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Interesting Mathematical Problems in Sciences and Everyday Life - 2011

SURVEY ON ISLANDS – a possible research topic, even for students

Eszter K. Horváth
Department of Algebra and Number Theory, University of Szeged, Hungary
Aradi vértanúk tere 1, Szeged, Hungary, 6720
horeszt@math.u-szeged.hu
http://www.math.u-szeged.hu

In recent algebra investigations, the maximum number of islands on some grid turned out to appear as interesting combinatorial problem. The aim of this paper is to overview the history and the results in this topic. The first result, the formula of Gábor Czédli appeared in [2], his proof was based on lattice theory. The investigation went on two different directions, nowadays both are living research areas.

1. **Lattice theory direction:** After discovering the relationship between the maximum number of rectangular islands and the weak bases of finite distributive lattices, new investigations and new results appeared within lattice theory.

2. **Elementary direction:** After discovering two elementary proof techniques in [1], several new questions and answers have been born, some of which can be asked and answered even by high-school level knowledge.

We are convinced that these theoretical investigations will interact each other in the future, providing further results and applications as well. There are already many easily understandable open problems in this topic, giving challenge to the researching mathematicians (as well as to the students). We would like to introduce the topic the reader, in order to study it further by choosing some paper from the References.

1. Introduction

In principle, we all know what islands are, see also Figure 1 and Figure 15. However, in nowadays life, digitalization plays role everywhere. So, we will consider the problem of islands on some grid. If we change the conditions, we obtain several interesting combinatorial problems.

Figure 1.

2. The definition of an island

First, we need a grid. Its cells constitute the so-called *board*. The cells have *neighbours*. Of course, we have to define exactly, which cells are considered neighbours. For example, on Figure 2, the neighbours of the blue cells are yellow.

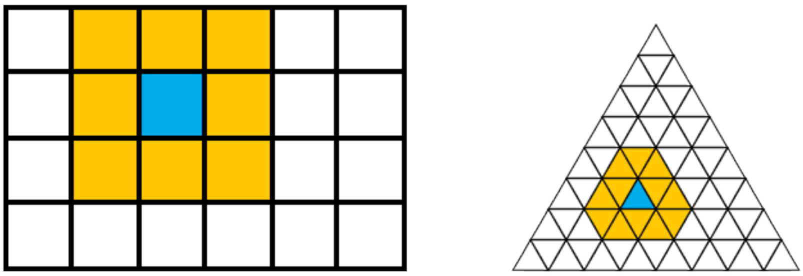

Figure 2.

We put real numbers into the cells. These are the so-called *heights*. We fix a shape, e.g. rectangle or triangle. We call this rectangle/triangle an *island* if its cells have greater heights then the heights in the neighboring cells. In Figure 3, one can see a rectangular island on the square grid with water level 4.

The maximum number of islands is itself an interesting question, furthermore in [6] there is a necessary and sufficient condition for a code to be instantaneous (prefix-free), and this condition uses the notion of a one-dimensional island.

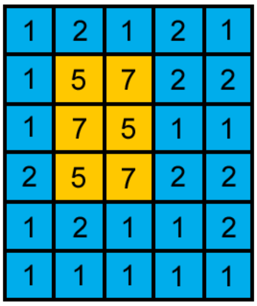

Figure 3.

In Figure 4, one can see a triangular island on the triangular grid with water level 4.

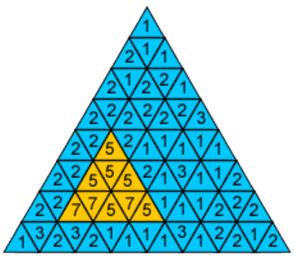

Figure 4.

In other words, we call a rectangle/triangle an *island*, if for the cell t, if we denote its height by a_t, then for each cell $\hat{t}$ neighbouring with a cell of the rectange/triangle T, the inequality $a_{\hat{t}} < min\{a_t : t \in T\}$ holds.

Of course, sometimes the shape above the water level is not a rectangle/triangle, but counting these islands turned out to be much more interesting mathematical problem.

3. Let’s count the islands!

Consider the following heights in Figure 5. How many rectangular islands do we have for all water levels? (For short: Hos many rectangular islands do we have?)

2	1	3	2
2	1	3	2
3	1	1	1

Figure 5.

If the water level is 0.5, then we have one rectangular island, see Figure 6.

