $t \geq 0$. Here, matrix functions F, G and Π are at least continuous (for our purposes, but one can permit lower regularity like in porous flow equations). Also $\dim(U) = m \times 1$, $U = [U^1, \ldots, U^m]$, $\dim(H) = m \times 1$, $\dim(\Pi) = m \times 1$, $\dim(G) = m \times n$, $G = (G_1, \ldots, G_n)$, and G_α is a row matrix.

Here and bellow, all operators acting on (x,t)-space are capitalized (Div, for example) while the ones action on x-space are not (div, for example). In the sequel, D denotes the differential regarded as a row operation, $\mathrm{D} = [\partial/\partial U^1, \ldots, \partial/\partial U^n]$.

The system (34) is said to be in a canonical (evolutionary) form if $H(U,x,t) \equiv U$.

Definition 12 *The system (34) is called hyperbolic in the t-direction if the following holds. For a fixed $U \in \Omega$ (physical domain) and $\nu \in S^{m-1}$ (the unit sphere), the matrix $\mathrm{D}H(U,x,t)$ (with dimension $n \times n$) is nonsingular, while the eigenvalue problem*

$$\Big(\sum_{\alpha=1}^{m} \nu_\alpha \mathrm{D} G_\alpha(U,x,t) - \lambda \mathrm{D} H(U,x,t) \Big) R = 0$$

has real eigenvalues

$$\lambda_1(\nu; U, x, t), \ldots, \lambda_n(\nu; U, x, t),$$

called characteristic speeds, and n *linearly independent eigenvectors*

$$R_1(\nu; U, x, t), \ldots, R_n(\nu; U, x, t).$$

A very important example is the symmetric hyperbolic system when $\mathrm{D}H(U, x, t)$ is symmetric positive definite matrix, while $\mathrm{D}G_\alpha(U, x, t)$, $\alpha = 1, \ldots, m$, are symmetric matrices. One of the main ideas here is to discover what is needed for (34) to satisfy the properties holding for symmetric system.

4.1.1. Entropy-entropy flux pairs

Let U be a strong (classical) solution to (34). If there exist a function $\eta = \eta(U(x, t), x, t)$ and an m-row vector $Q = (Q_1, \ldots, Q_m)(U(x, t), x, t)$ such that

$$\partial_t \eta(U(x, t), x, t) + \mathrm{div} Q(U(x, t), x, t) = h(U(x, t), x, t), \tag{35}$$

for an appropriate function h.

The function η is called an entropy for the system (34) and Q is called the entropy flux associated with η.

It is necessary for 35 to holds that there exists a matrix $B(U, x, t)$ such that

$$\mathrm{D}\eta(U, x, t) = B(U, x, t)^T \mathrm{D}H(U, x, t) \tag{36}$$

with

$$\mathrm{D}Q_\alpha(U, x, t) = \mathrm{D}\eta(U, x, t)\mathrm{D}G_\alpha(U, x, t), \ \alpha = 1, \ldots, m. \tag{37}$$

providing

$$\mathrm{D}^2\eta(U, x, t)\mathrm{D}G_\alpha(U, x, t) = \mathrm{D}G_\alpha(U, x, t)^T \mathrm{D}^2\eta(U, x, t), \ \alpha = 1, \ldots, m. \tag{38}$$

If the system is in a canonical form, then (36) reduces to $\mathrm{D}\eta = B^T$.

The relation (38) is crucial for the existence of an entropy: Once it is satisfied, a job of finding B from (36) and solving the second system of PDEs (38) is straightforward. It reduces to a system of $n(n-1)m/2$ PDEs, so is formally overdetermined unless $n = 1$ with m arbitrary and $n = 2$ with $m = 1$.

However, there are cases when (38) is satisfied. One important example is a symmetric (thus hyperbolic if given in canonical form) system: We can take $\eta = |U|^2$. We have even more: Suppose that a system of balance laws satisfies (38) and $\eta(U, x, t)$ is uniformly convex in U. Then the change $U^* = \mathrm{D}\eta(U, x, t)^T$ of variables renders the system symmetric. Therefore, if a system is in canonical form and has a convex entropy, then it is necessarily hyperbolic.

A system endowed with a convex entropy is called physical system.

Let us present some results which emphasize the importance of convex entropies even for strong solutions.

