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Interesting Mathematical Problems in Sciences and Everyday Life - 2011

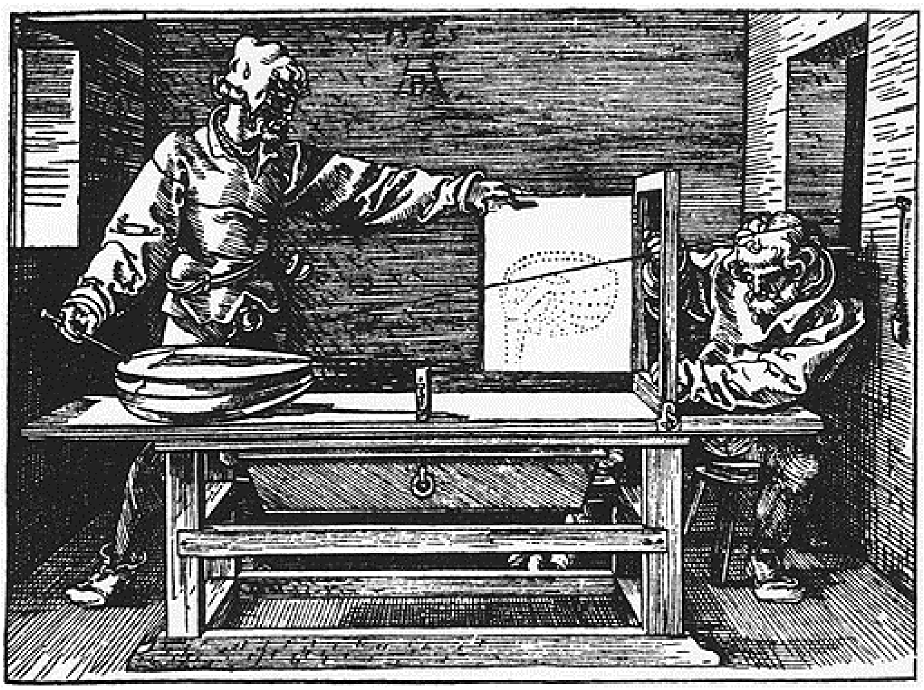
3D on paper and screen
Displaying three-dimensional figures

Axonometric and perspective representation

Lajos Szilassi

University of Szeged

E-mail: szilassi@jgytf.u-szeged.hu



A. Dürer's woodcut of the study of perspective

Abstract: The task of classical descriptive geometry is to produce precise, reconstructible drawings on three-dimensional figures; nowadays, in the case of drawings made by means of computer, this is often completed with movable, interactive parts. This work is a brief introduction to the axonometric and perspective representation, the tools for preparing descriptive images, keeping in view some principal aspects. We discuss both the ruler-and-compass construction methods and those in which computers are applied.

The emphasize is not on the tutorial aspects of a software made for such purpose, but rather, on its mathematical backgroumd.

Our aim is that to develop a certain awareness of our readers, in the sense that they not only look at either a hand-made or a computer-produced drawing with more appreciative eyes, but they themselves be able to prepare such drawings, or, with suitable programming knowledge, their own softwares as well.

There are files attached to almost all the drawings presented here, which were made by the dynamic geometry softwares, either Euklides or Euler3D.

These softwares can be downloaded, and basics for their use are also given here.

We suggest our readers to study this writing together with interactive use of these softwares.

1. The tools of descriptive geometry

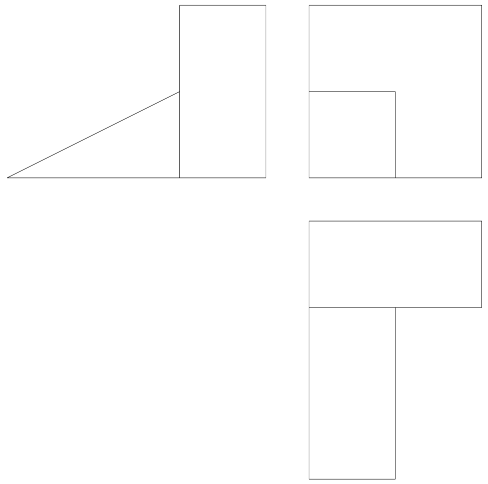
Nowadays the computers supply beautifully designed, moving, furthermore, interactively movable drawings of real, or just virtual, spatial figures. In most cases they unburden us of the construction of these drawings; at the same time, they also deprive us of the pleasure of drawing. One cannot be content with passive reception of the view; even if one cannot make such beautiful drawings, one should know what happens while a spatial figure is produced on the paper sheet or on the screen. We would like to develop such awareness of our readers, in the sense that they not only look at either a hand-made or a computer-produced drawing with more appreciative eyes, but they themselves be able to prepare such drawings. To this end, we will look over the basics of descriptive geometry.

The task of descriptive geometry is to produce a clear, uniquely reconstructible image of spatial figures, such that one should be able to read out from it both the structural and metric data of the given geometric figure. Its method is the projection: one project the geometric figure to be represented onto one or more (image) plane (the plane of the paper sheet or of the screen) by straight lines, which are parallel, or start from a common point.

If the primary aim of the representation is simpler constructibility, easy reconstructibility, or that the metric data should be faithfully represented, then the most suitable method is the Monge[1] projection. If one applies, as usual, only two image planes, than in order to uniquely represent e.g. a polyhedron (a solid body with planar side faces), one has to label the vertices, and also the two distinct images of each of them. Without such labeling, one

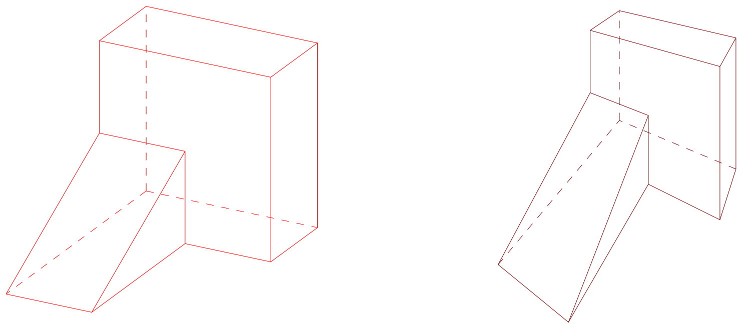
[1]Gaspard Monge (1746–1818) French mathematician, after the French Revolution Minister of the Navy, military organizer.

cannot always reconstruct tha spatial figure just from thes two images. A third, side-view image may substantially contribute to the unique reconstructibility of the figure.



The top view, front view and (right[2]) side view image of a polyhedron

If the primary aim of the representation is descriptiveness, then the most suitable method is the axonometric, or perspective representation.



The axonometric and perspective drawing of the polyhedron above

We would like to make the reader acquainted with the basics of the axonometric and perspective representation. One of the benefits of this may be some skill in preaparing such drawing. On the other hand, the drawings in the textbooks and on the screen of the computers are often axonometric or perspective drawings, thus an appreciative, or, sometimes, critical attitude towards such drawings is advisable. We will discuss the advantages and disadvantages of the different methods of representations, mostly via comparing the drawings made by these methods.

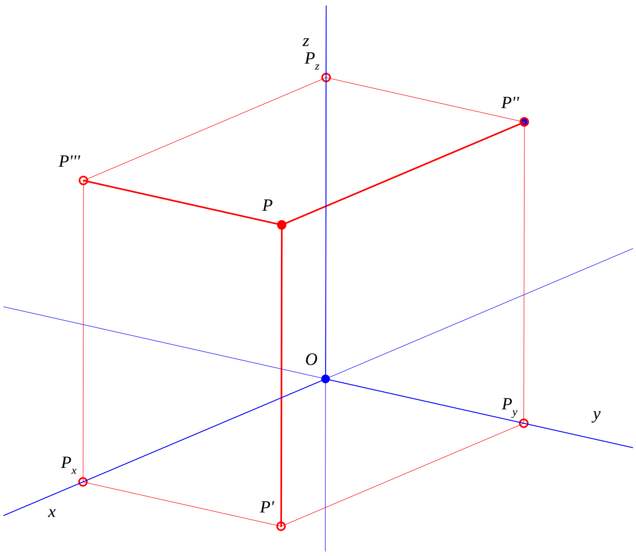
[2] The attribute „right side” refers to the fact that the figure is projected from the direction of ones right hand to the image plane located on left to the first two image planes; thus having unfolded the image planes to the plane of our sheet of paper, it will be located on the left hand side of our drawing.

2. Axonometry

2.1. The basics of the axonometric representation

The essence of the axonometric representation is that the spatial figure to be represented is placed in a spatial orthogonal coordinate system, then it is mapped by a parallel projection, along with the coordinate system, onto a plane, the so-called axonometric image plane.