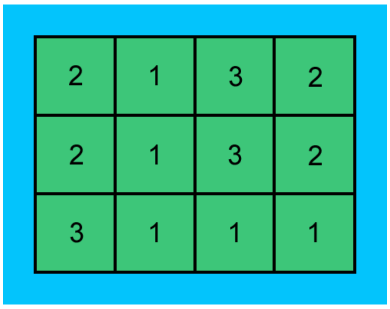

Figure 6.

If the water level is 1.5, then we have two rectangular island, see Figure 7.

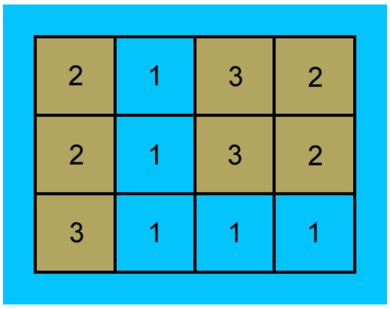

Figure 7.

If the water level is 2.5, then we have the following two rectangular islands:

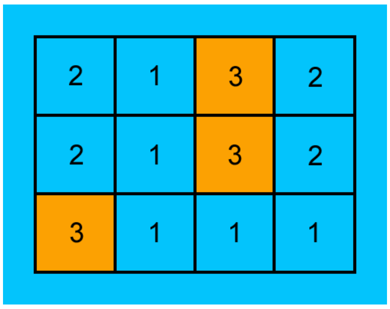

Figure 8.

If we increase the water level above 3, then everything is under the water, i.e. we do not have any island. Altogether, for these heights, we have $1 + 2 + 2 = 5$ rectangular islands.

Can we create more rectangular islands on the same grid (with different heights)?

Yes! With the following heights, we have $1 + 2 + 4 + 2 = 9$ rectangular islands, see Figure 9.

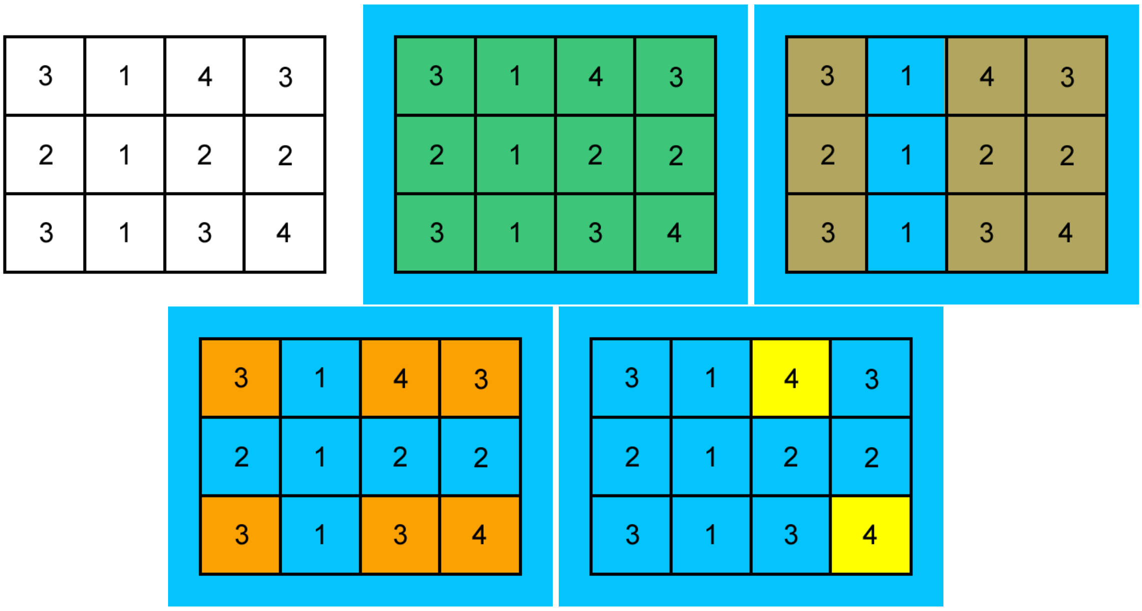

Figure 9.

However, making more rectangular islands is *impossible* !!!

It is the result of Gábor Czédli in [2] that the maximum number of rectangular islands on the square grid is

$$f(m,n) = \left[\frac{mn + m + n - 1}{2}\right].$$

In [1] it is proved that this formula is true not only with the neigborhood relation defined by Figure 2, but also with four neugbours (symmetrically).