In the sequel we shall consider only homogenous (weakly) hyperbolic systems of conservation laws

$$\partial_t H(U(x, t)) + \mathrm{div} G(U(x, t)) = 0, \tag{39}$$

even though the analysis can be extended in a routine way to the general case (34) when H and G are smooth enough (C^1 will suffice). Assume also that the system is given in a canonical form, $H(U) \equiv U$. The hyperbolicity means that

$$\Lambda(\nu; U) = \sum_{\alpha=1}^{m} \nu_\alpha \mathrm{D}G_\alpha(U)$$

has all real eigenvalues $\lambda_1(\nu; U), \ldots, \lambda_n(\nu; U)$ and n linearly independent eigenvectors $R_1(\nu; U), \ldots, R_n(\nu; U)$. If the system is only weakly hyperbolic, then all the eigenvalues are real, but there are less than n linearly independent eigenvectors. The system is strictly hyperbolic if there are n real distinct eigenvalues (and thus the same number of linearly independent eigenvectors).

Theorem 13 *Assume that the system of conservation laws (39) with $H(U) \equiv U$ is endowed with an entropy $\eta \in C^3$ with $\mathrm{D}^2\eta$ positive definite on the physical domain Ω. Assume that $G \in C^{l+2}$ and $U_0 \in C^1$ with values in a compact subset of Ω while $\nabla U_0 \in H^l$, $l > m/2$. Then there exists $T_\infty \leq \infty$ such that there exists a unique classical solution U to (39) with $U|_{t=0} = U_0$ on $[0, T_\infty)$. Moreover, such T_∞ is maximal: If $T_\infty < \infty$, then the gradient of U explodes (so called "gradient catastrophe") and/or the range of U escapes from any compact subset of Ω (i.e. U explodes).*

The theorem simply says that the necessary condition for existence of a classical local unique solution to a hyperbolic system is that it is physical.

Generally speaking a classical existence problems do not occur for general hyperbolic systems only in the cases of scalar conservation law (in any space dimension) and 1-D 2×2 systems.

4.2. Entropy examples

– Scalar CLs (Kruskov's entropies)

– 2×2 systems with one space dimension: (38) is a single linear (hyperbolic) PDE.

– The most important physical models (gas dynamics, MHD, elasticity, phase flow, mixtures,...) has entropies (but not always convex).

4.3. Elementary waves for conservation laws in one space dimension

One can find very useful the class of functions with finite total variation, where

Definition 14 Total variation *of a function v is defined by*

$$\mathrm{TV}(v) = \sup \sum_{j=1}^{N} |v(\xi_j) - v(\xi_{j-1})|, \tag{40}$$

where the supremum is taken by all partitions of the real line

$$-\infty = \xi_0 < \xi_1 < ... < \xi_N = \infty.$$

One can write 40 in the form

$$\mathrm{TV}(v) = \mathrm{limsup}_{\varepsilon \longrightarrow 0} \frac{1}{\varepsilon} \int_{-\infty}^{\infty} |v(x) - v(x-\varepsilon)|.$$

Let

$$\begin{gathered} \frac{\partial}{\partial t} u_1 + \frac{\partial}{\partial x} f_1(u_1, \dots, u_n) = 0 \\ \vdots \\ \frac{\partial}{\partial t} u_n + \frac{\partial}{\partial x} f_n(u_1, \dots, u_n) = 0 \end{gathered} \tag{41}$$

be $n \times n$ one-dimensional conservation laws system, where

$$u = (u_1, \dots, u_n) \in \mathbb{R}^n, \ f = (f_1, \dots, f_n) : \mathbb{R}^n \longrightarrow \mathbb{R}^n.$$

Denote by $A(u) := Df(u)$ Jacobi matrix of f at a point u. The above system reads (using vector notation)

$$u_t + f(u)_x = 0. \tag{42}$$

If a solution is smooth enough (C^1), then quasilinear form

$$u_t + A(u)u_x = 0 \tag{43}$$

defines the equivalent system.

Let us repeat that the system is called *strictly hyperbolic* if all characteristic values of $A(u)$ are real and distinct. They are ordered in the following way

$$\lambda_1(u) < \lambda_2(u) < .... < \lambda_n(u).$$

If there exist n linearly independent characteristic vectors, the system is called hyperbolic.

Left-hand sided $l_1(u), ..., l_n(u)$ and right-hand sided $r_1(u), \dots, r_n(u)$ characteristics vectors are determined in a way that it holds

$$l_i(u)r_j(u) = \begin{cases} 1, & i = j \\ 0, & i \neq j. \end{cases}$$

To avoid technical complications we consider 1-D system

$$\partial_t U(x,t) + \partial_x F(U(x,t)) = 0, \tag{44}$$

with F be a C^3 map from $\Omega \subset \mathbb{R}^n$ into $\mathbb{R}^n$.