Let three pairwise perpendicular directed straight lines, *x*, *y*, *z*, be given in the space, incident to the origin, such that they form a so-called right-handed system[3] in this order. Place in this so-called spatial orthogonal coordinate system a point P. Project it orthogonally onto the planes (*x*,*y*), (*y*,*z*) and (*x*,*z*). Then project the points *P'*, *P''* and *P'''* obtained in this way respectively, along with the point *P* and the axes, onto the plane of our sheet of paper (or screen) by a parallel projection, i.e. onto the axonometric image plane. (This latter cannot be parallel with the direction of the projection.) We call the images of the coordinate axes the axonometric frame of axes. Given any two of the points *P*, *P'*, *P''* and *P'''* in due manner, the other two can be constructed. "In due manner" means in this case that the lines of recall of the points must be parallel with the corresponding coordinate axes. Thus, for example $PP'P_yP''$ is a parallelogram whose sides are parallel to the axes *X* and *Z*, respectively (P_y is the straight line going through *P* and parallel with the *(x,z)* plane). This requirement is essentially the same as that in the Monge projection the lines of recall must be perpendicular to the line of intersection of the image planes.



The axonometric image of a point

[3] "Right-handed" means that if we stretch the first finger, the central finger and the thumb of our right hand in (approximately) mutually perpendicular directions such that first finger points along the positive direction of the axis *y*, the thumb is along the direction of the axis *x*, then the central finger points along the positive direction of the axis *z*. We note that this is a non-mathematical convention, just as we consider the counter-clockwise rotation in the plane positive.

The axonometric image of the point P can also be uniquely determined from the points P_x , P_y , P_z , the axonometric images of the projections of P onto the coordinate axes. In turn, these latter points can also be uniquely determined if the axonometric images of unit segments parallel with the coordinate axes are given. Essentially, the coordinates of the point P in the spatial orthogonal coordinate system are (P_x, P_y, P_z). One can say that the axonometric image of a point P given by its Cartesian coordinates is uniquely determined. On the other hand, can they be reconstructed as well? The answer to this question is given by the most fundamental relationship of the axonometric representation:

POHLKE'S[4] THEOREM:

Let three line segments be given in the plane with common endpoints, of arbitrary length and direction. Then there are three pairwise perpendicular line segments of equal length in the space (such as for example three edges starting from a vertex of a cube) such that they can be mapped by a suitable parallel projection onto the given coplanar line segments.

By parallel projection, the images of parallel lines are parallel (or coincident); hence any drawing that consists of three parallelograms, pairwise having a side in common, can be considered as an image of a cube obtained by parallel projection. In most cases such a projection is of course not orthogonal to the axonometric image plane (the plane of our sheet of paper). This causes a problem when considering such an image. For, a drawing is usually seen more or less in perpendicular direction. However, one should have look at it possibly at an oblique angle, and from a suitable direction, in order to see the image "properly".

The axonometry is called an *obliqe* (or *clinogonal*) *axonometry* if the direction of projection is not perpendicular to the image plane. One can easily draw a polyhedron in oblique axonometry, since one should only ensure that parallelity and length ratios are preserved on the image. On the other hand, one can seldom be satisfied with such a drawing, since it is not easy to find the direction in which it can be considered realistic.

Choose a spatial orthogonal coordinate system such that its axes are the edge lines of a cube, and let the edges of the cube be of unit length. Then, by the theorem above, three arbitrarily chosen line segments in the plane with common endpoints correspond to the axes of this coordinate system, and are the images of the unit line segments on these axes. The ratio of the length of these segments and of the edges of the cube is called the *foreshortening* along the axes x, y and z, respectively. (This ratio can sometimes be larger than one; think of the evening shadow of a stick, which can be longer than the stick itself.)

There is a so-called affine relationship between a planar figure and its parallel projection. For example, the affine images of the unit segments on the axes x and y starting from the origin are the (arbitrarily given) images of these segments. These together uniquely determine the affine map between the xy plane (in the space) and its axonometric image. Thus the axonometric image of a figure can be uniquely constructed from its Monge projection.

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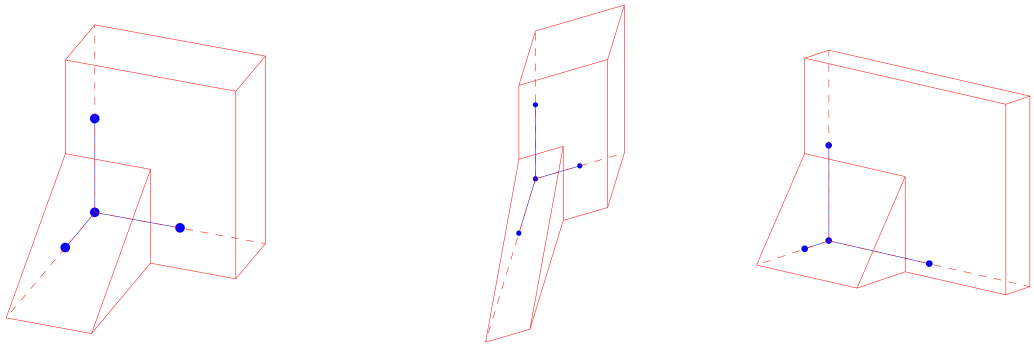
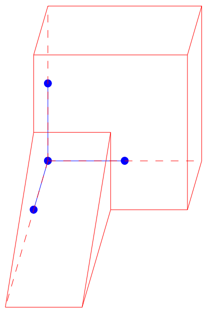
[4] Karl Pohlke (1810-1876) German professor of descriptive geometry has formulated the theorem named after him in 1853, but he published it only in his book *Darstellende Geometrie,* 1860 (without proof).



The images of a concave polyhedron in oblique axonometry

2.2. *The cavalier perspective*

There is a frequently used particular case of the oblique axonometry, the so-called *cavalier perspective*, or *cavalier projection* (perhaps it is too frequently used). In this case the (*xz*) plane of the coordinate system is parallel with the axonometric image plane. Thus the foreshortening on the axes *y* and *z* is 1, while on the axis *x* (the direction of which can be arbitrary) it can be either smaller or larger then 1. (Usually it is chosen ½ or 2/3, but this is just a convention.)



Cavalier axonometry

In this way of representation the direction of the projection is necessarily oblique; for, if it were orthogonal, then we had merely a front view projection, which is of course not sufficient for the reconstruction of the original figure.

The advantage of the cavalier perspective is that the details lying in the plane parallel with the *(y,z)* plane (it is said the front plane) are congruent with their axonometric images. It was initially used for military fortifications. In French, the “cavalier” (literally rider, horseman) is an artificial hill behind the walls that allows to see the enemy above the walls. The cavalier perspective was considered originally the way the things were seen from this high point.

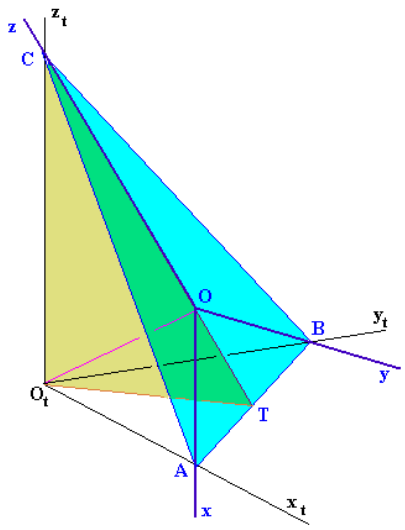
2.3. *Orthogonal axonometry*

For the representation of a solid geometric figure the **orthogonal** (perpendicular) axonometry seems the most suitable tool, which projects the spatial coordinate frame, along with the figure placed in it, perpendicularly onto the axonometric image plane. In this representation (as in all other axonometric representation) the images of parallel and equal line segments are also parallel and equal; thus, for example, they are particularly suitable to represent the relationship between the edges of a polyhedron. At the same time, since the drawings are obtained by orthogonal projection, when the images are seen in perpendicular direction, they almost look like the original solid figure. “Almost”, since actually a *perspective image* is produced in our eyes, in which parallel segments look not parallel (although from larger distance this “convergence” is imperceptible). On the other hand, our eyes ensure spatial vision, which a single drawing can never provide. We return to this later on. Now let us get acquainted with the orthogonal axonometry, on the one hand, in order to be able to make such drawings by ourselves. On the other hand, it is worthwhile to bear in mind that most of the images made by computer are produced in this way; thus it is appropriate to consider such a drawing “understandingly”.

Producing an orthogonal axonometric image by construction

It is no matter from the point of view of the producing image where to place the spatial coordinate frame, along with the figure to be projected: it can be either behind or in front of the axonometric image plane. Now place it behind that, so that the axonometric image plane intersects the coordinate axes in the points *A*, *B* and *C*; these points are the vertices of the so-called trace triangle.