4. How to prove the right conjecture about the maximum number of islands?

We prove the above formula of Gábor Czédli [2] for rectangular islands, however the methods are applicable for further cases as well.

4.1. We have at least this number of rectangular islands (lower estimate, lower bound)

We prove that the maximum number of rectangular islands is at least $f(m,n) = \left[\frac{mn+m+n-1}{2}\right]$ in such a way that we show this number of rectangular islands.

We prove by induction on the number of cells that $f(m,n) \geq \left[\frac{mn+m+n-1}{2}\right]$.

If $m = 1$, then $\left[\frac{n+1+n-1}{2}\right] = n$; if we put $1, 2, 3, \ldots, n$ into the cells, in this order, then we will have n rectangular islands.

If $n = 1$, then the same argument shows that we are able to create m rectangular islands.

If $m = n = 2$, then consider Figure 10, the number of rectangular islands is 3, as required :

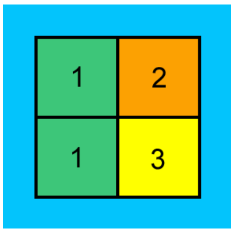

Figure 10.

Let $m, n > 2$.

We put maximally many islands into the rectangles of sizes $(m-2) \times n$, and $1 \times n$. Between these rectangles, we put one row of cells with heights smaller then the minimum of the heights in the two rectangles, furthermore we put even smaller heights outside of our big rectangle. Followingly, we apply the induction hypothesis, i.e. the inequation for smaller rectangles:

$$f(m,n) \geq f(m-2,n) + f(1,n) + 1 \geq \left[\frac{(m-2)n+(m-2)+n-1}{2}\right] + \left[\frac{n+1+n-1}{2}\right] + 1 =$$
$$= \left[\frac{(m-2)n+(m-2)+n-1+2n}{2}\right] + 1 = \left[\frac{mn+m+n-1}{2}\right].$$

4.2. We cannot create more islands (upper estimate, upper bound)

4.2.1. A method: lattice method

The original method, which produced the result first, was based on lattice theory, using the result of [4] that any two weak bases of a finite distributive lattice have the same number of elements. This proof can be read in [2], and provided several research directions in lattice theory, e.g. [3], [5], [7]. Some time later, two other proving methods appeared in [1]. In this paper we present only these two proofs (method B and method C, as follows).

4.2.2. B method: induction

If $m = n = 1$, then the statement is obviously true. Let $m > 1$ or $n > 1$. The induction hypothesis: if $u < m$ of $v < n$, then for the rectangle R of size $u \times v$, $f(R) = f(u, v) \leq \frac{1}{2}(u+1)(v+1) - 1$.

Let $\mathcal{I}^*$ denote such system of rectangular islands that contains maximally many rectangular islands. Denote by $\max \mathcal{I}^*$ the set of all maximal rectangular islands for $\mathcal{I}^*$, i.e. those set of rectangular islands that have only one bigger rectangular island. For rectangle of size $u \times v$ the number of grid points is $\|R\| = (u+1)(v+1)$. Now

$$f(m, n) = 1 + \sum_{R \in \max \mathcal{I}^*} f(R) \leq 1 + \sum_{R \in \max \mathcal{I}^*} \left(\tfrac{1}{2}\|R\| - 1\right) =$$
$$= 1 - |\max \mathcal{I}^*| + \tfrac{1}{2} \sum_{R \in \max \mathcal{I}^*} \|R\| \leq 1 - |\max \mathcal{I}^*| + \tfrac{1}{2}(m+1)(n+1).$$

so we obtained

$$f(m, n) \leq 1 - |\max \mathcal{I}^*| + \tfrac{1}{2}(m+1)(n+1).$$

If we have at least two maximal rectangular islands, then the proof is ready.

If we have only one maximal rectangular island, then one of the following inequalities are true:

$$f(m, n) \leq 1 - |\max \mathcal{I}^*| + \tfrac{1}{2}m(n+1) = 1 - 1 + \tfrac{1}{2}m(n+1) \leq \tfrac{1}{2}(m+1)(n+1) - 1.$$

$$f(m, n) \leq 1 - |\max \mathcal{I}^*| + \tfrac{1}{2}(m+1)n = 1 - 1 + \tfrac{1}{2}(m+1)n \leq \tfrac{1}{2}(m+1)(n+1) - 1.$$

If we have no maximal rectangular island, then we have only one rectangular island, so the proof is also ready.