4.3.1. Riemann invariants

Definition 15 *An i-Riemann invariant of (44) is a smooth scalar-valued function such that*

$$\mathrm{D}w(U)R_i(U) = 0, \ U \in \Omega.$$

We say that the system (44) has a coordinate system of Riemann invariants if there exist n scalar-valued functions $(w_1, \dots, w_n)$ on Ω such that w_j is an i-Riemann invariant of the system for $i, j = 1, \dots, n$, $i \neq j$.

Immediately we have the following theorem.

Theorem 16 *The functions $(w_1, \dots, w_n)$ form a coordinate system of Riemann invariants for (44) if and only if*

$$\mathrm{D}w_i R_j(U) \begin{cases} = 0, & \text{if } i \neq j \\ \neq 0, & \text{if } i = j. \end{cases}$$

In other words, $\mathrm{D}w_i$ is a left i-th eigenvector of the matrix $\mathrm{D}F$.

It is convenient to normalize eigenvectors $R_1, \dots, R_n$ if the Riemann coordinate system exists such that

$$\mathrm{D}w_i R_j(U) = \begin{cases} 0, & \text{if } i \neq j \\ 1, & \text{if } i = j. \end{cases}$$

Multiplying i-th equation of the system (44) by $\mathrm{D}w_i$, $i = 1, \dots, n$ we get

4.3.2. Shock waves

Like in the case of $n = 1$ we shall suppose that $x = \gamma(t)$ defines a discontinuity curve of piecewise smooth solutions $u_l(x,t)$ and $u_d(x,t)$, i.e.

$$u(x,t) = \begin{cases} u_l(x,t), & x < \gamma(t) \\ u_d(x,t). & x > \gamma(t) \end{cases}$$

In order that u defines a weak solution one has to find γ from Rankin-Hugoniot conditions for system

$$\dot{\gamma} \cdot (u_d - u_l) = f(u_d) - f(u_l). \tag{45}$$

Now, u_d, u_l, $f(u_d)$ and $f(u_l)$ are n-dim vectors. That means that a discontinuity curve $x = \gamma(t)$ can not be found in a direct way like in the case of a single equation. That is, it is not true that for each pair of constant initial vectors u_l, u_d there exists a shock wave solution (like in the case of a single equation).

Denote by

$$A(u, v) := \int A(\theta u + (1-\theta)v)d\theta$$

averaged matrix, where $\lambda_i(u, v)$, $i = 1, \ldots, n$, are its characteristic values. Then (45) can be written in the equivalent form

$$\dot{\gamma} \cdot (u_d - u_l) = f(u_d) - f(u_l) = A(u_d, u_l)(u_d - u_l). \tag{46}$$

In the other word, RH conditions hold if (u_d, u_l) is a characteristic vector of the averaged matrix $A(u_d, u_l)$, and speed $\dot{\gamma}$ equals its characteristic value.

4.3.3. Rarefaction waves

Let us find solutions of the form $u = u\left(\frac{x}{t}\right)$ (selfsimilar solutions) for system (43):

$$u_t + A(u)u_x = -\frac{x}{t^2}u'(y) + \frac{1}{t}A(u(y))u'(y) = 0,$$

where $y = \frac{x}{t}$. From the last equation it follows

$$A(u)u' = yu',$$

what means that u' is equal to the right-hand sided characteristic vector r_i and $y = \lambda_i$, for $i = 1, \ldots, n$.

4.3.4. Entropy conditions

As one could see, even for the case $n = 1$ there is a problem of uniqueness for weak solutions. In order to chose physically relevant solution we will use so called *entropy conditions*. The solution which satisfies it is called *admissible*.

Entropy conditions 1 – vanishing viscosity. A weak solution u to (41) is admissible if there exists a sequence of smooth solutions u_ε to

$$u_{\varepsilon t} + A(u_\varepsilon)u_{\varepsilon x} = \varepsilon u_{\varepsilon xx}$$

which converges to u in L^1 as $\varepsilon \to 0$.