Denote x_t, y_t, z_t, O_t the spatial coordinate frame and its origin, and denote *x*, *y*, *z*, *O*, respectively, the orthogonal projections of these points on the plane *ABC*. Since the direction of projection is perpendicular to the image plane (the plane of the triangle $ABC\Delta$), then $O_tO \perp ABC\Delta$. Hence $CO_tT\Delta \perp ABC\Delta$, where $T = CO \cap AB$. On the other hand, since $CO_t = z_t \;\perp\; (x_t y_t) = ABO_t\Delta$, thence $CO_tT\Delta \perp ABO_t\Delta$. Thus, since both $ABC\Delta$, and $ABO_t\Delta$ is perpendicular to $CO_tT\Delta$, thence so is their line of intersection; hence $CO_tT\Delta \perp AB$. This means that all the straight lines of $CO_tT\Delta$ are perpendicular to *AB*; thus, $CO \perp AB$. Likewise, one can see that $AO \perp BC$ and $BO \perp CA$ also holds.



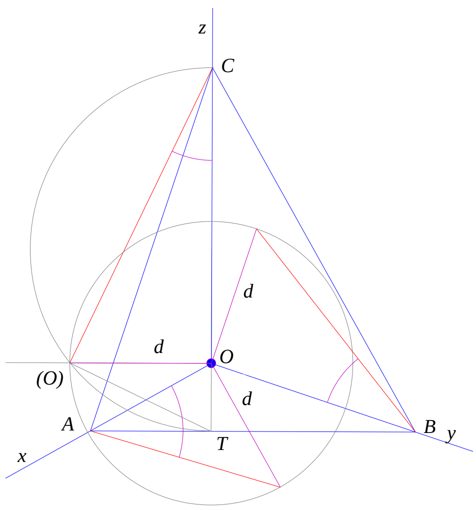


Image plane angles in orthogonal axonometry

Our result means that the lines of altitudes of the $ABC\Delta$ will be the projections of the orthogonal projections of the axes, and its orthocentre will be the point *O*.

Accordingly, an orthogonal axonometric system can be given by

- the images of the three axes;
- the acute trace triangle *ABC*;

- the obtuse triangle *AOB*.

In all these three cases one obtains the trace triangle *ABC*, whose orthocentre is *O*.

Our task is to construct from these data the foreshortenings of the orthogonal axonometry, i.e. the images of the unit line segments on the spatial coordinate axes. Instead, we shall construct the image plane angles, i.e. the angles between the spatial axes and the image plane. For, if one wants to measure a segment on any of the axes, it will suffice to measure this segment on one of the sides of the image plane angle belonging to the given axis; then the orthogonal projection of this segment on the other side will just be a segment whose length provides the desired foreshortening.

The image plane angle belonging to the axis z_t will be the angle CO_tT of the triangle TCO_t. To construct this, it suffices to rotate the right triangle $CO_tT\Delta$ into the image plane (i.e. our sheet of paper). Its hypotenuse *CT*, as well as the orthogonal projection of its leg is just on our drawing. The triangle $C(O)T\Delta$ obtained by constructing the Thales' circle of CT contains the desired $(O)CO\angle$; on the other hand, as a "by-product", we obtained the distance $d = (O)O = O_tO$, which is usually called the *distance of the orthogonal axonometry*.

The same construction could also be performed for obtaining the image plane angles belonging to the other two axes. But, since the distance *d* is already known, one can easily construct the right triangles, whose one leg is *AO*, respectively *BO*, and the leg opposite to the desired image plane angles is *d*.

By the time we performed this construction, our drawing became too crowded with lines, hence it is not suitable to give in the same drawing the axonometric image of the figure to be represented. Thus it is worth to use a separate sheet, on which the axonometric images, as well as the plane angles are copied, in order to concentrate just to the figure to be represented (in our case, a polyhedron).

The purpose of the construction above was in fact to obtain the foreshortenings along the coordinate axes. Knowing them, the geometric figure (the polyhedron) can be drawn in the same way as in oblique axonometry, but in this case our drawing will seem more realistic. Since essentially we constructed the plane angles of the coordinate axes, this can be exploited to the purpose on "placing" on an axis not only the unit segment, but an arbitrary segment as well.

The foreshortenings of the coordinate frame can also be obtained in other way, with a simpler construction, at the expense of somewhat deeper consideration. In addition to constructing these segments, we also give a transformation formula. Given a point by its coordinates, this formula yields the coordinates of its image (in the plane of the computer screen); thus our readers with some skill in programming may produce the axonometric image of a polyhedron on the screen as well.

Producing the axonometric image by computer

The essence of the consideration is that given the unit segments O_tX_t, O_tY_t, O_tZ_t on the axes of the spatial orthogonal coordinate system, one positions them with respect to the image plane; then one constructs their orthogonal projections onto this plane. Besides, we calculate coordinates of the endpoints of these segments with respect to coordinate system of the screen.

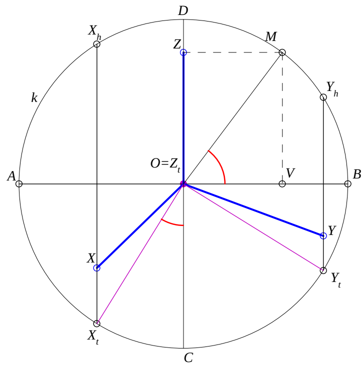
What is it with respect to which a spatial figure is moving on the screen of the computer (even if actually we move it)? The programmers consider a coordinate system fixed to the screen (in other words, to the image plane), as well as a moving spatial coordinate system in which the figure (actually, a polyhedron) in question is described. When producing an axonometric image (either by constructing by ruler and compass, or by calculation with computer), one has to determine, for a point given in the moving coordinate system, its image in the fixed coordinate system.

Let be denoted the axes of the fixed coordinate system (i.e., those of the image plane) by u and v (the horizontal and vertical axis, respectively). Temporarily, also consider this coordinate system spatial, thus denote its third axis, perpendicular to the image plane, by w (this is directed towards us if the system (u, v, w) is right-handed).

We first locate O_t, X_t és Y_t in the plane of our sheet (screen) so that the origin of the two coordinate systems coincide, the point Y_t lies on the axis u and the point X_t lies on the axis v. In this case the vector O_tZ_t is also perpendicular to the image plane, and is directed towards us, provided that our spatial coordinate system is right-handed, too. Then, the coordinates of te points under investigation are:

$$\mathrm{X_t} = (\ 0, -1, 0).$$
$$\mathrm{Y_t} = (\ \ 1, \ \ 0, 0).$$
$$\mathrm{Z_t} = (\ 0, \ \ 0, 1).$$

The point X_t still coincides with the point C of the image plane, and Y_t with B, where A, B and C, D are the points lying on the axis u and v, respectively, at a unit distance from the origin.



Applying two rotations, any other position of the spatial coordinate system with respect to the image plane can be attained, different from that in which the image of the axis z is the vertical axis of the image plane. First, rotate the this latter system about its own axis z_t=O_tZ_t by an angle $\vartheta = \mathrm{COX_t}$. Since we use a spatial rotation, it is a bit more difficult to determine the direction of the rotation. We use the convention that the rotation is considered positive if it positive as seen from the direction opposite to the vector $\overrightarrow{\mathrm{OZ_t}}$ (i.e., the vector of the axis of rotation). In this case this means that, while the axis z_t is kept fixed, we see the points X_t and Y_t moving in negative direction when ϑ is increased. (We fixed this direction and angle, since most of the softwares which move spatial figures do the same.)

Then the endpoints of the vectors of the spatial unit vectors are:

$$\begin{aligned} X_t &= (-\sin\vartheta,\ -\cos\vartheta,\ 0). \\ Y_t &= (\ \ \cos\vartheta,\ -\sin\vartheta,\ 0). \\ Z_t &= (\quad 0, \quad\quad 0,\ 1). \end{aligned}$$

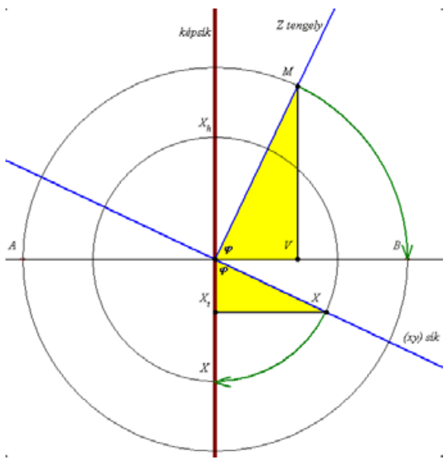
Let the other rotation be a rotation by an angle φ about the axis $u=OB$. During such a rotation the point Z_t moves along a circle which is perpendicular to the axis u, and hence to the image plane; the diameter of this axis is CD. At the same time, the point X_t moves along a circle of radius $\cos\vartheta$ in a plane parallel to the former, and Y_t moves along a circle of radius $\sin\vartheta$ (the diameter is X_hX_t, and Y_hY_t respectively).

By a ruler-and-compass construction, or, for example, by the dynamic geometry software EUKLIDES, one easily obtains the (orthogonal axonometric) image of our spatial coordinate system. Let $\varphi = BOM\square$, where M is an arbitrary point of the circle k. The image of the point Z_t, lying on the axis v, is the perpendicular projection of the point M on CD; it is Z.