4.2.3. C method: tree-graph method

Our rectangular islands constitute tree graph by inclusion, see Figure 11 and 12.

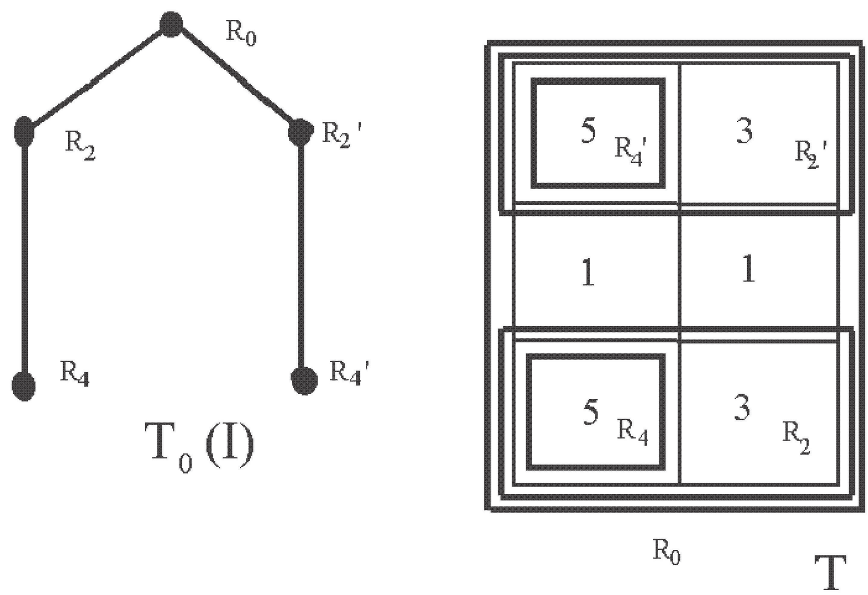

Figure 11.

First, we need a Lemma.

Lemma

Let T be a rooted tree such that any non-leaf node has at least 2 sons. Let ℓ be the number of leaves in T. Then $|V| \leq 2\ell - 1$.

Proof

Let us direct the tree-graph from its root to the direction of its leaves. Then for the in-degrees and out-degrees the following equation holds:

$$\sum D^+ = \sum D^-.$$

Now

$$\sum D^+ = |V| - 1$$

because each nod has father except for the root. Moreover:

$$\sum D^- \geq 2(|V| - \ell),$$

because all non-leaf nod has at least two sons. So

$$\sum D^+ = |V| - 1 = \sum D^- = 2(|V| - 1),$$

i.e. we obtained

$$2\ell - 1 \geq |V|,$$

the proof of Lemma is ready.

As we mentioned, our rectangular islands constitute tree-graphs by inclusion. For having at least binary tree-graph in all cases (condition of Lemma), we introduce the so-called ”dummy island” in case the island shrinks when the water level increases, e.g. see Figure 11-12, Figure 1, [1] and [18] for more details. (In Figure 15, when water level increases, two islands appear above the big island, so there is no need for dummy island.)

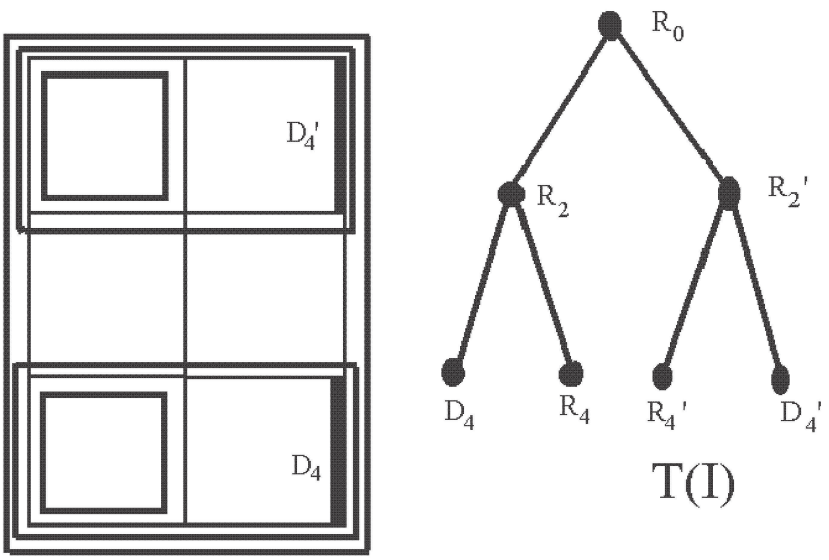

Figure 12.