Entropy conditions 2 – entropy inequality. C^1–function $\eta : \mathbb{R}^n \to \mathbb{R}$ is called *entropy* for system (41) with appropriate *entropy flux* $q : \mathbb{R}^n \to \mathbb{R}$ if

$$D\eta(u)Df(u) = Dq(u), \;\; u : \mathbb{R}^n \to \mathbb{R}^n. \tag{47}$$

Note that (47) implies

$$(\eta(u))_t + (q(u))_x = 0,$$

for $u \in C^1$ as a solution to (41). When one substitutes $u_t = -Df(u)u_x$ into the above equation,

$$D\eta(u)u_t + Dq(u)u_x = D\eta(u)(-Df(u)u_x) + Dq(u)u_x = 0.$$

A weak solution u to (41) is admissible if

$$(\eta(u))_t + (q(u))_x \leq 0$$

in a distributional sense, i.e.

$$-\int \eta(u)\varphi_t + q(u)\varphi_x \geq 0,$$

for every $\varphi \geq 0$, $\varphi \in C_0^\infty(\mathbb{R} \times [0, \infty))$.

Thus,

$$D\eta(u)u_t + Dq(u)u_x = 0$$

outside a discontinuity, and

$$\dot{x}_\alpha(\eta(u(x_\alpha+)) - \eta(u(x_\alpha-))) \geq q(u(x_\lambda+)) - q(u(x_\alpha-))$$

on the discontinuity curve $x = \dot{x}_\alpha(t)$.

Entropy condition 3 – Lax condition. Shock wave connecting states u_l i u_d and has a speed $\dot{\gamma} = \lambda_i(u_l, u_d)$ is admissible if

$$\lambda_i(u_l) \geq \lambda_i(u_l, u_d) = \dot{\gamma} \geq \lambda_i(u_d). \tag{48}$$

Because of the ordering of characteristic values

$$\lambda_j(u_l) > \dot{\gamma},\ j > i$$
$$\lambda_j(u_d) < \dot{\gamma},\ j < i.$$

Such a wave is called i-th shock wave.

4.3.5. Rarefaction (RW) and shock wave (SW) curves

Fix $u_0 \in \mathbb{R}^n$ and $i \in \{1, \ldots, n\}$. Integral curve for vector field r_i through u_0 is called *i-th rarefaction curve* (RW_i). One can get it explicitly by solving the Cauchy problem

$$\frac{du}{d\sigma} = r_i(u),\ u(0) = u_0. \tag{49}$$

That curve will be denoted by

$$\sigma \mapsto R_i(\sigma)(u_0).$$

Due to the above definition, u_0 can be joined with $u \in RW_i(u_0)$ by a single rarefaction wave.

Note that a curve parameterization depends on a choice of r_i. If $|r_i| \equiv 1$ then the curve is parametrizated by its length.

Fix $u_0 \in \mathbb{R}^n$ again. Let u be a right-hand state which can be joined to u_0 with i-th shock wave. (One uses RH conditions and Lax condition (48)). So, vector $u - u_0$ is a right-hand sided i-th characteristic vector for $A(u, u_0)$. By basic theorem of linear algebra this is true if and only if $u - u_0$ is orthogonal to l_j, $j \neq i$ (j-th left-hand sided characteristic vector for $A(u, u_0)$) i.e.

$$l_j(u, u_0)(u - u_0) = 0,\ \forall j \neq i,\ \dot{\gamma} = \lambda_i(u, u_0). \tag{50}$$

One can see that (50) is the system of $n-1$ scalar equation with n unknowns (components of $u \in \mathbb{R}^n$). Linearizing (50) in a neighborhood of u_0 one gets linear system

$$l_j(u_0)(w - u_0) = 0,\ j \neq i.$$

It has a solution $w = u_0 + Cr_i(u_0)$, $C \in \mathbb{R}$. By Implicit Function Theorem, a set of solutions forms a regular curve (C^1–class) in a neighborhood of u_0 with a tangent vector r_i in the point u_0. That curve is called the *curve of i-th shock wave* and denoted by

$$\sigma \mapsto S_i(\sigma)(u_0).$$

Both of the above curves exist in a neighborhood of u_0 (if f is smooth enough), and it can be proved that they have the same tangent in the point u_0 parallel to $r_i(u_0)$.