Since the plane (xy) rotates together with the axis z, the perpendicular projection of each point of the plane (xy) gets closer or farther to the axis u of the image plane at the same ratio as the perpendicular projection of the point M on the axis u (i.e., V) to the origin:

$$\frac{X_hX}{XX_t} = \frac{Y_hY}{YY_t} = \frac{AV}{VB}\ .$$

This can be made clearer by a drawing which shows the image plane from ”side-face”:



Accordingly, the endpoints of the spatial unit vectors in the coordinate system az (u, v, w) are:

$$\begin{aligned} X_t &= (-\sin\vartheta,\ -\cos\vartheta\cdot\cos\varphi,\ \cos\vartheta\cdot\sin\varphi). \\ Y_t &= (\ \ \cos\vartheta,\ -\sin\vartheta\cdot\cos\varphi,\ \ \sin\vartheta\cdot\sin\varphi). \\ Z_t &= (\quad 0, \quad\quad \sin\varphi, \quad\quad \cos\varphi). \end{aligned}$$

Here the first two coordinates are the coordinates of axonometric image of the unit vectors. Given a point $P_t(x, y, z)$, its axonometric image $P(u, v)$ can be calculated by the follolwing formula:

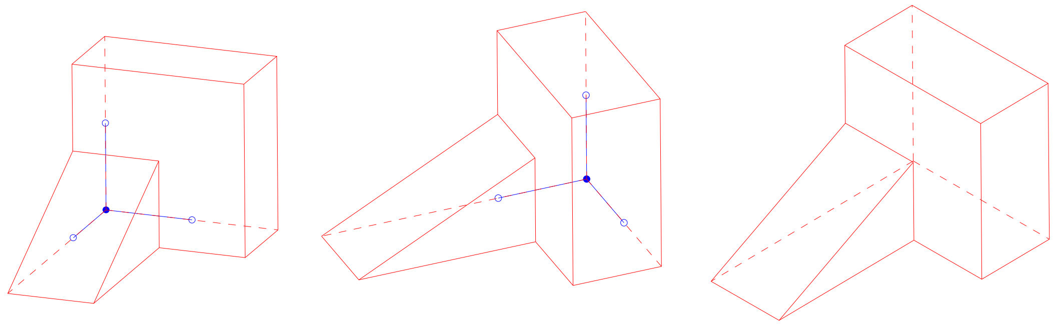
$$u = -x \cdot \sin \vartheta + y \cdot \cos \vartheta.$$
$$v = -x \cdot \cos \vartheta \cdot \cos \varphi - y \cdot \sin \vartheta \cdot \cos \varphi + z \cdot \sin \varphi.$$

As a ”side-product”, we obtained how far the point *P* is from the image plane:

$$w = x \cdot \cos \vartheta \cdot \sin \varphi + y \cdot \sin \vartheta \cdot \sin \varphi + z \cdot \cos \varphi.$$

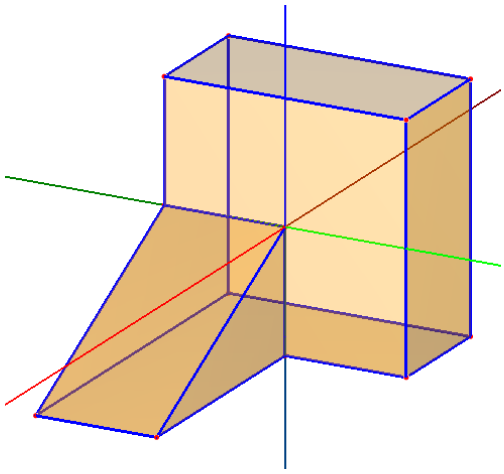
This data is needed when want to draw on the screen a perspecive image.

The users of the attached two EUKLIDES files may observe that in the first one the rotation of the figures about the axis *z* can be achieved by moving the point *F*, while tilting, that is, rotating about the axis *u* of the screen, is achieved by moving the point *V*. To this purpose, we used only horizontal movement of the point *F*, and vertical movement of the point *V*. These can also be drawn together.



The images of a polyhedron in orthogonal axonometry.

This is made in the other file, where both rotations can be achieved by moving a single point *M*. The same is done by most softwares which move spatial figures, included Euler 3D, too, where the figure can be moved using pushed mouse button.

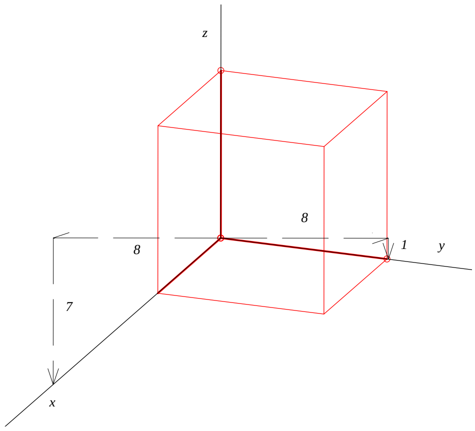


The image of a polyhedron in orthogonal axonometry made by Euler 3D.

Moving the Euler 3D drawing above, one may observe that in the bottom row there are two parameters changing. These are just the values of the angles φ and ϑ given in degree. The same coordinates are used in geography as well.

One may say that an orthogonal axonometric image in general position is most suitable for visualizing solid geometric figures. A ”professional” graphic artist, who for example undertakes making figures of textbooks, can be expected to prepare such a drawing, but a mathematics teacher cannot be be expected to do so. A simpler way for the latter is to use the ”technical axonometry”. It is a special case of the oblique Axonometry, which which surprisingly well approximates a drawing made in orthogonal axonometry with axes in of prescribed direction.

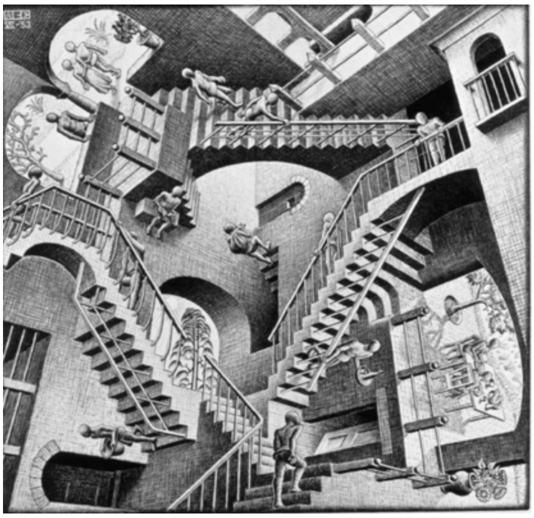
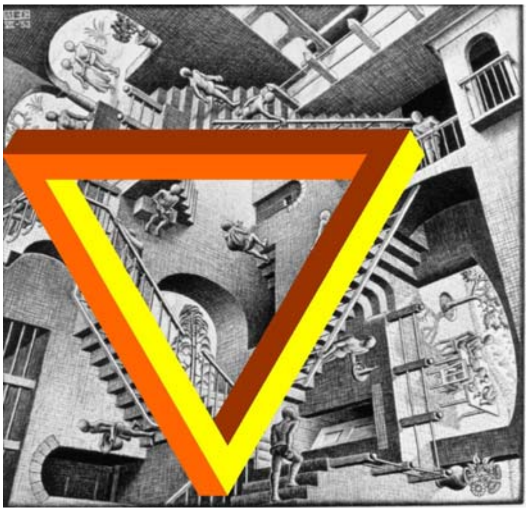
In this axonometry the slope of the axis x is $\frac{7}{8}$ and that of the axis y is $-\frac{1}{8}$. With this setting of axes, the foreshortenings along the axes y and z in the orthogonal axonometry are almost the same, and, along the axis x it is approximately half of the former. Thus one does not make great error if the foreshortenings are taken as $q_x = \frac{1}{2}$; $q_y = q_z = 1$, respectively.



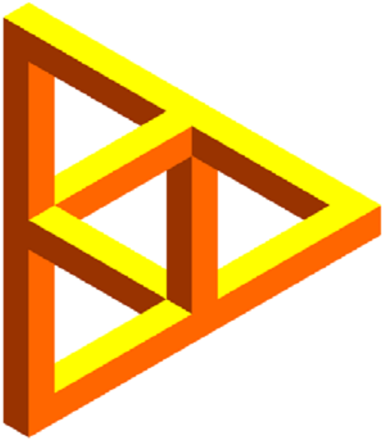
Cube represented in orthogonal axonometrxy as well as in technical axonometry

The isometric axonometry

The construction of a drawing in orthogonal axonometry provides a ”right” representation for almost all positioning. But there are also special cases of the orthogonal axonometry. One can most easily construct a drawing in *isometric axonometry*, where the images of the axes subtend an angle of 120 degrees pairwise with each other. This results in the same foreshortening along each axis, thus one does not need to deal with its construction. For example, in this way of representation the images of the closest and farthest vertex of a cube coincide; this may provide an idea of drawing an ”impossible” triangle, which one can meet in the art of Escher.

M.C. Escher: Relativity

Escher's triangle and a cubein the isometric axonometry

One may criticize the orthogonal axonometry because of lack of "perspectivity", that is, the straight lines that are parallel to each other in reality do not converg to each other. This problem can be solved by the perspective representation. In the next section we get acquainted with its basics, together with its advantages and disadvantages as well.

3. Perspectivity

3.1. Constructing a perspective image

The perspective representation also uses a single image plane, called the *perspective image plane*, where the projecting lines are incident to a common point, the *centre of projection*.

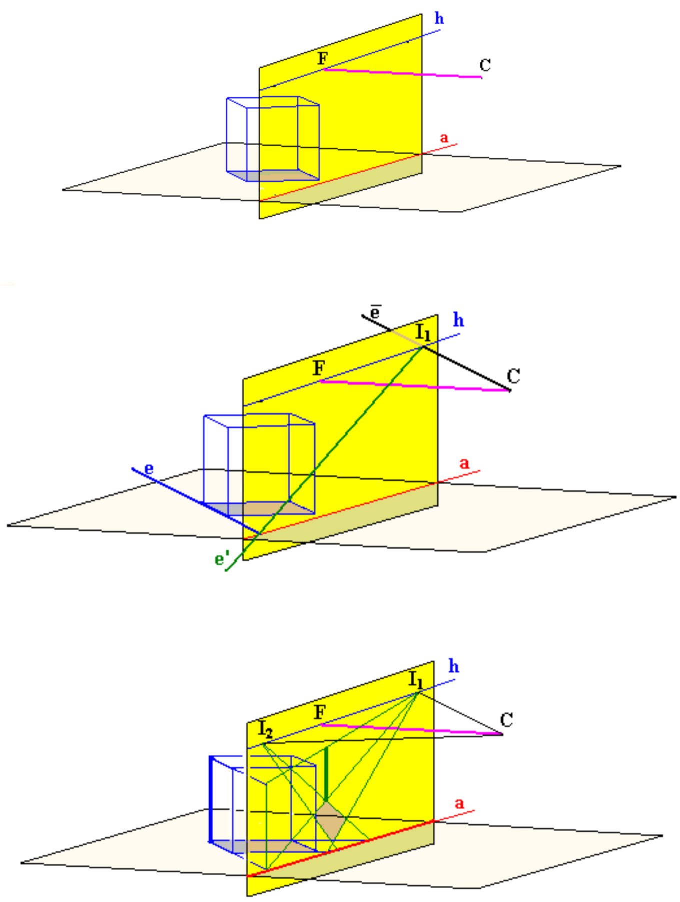
Consider the perspective image plane π vertical. The object to be represented (in our case, a cube) is placed on a "horizontal" plane, thus, on a plane perpendicular to the former. This is called the *base plane*. Furthermore, the object is within the half-space of the image plane opposite to the centre, so that, if possible, it has no face parallel to the image plane.

The centre *C* is given by its perpendicular projection *F* on the image plane, and by its distance (*d*=*CF*) from the image plane, called the *focal length*; *F* is called the *principal point*. Moreover, the base plane is given by its line of intersection with the image plane, which is called the *base line* and is denoted by *a*. The plane going through the centre and parallel to the base plane is called the *horizon plane*, and its line of intersection with the image plane is the *horizon line h*. This latter is parallel to the base line and incident to the principal point.

The height from which the figure placed on the base plane is seen is determined by the distance of the centre from the base plane. On the perspective image this is equal to the distance *d*(*a*,*h*) of the base line and the horizon line, which is called the *height of the horizon.*

The image *e'* of a line *e* lying in the base plane is the intersection of the plane (*Ce*) with the image plane. There is a special point of the line *e*', the so-called the vanishing point (I_1), which is the point of intersection of the image plane with the line going through *C* and parallel to *e*: $I_1 = \bar{e} \cap \pi$. All the lines of the space parallel to *e* has the same vanishing point; in other words, all the lines parallel to *e* intersect each other in this point. The vanishing points of the lines lying in the base plane or parallel to it are incident to the so-called *horizon line* (*h*), which is the line of intersection of the image plane with the plane going through the centre and parallel to the base plane.

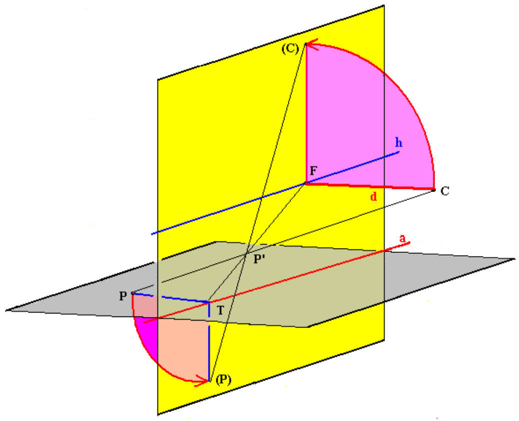
The edges of the cube determine three distinct directions, one of which in the present case is parallel to the perspective image plane; therefore, there is no vanishing point belonging to this direction.



The mutual position of the perspective image plane, the centre, base plane and of the drawn figure.

The vanishing points I_1, I_2 belonging to the other two directions form a right triangle together with C, such that its altitude belonging to the hypotenuse is the focal length $d=CF$.

Let us rotate the base plane about the base line into the perspective image plane (i.e., the plane of our paper sheet). Also, rotate the horizon plane about the horizon line in the same direction. During this rotation, a point P of the base plane describes the circular arc $P(P)$ with centre T, while the centre C describes the circular arc $C(C)$ with centre F. The legs of these arcs are parallel: $PT \| CF$ and $(P)T \| (C)F$; thus these arcs are in homothetic position.



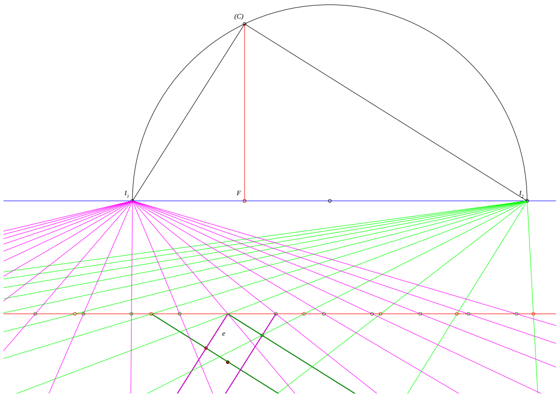
One may equally well say that there is a central collineation[5] between the perspective image of the base plane and its rotate about the base plane. The centre of this collineation is the rotate of the centre of the representation about the horizon line; its axis is the base line; in addition, its vanishing line is the horizon line.

Conversely: if one produces the perspective image of the base plane, or a figure within it, from the rotate of the base plane (respectively a figure in it) into the image plane, then the vanishing line of the central collineation in question is naturally the horizon line.

This provides the possibility to construct the so-called ”Möbius[6] net” provided that the rotated image of a square lying in the base plane, as well as the rotate of C about the horizon h is given. This net is the perspective image of a square grid. For, if one knows the points of intersection of the cube edges lying in the base plane (i.e. the intersection points of these edges with the base line), then ”scaling” the base line with the segments determined by them, one just has to connect the points in the base plane with the corresponding vanishing point.

[5] Central collineation is a projective geometric transformation. Here we do not discuss it and possibilities of its practical application.

[6] August Ferdinand Möbius (1790–1868). Astronomer, professor of mathematics at the University of Leipzig. His name is best known today for his topological construction, the so-called Möbius strip.

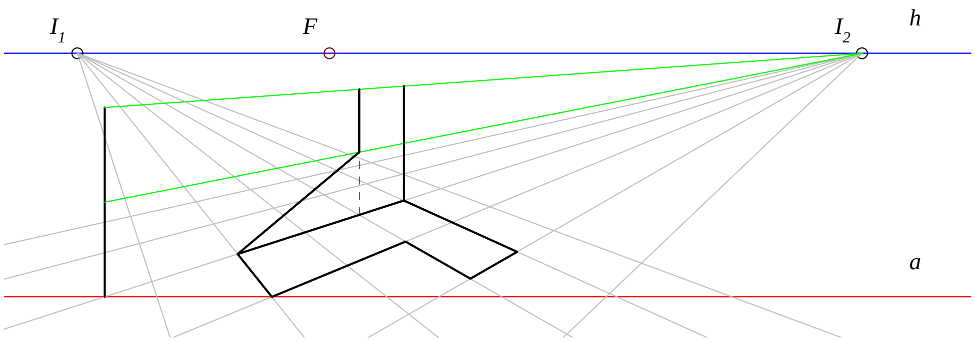


The Möbius net

The image of the Möbius net (and, together with it, of any spatial figure) is determined by the following data:

- the distance of the principal point from the base line;
- the focal length $d=CF$, increasing or decreasing which the vanishing points get farther or closer to each other, respectively;
- the angle subtended by the edges of the square in the base plane an by the base line; this also has an effect on the distance of the two vanishing points from each other. (Think of the fact that the triangle CI_1I_2 has to be constructed in fact from the altitude $d=(C)$F belonging to its hypotenuse, and from the direction of one of its legs!)