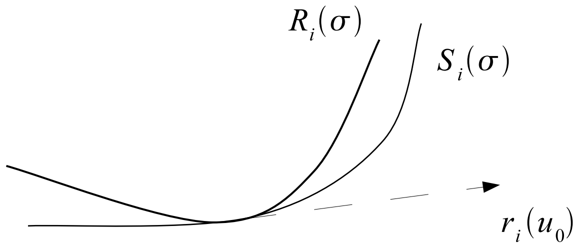

Figure 15: Shock wave and rarefaction curves

4.3.6. Riemann problem

Definition 17 *We say that i-th characteristic field is* genuinely nonlinear *if*

$$D\lambda_i(u)r_i(u) \neq 0.$$

If

$$D\lambda_i(u)r_i(u) \equiv 0,$$

then i-th field is said to be linearly degenerate

Note that in the case when i-th field is genuinely nonlinear one can chose the orientation of r_i (by changing its sign, eventually) such that

$$D\lambda_i(u)r_i(u) > 0.$$

Let us suppose that the above relation holds in the sequel for genuinely nonlinear fields.

In the rest of the paper we shall use the following general assumption:

System (41) is strictly hyperbolic with smooth coefficients. For each $i \in \{1, \dots, n\}$, i-th characteristic field is either genuinely nonlinear or linearly degenerate.

Centered rarefaction wave. Let i-th field be genuinely nonlinear and suppose that u_d lies on a positive part of RW curve starting from u_l, i.e. $u_d = R(\sigma)(u_l)$ for some $\sigma > 0$.

Theorem 18 *Let us define*

$$\lambda_i(s) = \lambda_i(R_i(s)(u_l))$$

for every $s \in [0, \sigma]$.

Because of genuine nonlinearity, mapping $s \mapsto \lambda_i(s)$ is strictly increasing. Let $t \geq 0$. Function

$$u(x,t) = \begin{cases} u_l, & x < t\lambda_i(u_l) \\ R_i(s)(u_l), & x = t\lambda_i(s) \\ u_d = R_i(\sigma)(u_l), & x > t\lambda_i(u_d), \end{cases} \tag{51}$$

where $\frac{x}{t} = y = \lambda_i(s)$, $s \in [0, \sigma]$, is piecewise smooth solution to Riemann problem

$$u_t + f(u)_x = 0$$
$$u\Big|_{t=0} := u_0 = \begin{cases} u_l, & x < 0 \\ u_d, & x > 0. \end{cases}$$

Proof. One can easily see that

$$\lim_{t\to 0}\|u(x,t)-u_0\|_{L^1}=0.$$

Besides that, (41) trivially holds true for $x<t\lambda_i(u_l)$ and $x>t\lambda_i(u_d)$, because $u_t=u_x=0$. Suppose $x=t\lambda_i(s)$, for some $s\in(0,\sigma)$. Since $u\equiv\mathrm{const}$ along each halfline $\{(x,t):\ x=t\lambda_i(s)\}$, there holds

$$u_t(x,t)+\lambda_i(s)u_x(x,t)=0. \tag{52}$$

Since

$$\begin{aligned}u_x&=\frac{\partial u}{\partial x}=\frac{dR_i(s)(u_l)}{ds}\\&=\Big(\frac{d\lambda_i(s)}{ds}\Big)^{-1}\frac{d\lambda_i}{\mathrm{dx}}=r_i(u)\Big(\frac{d_i(s)}{ds}\Big)^{-1}\frac{1}{t},\end{aligned}$$

u_x is a characteristic vector for $A(u)$ when $\lambda_i(s)=\lambda_i(u(t,x))$, i.e.

$$A(u)u_x=\lambda_i u_x.$$

Note that assumption $\sigma>0$ is crucial for the above construction of a solution. If $\sigma<0$, (51) would define a triple valued function in the area $\frac{x}{t}\in[\lambda_i(u_d),\lambda_i(u_l)]$.

Shock waves. Let i-th characteristic field be genuinely nonlinear and let u_d be connected with u_l by i-shock wave, $u_d=S_i(\sigma)(u_l)$. Then $\lambda:=\lambda_i(u_d,u_l)$ is the speed of that wave and

$$u(x,t)=\begin{cases}u_l, & x<\lambda t\\ u_d, & x>\lambda t\end{cases} \tag{53}$$

is piecewise constant solution to the above Riemann problem.

Note that in the case $\sigma<0$ that solution is admissible in the Lax-sense, because

$$\lambda_i(u_d)<\lambda_i(u_l,u_d)<\lambda_i(u_l).$$

For $\sigma>0$, one would have

$$\lambda_i(u_l)<\lambda_i(u_d)$$

and Lax condition could not be satisfied.