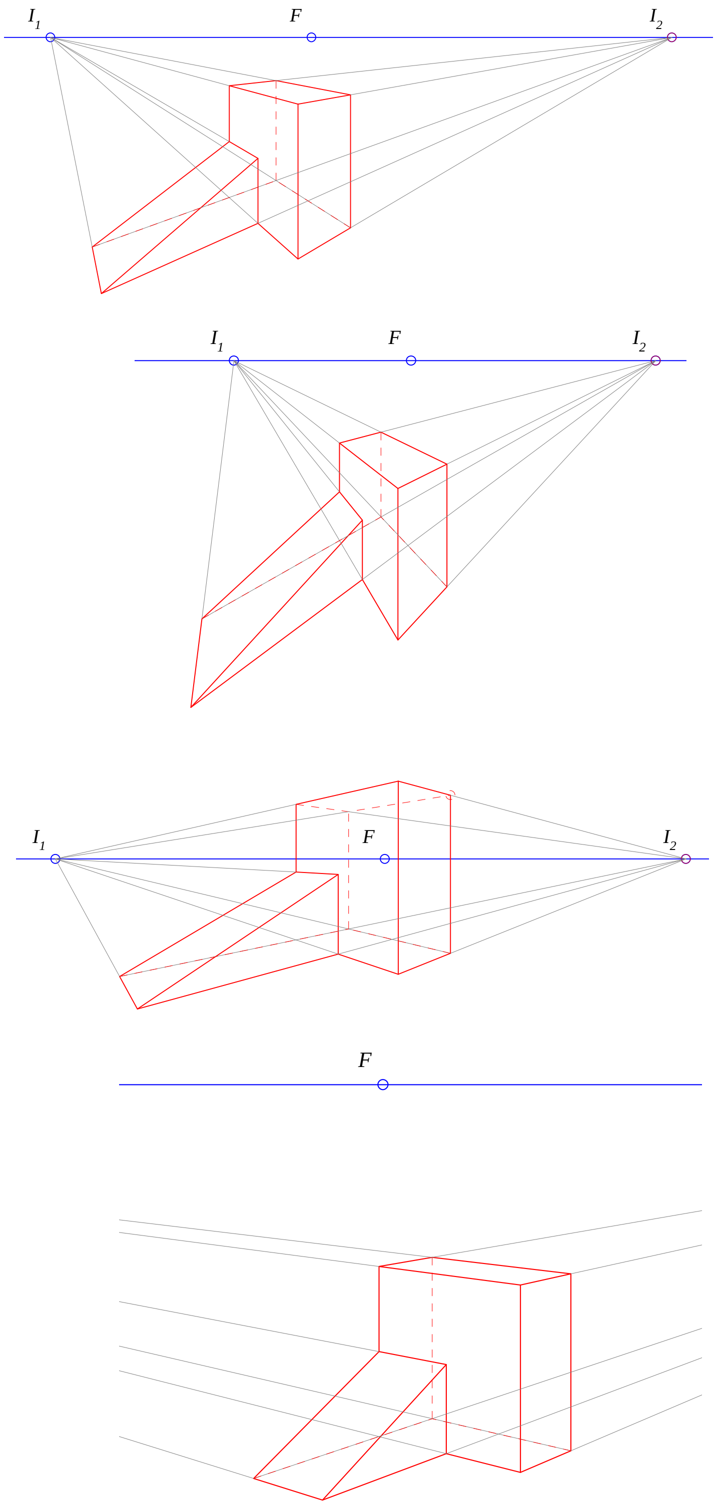
By drawing a sufficiently large part of the Möbius net, one easily obtains the ”ground-plan” of our figure to be represented. One should construct the (suitably foreshortened) image of vertical segments lying on a line perpendicular to the base plane and intersecting it in the suitable grid point. These segments are seen in their original size only if they lie just in the base plane. Hence a line located elsewhere has to be ”carried out” in the image plane by a parallel projection (this means in fact a projection from a suitable vanishing point), then one has to put on there the real length of these segment, and to project them back to its place determined by the Möbius net. (The points of the base plane are carried to the base line through such a projection, thus in the image plane the distance of such a point from the base plane will be its–real– distance from the base line.)



The role of the focal length and the principal point in the perspective representation

Let us investigate some perspective image made from a well-known figure. It can easily be constructed from the drawing that in reality where is the centre C in the space from which the original figure was seen: it is above the principal point F, at a distance from where the segment I_1I_2 can be seen at a right angle. This is not a large distance as compared to the size of the drawing. One could see the drawing realistic if one tried to look at it from that point. However, one cannot see sharply from such a distance.

This is the largest drawback of the perspective representation: one can rarely look at a perspective image from the centre of projection. But seen from elsewhere, the perspective is ”distorted”.



The perspective image depends only on the mutual position of the represented object and the centre.

This is the largest drawback of the perspective representation: one can rarely look at a perspective image from the centre of projection. But seen from elsewhere, the perspective is ”distorted”.

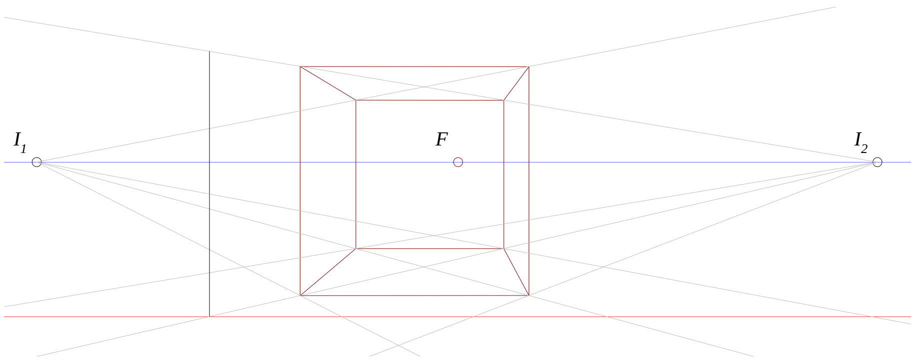
With decreasing the distance, one obtains more and more surprising images; this does not mean, however, that one did not construct the image well, but just that it is seen not from the proper point.

If one took the centre to a distance from where a drawing is usually seen, then the vanishing points would get very far from each other as compared to the size of the drawing, so that they could not hold on our sheet. An important point of learning to draw is just to attain such a ”perspectivity experience”. Here one has also to be skilled in applying the laws of the perspectivity not only when using ruler and compass (or actually, computer), but when making free-hand drawing, and, accordingly, to properly use the vanishing points even if they are not really present on the drawing sheet.

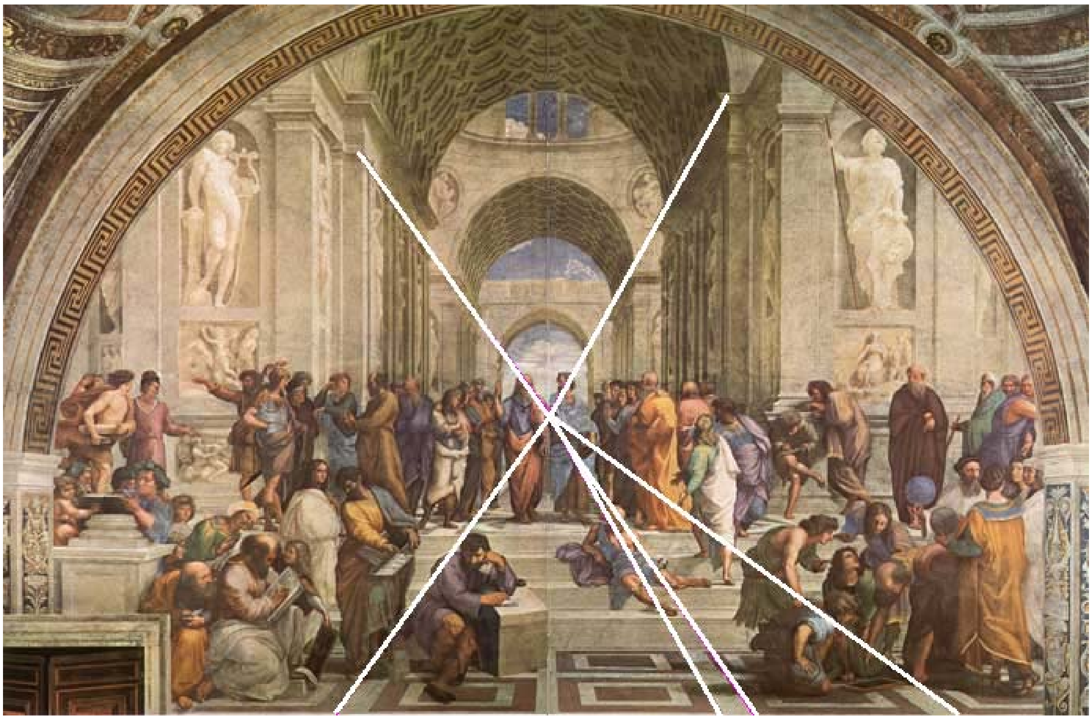
Drawings with one, two or more vanishing points

The drawings above are perspective images with two vanishing points: of the spatial orthogonal coordinate system, in which we positioned the figure to be represented, two axes intersect the perspective image plane, but the third do not. In fact, one can find as many vanishing points for a perspective image as there are lines in different directions not parallel with the image plane.

In the figure below the one-point perspective image of a cube is seen, such that it is larger than the height of the horizon.

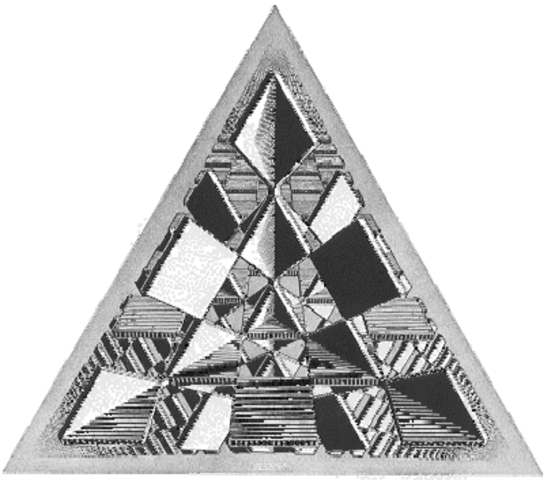
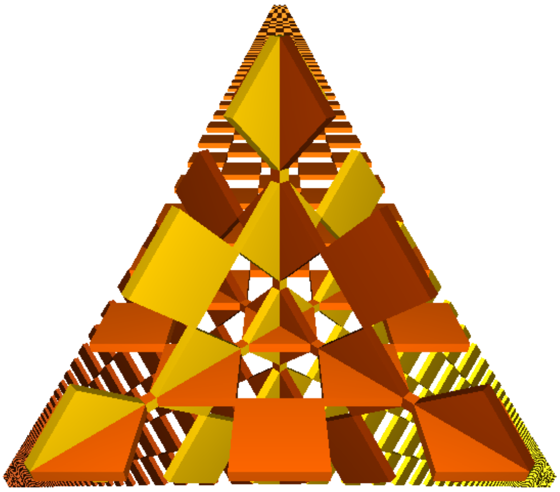


We would not undertake to discuss the role and importance of perspective in art and history of art; nevertheless, we call the attention of our readers to The School of Athens by Raffaello, which is a wonderful application of the perspective representation with one vanishing point. The only vanishing point of the fresco can be found at the point of contact of the bodies of the two great philosophers, Plato and Aristotle.



The School of Athens by Raffaello

Our other example is concerned with the representation using three vanishing points. Here it is worth to meditate on that how important is the question „Which is the way of representation, which is the point from where the picture can be seen just like this?" Based on the artistic construction by Escher, the drawing below was made by means of computer:

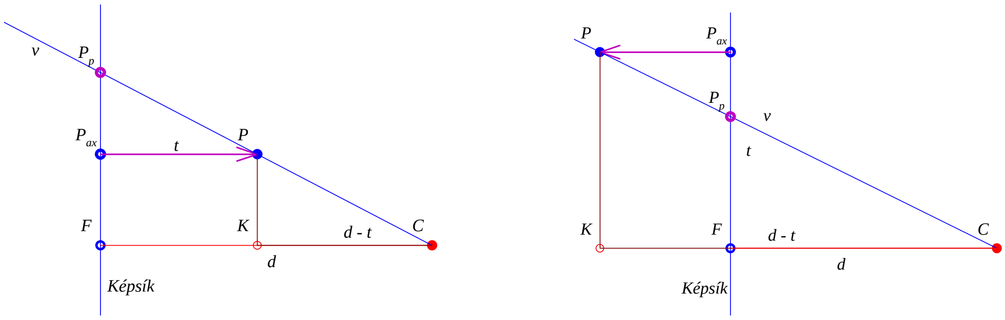
M. C. Escher: Three intersecting planes (woodcut)
and [its interactive realization with Euler3D programme]

The "secret" of this work is that the artist applied a very close perspective, with large optical angle.

3.2. Making a perspective image by computer

Let us consider how to make a perspective image by computer (assumed that we could not leave it to an accomplished program). Having prepared the axonometric image of a point placed in a spatial orthogonal coordinate sytem, to be precise, calculated its coordinate with respect to the coordinate system of the screen, let us see how one should modify these coordinates in order to obtain a perspective image. First we retrict ourselves to the case when the principal point coincides just with the origin the spatial and the planar coordinate system have in common.

Let us look at the image plane from ”side-face”. Let $d=CF$ be the distance between the centre C and the image plane (i.e., the focal length), where F is the principal point. Let t be the (signed) distance of the point P from the image plane, such that it is taken positive if P and C is located within the same half-space determined by the image plane. The image of P in orthogonal axonometry is its perpendicular projection to the image plane: P_{ax}, while its perspective image is the intersection point of the image plane with the projecting ray through C: P_p. This latter exists of course only in the case if $t<d$, that is the ray $[CP)$ from C intersects the image plane. In addition, denote K the perpendicular projection of P on the line (CF).



The points listed above are all in the plane (CFP). Since the triangles P_pFC és PKC are similar to each other, the relationship between their sides P_pF and $P_{ax}F$ $(=PK)$ can easily be written:

$$\frac{P_pF}{FC}=\frac{PK}{KC}=\frac{P_{ax}F}{KC} \quad\Rightarrow\quad P_pF=P_{ax}F\cdot\frac{d}{d-t}\,.$$

This relation is independent of the location of the point P in one or other half-space of the image plane, for t is taken to be a signed distance.

One can read out from either the construction or from the formula above that the points lying in the plane through C and parallel with the image plane have no (finite) images.

Using our formulas for the orthogonal axonometric projection of the point given by coordinates $P(x,y,z)$ (and taken into consideration the equality $t=w$), the perspective image of the same point can be calculated by the formulas:

$$u=\frac{d\cdot(-x\sin\vartheta+y\cos\vartheta)}{d-(x\cos\vartheta\sin\varphi+y\sin\vartheta\sin\varphi+z\cos\varphi)}$$

$$v=\frac{d\cdot(-x\cos\vartheta\cos\varphi-y\sin\vartheta\cos\varphi+z\sin\varphi)}{d-(x\cos\vartheta\sin\varphi+y\sin\vartheta\sin\varphi+z\cos\varphi)}\,,$$

where the centre of perspectivity is ”above” the origin, at a distance d. These formula can be extended by two further parameters if the principal point is carried over from the origin to the point $F(f_u;\,f_v)$ of the image plane. This is needed in the case when, for example, we want to ”have a better view” on the figure to be represented. In this case the point $P(x;y;z)$ has to be shifted to the principal point by a vector $\overrightarrow{OF}=(f_u;\,f_v)$, then, having calculated its perspective image there, it has to be shifted back to its original location by the opposite of the former vector:

$$u = f_u + \frac{d \cdot \left(-x \sin\vartheta + y\cos\vartheta - f_u\right)}{d - \left(x\cos\vartheta \sin\varphi + y\sin\vartheta\sin\varphi + z\cos\varphi\right)}$$

$$v = f_v + \frac{d \cdot \left(-x\cos\vartheta\cos\varphi - y\sin\vartheta\cos\varphi + z\sin\varphi - f_v\right)}{d - \left(x\cos\vartheta\sin\varphi + y\sin\vartheta\sin\varphi + z\cos\varphi\right)}\ .$$

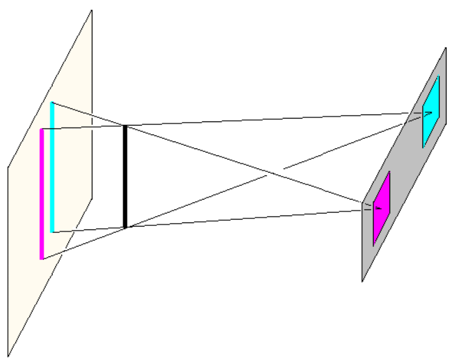
Using these relationships, one can easily produce either an orthogonal axonometric or perspective (wireframe) image of any polyhedron, with all software which is able to draw a line segment given by its endpoints.