Contact discontinuities. Suppose that i-th characteristic field is linearly degenerate and $u_d=R_i(\sigma)(u_l)$ for some σ. By the assumption, λ_i is constant along that curve. Putting $\lambda:=\lambda_i(u_l)$, one can see that piecewise constant function given by (53) solves the above Cauchy problem, because RH condition is satisfied at discontinuity curve.

$$\begin{aligned}f(u_d)-f(u_l)&=\int_0^\sigma Df(R_i(s)(u_l))r_i(R_i(s)(u_l))\mathrm{ds}\\&=\int_0^\sigma \lambda_i(R_i(s)(u_l))r_i(R_i(s)(u_l)\mathrm{ds}\\&=\lambda_i(u_l)\int_0^\sigma\frac{\mathrm{d}R_i(s)(u_l)}{\mathrm{ds}}\mathrm{ds}=\lambda_i(u_l)(R_i(\sigma)(u_l)-u_l).\end{aligned}$$

We have used here that

$$\begin{aligned}\frac{d}{ds}\lambda_i(R_i(s)(u_l))&=D\lambda_i(R_i(s)(u_l))\frac{\mathrm{d}R_i(s)(u_l))}{\mathrm{d}s}\\&=(D\lambda_i r_i)\Big(R_i(s)(u_l)\Big)=0,\end{aligned}$$

as well as the definition of linear degeneracy.

In that case Lax conditions hold thus regardless to the sign of σ, because

$$\lambda_i(u_d) = \lambda_i(u_l, u_d) = \lambda_i(u_l).$$

From the above calculations one can deduce that

$$R_i(\sigma)(u_0) = S_i(\sigma)(u_0),$$

for every σ.

4.3.7. General solutions

As we have seen before, the set of points $\{u_d : u \in \mathbb{R}^n\}$ which could be connected with a left-hand side state of Riemann problem is just a curve. In order to connect two arbitrary points u_l, $u_d \in \mathbb{R}^n$ with an entropic solution of Riemann problem one can insert at most $n-1$ vectors

$$u_l, u_1, u_2, \ldots, u_{n-1}, u_d$$

such that between each pair (u_l, u_1), (u_1, u_2), ..., (u_{n-1}, u_d) there is one of the previously described elementary waves.

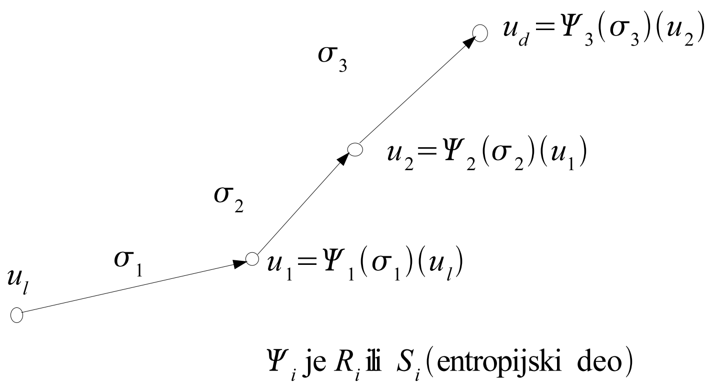

Figure 16: Sketch of a solution to Riemann problem

If the initial condition belongs to L^∞, then we shell approximate it by piecewise constant function. So there are a lot of Riemann problems which have to be simultaneously solved. One by one solution in the form of elementary waves can be easily find, but the main problem is how to deal with a huge number of mutual wave interactions.

We shall describe two procedures for that purpose.

(1) **Glimm scheme**. Before the first interaction of the initial elementary waves, one approximates a solution with new piecewise constant function again. That function becomes a new initial data and procedure is repeated as many times as needed. Rarefaction wave is approximated by a fan of non-admissible shock waves in this procedure.

The procedure will converge for small enough variation of initial states, i.e. total variation of the initial data is small enough.

There are a lot of technical problems concerning the above scheme, so a lot of effort was given to find a new procedure, the following one.

(2) **Front-tracking method)**. Again, rarefaction wave is approximated with a fan of non-entropic shock waves. But now waves are permitted to interact. In a point of interaction there is a new Riemann problem. One can solve it accurately or approximatively. In the later case, one constructs non-physical shock wave with small amplitude, but with the larger speed of all possible waves in order to prevent blow-up effect.

After that one can again use the same method for later interactions.

Again, this procedure will converge when total variation of the initial data is small enough.

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