The programs that are (also) suitable for visualizing solid geometric objects, like for example MAPLE, MATHEMATICA, or the Euler 3D used here, unburden us of these calculations; furthormore, they also visualize the faces of a polyhedron, thus solving the hidden-line problem as well. Preparing a program which solves this task using the formulas above is still not useless. For, such a program provides a good possibility for experimenting, especially in producing perspective images. One can easily make ”good” or ”less good” images by a suitable, or, actually, a ”wild” choice of the principal point and the focal length.

Stereogram: the drawing providing spatial illusionzere

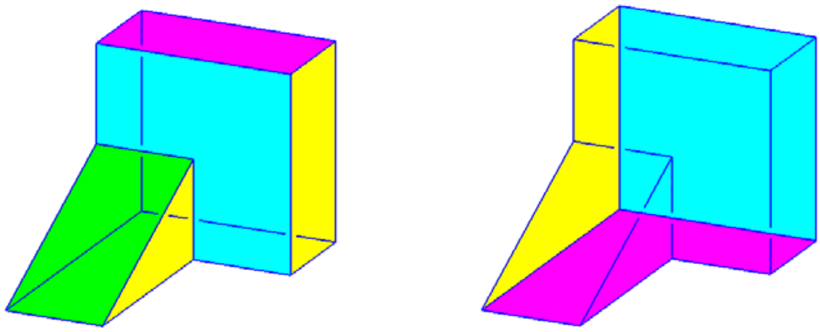
There may be other by-product of this work, too. If one is able to produce wireframe perspective image, then it is easy to make a so-called stereogram from it, which provides a ”real” three-dimensional image. To this end, one has to apply two different centres at a suitable distance from each other.

The objects are seen ”in space”, and we are able to distinguish their points closer and farther to us, on account of the fact that there are two distinct perspective images formed in our eyes, since these centres (i.e. our eyes) are at a distance of 7-8 cm from each other. The difference between these two images is larger or smaller when one look at an object closer or farther, respectively. This difference is processed by our brain so that, due to it, we are able to perceive the distance of the object from our eyes. Now one has to superpose on a single image plane a cyan perspective image made from the one centre, and a red one made from the other centre. Looking at such an image through red-cyan glasses, our eyes distinguish the two images, and our brain composes the spatial object for us.



4. Comparison of the methods of representation

Let us return to Pohlke's theorem. The theorem merely states that there *exists* a cube such that the parallel projection of its three edges starting from a common vertex are three arbitrary line segments in the image plane starting from a common point. Of course, this cube is not unique; for, each cube shifted in the direction of projection has the same image (the same is true for an arbitrary object, of course). But, apart from this fact, no spatial figure can be reconstructed merely from its wireframe axonometric image. For, the visibility of the figure can be indicated in two ways. The two images so obtained represent not the same figure, but one is the mirror mage of the other, with respect to a plane that is perpendicular to the direction of projection. In orthogonal projection this plane may be the image plane as well. Given a wireframe drawing, the reconstructibility can be made unique by a suitable choice of the hidden lines, or by giving the direction of the axes. Programs providing visualization of solid geometric figures, like for example Euler 3D, solve the hidden-line problem without the user's intervening. This requires serious mathematical considerations.



Two different figures with identical axonometric image

In contrast, one cannot decide arbitrarily the hidden-line problem in the case of the perspective images. For, such an image provides some information concerning that given parallel segments of equal length, which is the closer one. Actually, this is seen larger. Using hidden lines contradicting to this rule is a serious professional error.

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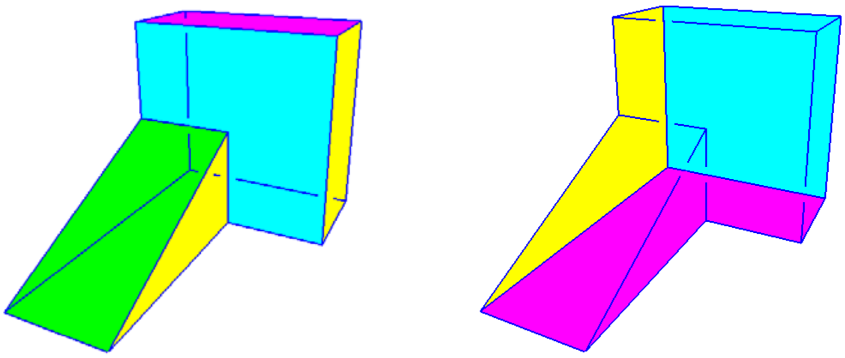
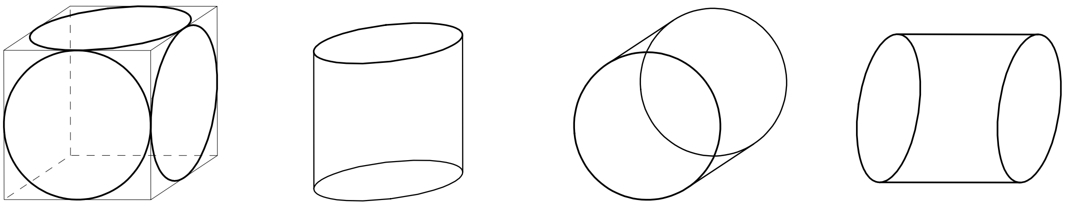


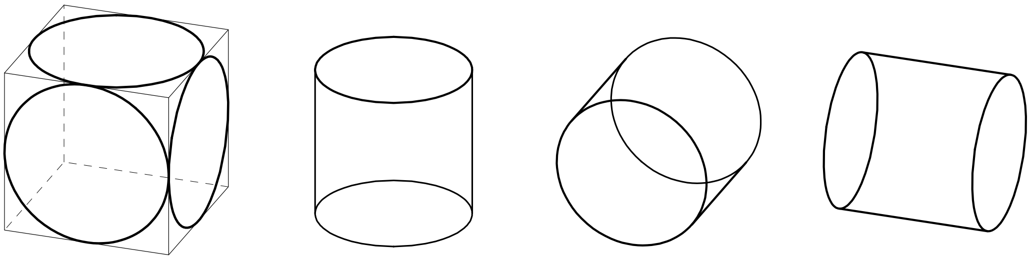
Perspective images with right and erroneous hidden lines

The cavalier perspective – although cubes, or figures composed of cubes, are easy to represent by it – results in an image as a cube is never seen actually. Draw the incircles of the square faces of a cube represented in cavalier perspective. The image of the circle in the front plane is a circle, but those of the other two are ellipses. In this representation a cylinder inscribed in a cube does not give the impression as if it were a right circular cylinder; this is even more so if the circumscribed cube is not given in the drawing.



Cylinders inscribed in a cube in cavalier perspective

Let us draw the same configuration in orthogonal axonometry.

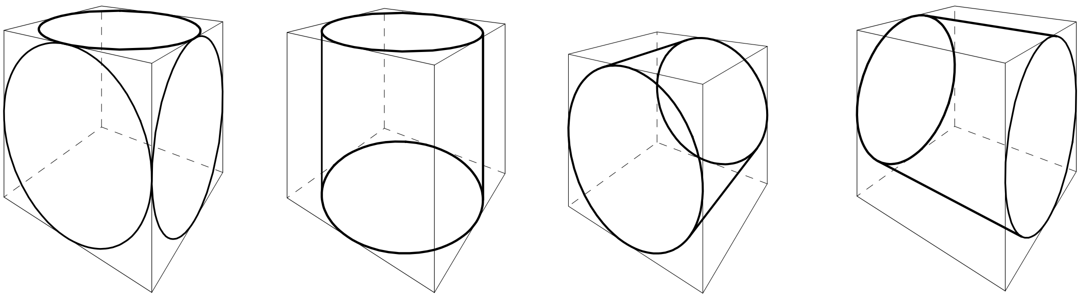


Cylinders inscribed in a cube in orthogonal axonometry

Observe that for any position of the right cylinder drawn in orthogonal axonometry, the extremal ruling of the image is always tangent to the ellipse which is the image of either the bottom or the top circle of the cylinder, so that the point of tangency is just the endpoint of the ellipse.

Consider any image in orthogonal axonometry of a cylinder inscribed in a cube. Then the four focal points of the ellipses representing its top and bottom bases form a square.

At last let us draw the same configuration in (a near) perspective.



Perspective image of cylinder inscribed in a cube.

For a circle, not only its axonometric image, but the perspective as well, is an ellipse , if the the point obtained as the perpendicular projection of the centre of representation on the plane of the circle falls outside of the circle. We remark, however, that while in axonometry the image of centre of the circle (in space) will be the centre of the corresponding ellipse, this is not the case in perspective representation. For, the axonometric image is provided by an affine mapping, which preserves the centre, but in central collineation (which provides the perspective image) the image of a circle is an ellipse such that its centre is the pole of the vanishing line of the collineation, with respect to the circle.

Here we restricted ourselves to the toolkit of descriptive geometry; thus we did not mention the very effective tools of computer graphics, like shadow mapping, reflection mapping, etc., which are used for artistic representation of solid geometric figures. We hope nevertheless that our contribution will help the reader to look on such works with more appreciative eyes.