We denote by s the number of the minimal rectangular islands, by d the number of dummy islands. By Figure 11 and 12, each minimal rectangular island covers at least four grid-points, each dummy island covers at least two grid-points.

This way we have covered not more than all the grid-points of the square grid, i. e.:

$$4s + 2d \leq (n+1)(m+1).$$

The number of leaves of $T(\mathcal{I})$ is $\ell = s + d$. Hence by Lemma the number of islands is

$$|V| - d \leq (2\ell - 1) - d = 2s + d - 1 \leq \frac{1}{2}(n+1)(m+1) - 1.$$

.

5. Triangular islands

Denote by $tr(n)$ the maximum number of triangular islands in the equilateral triangle of sidelength n. The following lower and upper bounds are proved in [9]:

$$\frac{n^2 + 3n}{5} \leq tr(n) \leq \frac{3n^2 + 9n + 2}{14}.$$

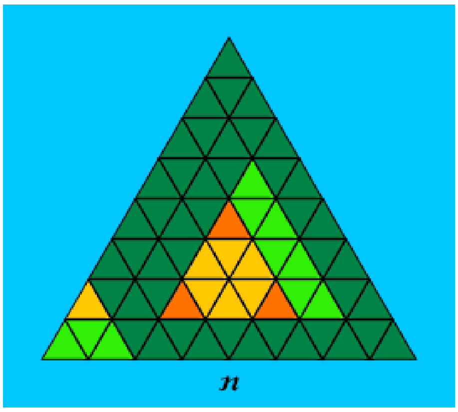

Figure 13.

Hovewer, up to now, nobody succeeded to obtain exact formula for the maximum number of triangular islands on triangular grid.

6. Square islands

It is proved in [10] that for the square grid of size $(m-1) \times (n-1)$ the maximum number of square islands $sq(m,n)$:

$$\frac{1}{3}(mn - 2m - 2n) \leq sq(m,n) \leq \frac{1}{3}(mn - 1).$$

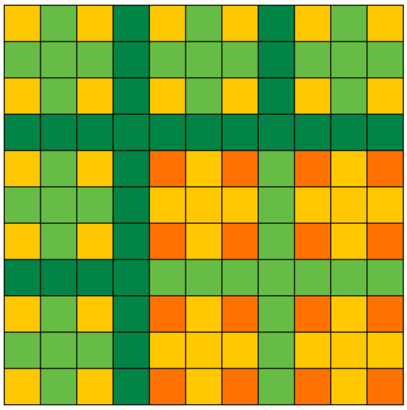

Figure 14.

However, similarly to the triangular case, the lower and upper bounds are close, but exact formula for this square case is also not known.

7. Exact results

The proofs of the following statements can be found in [1].

7.1. Peninsulas (semi islands)

Consider again rectangular islands on the square grid. Surprisingly, if we count the maximum number of such islands that reaches at least one side of the board is:

$$p(m,n) = f(m,n) = [(mn + m + n - 1)/2].$$

7.2. Cylindric board, rectangular islands

Let us put square grid onto the surface of a cylinder. Denote by $h_1(m,n)$ the maxium number of rectangular islands: If $n \geq 2$, then $h_1(m,n) = [\frac{(m+1)n}{2}]$.

7.3. Cylindric board, cylindric and rectangular islands

On cylindric board, it is possible to create cylindric islands. Denote by $h_2(m,n)$ the maxium number of rectangular or cylindric islands on the surface of a cylinder of height m. Of course, in this case, the maximum number of cilindric islands is more than in the previous case: If $n \geq 2$, then $h_2(m,n) = [\frac{(m+1)n}{2}] + [\frac{(m-1)}{2}]$.

7.4. Torus board, rectangular islands

Let us fold a torus from a rectangle of size $m \times n$. Denote by $t(m,n)$ the maximum number or rectangular islands on this board. If $m, n \geq 2$, then $t(m,n) = [\frac{mn}{2}]$.

7.5. Islands in Boolean algebras

The board consists of all vertices of a hypercube, i.e. the elements of a Boolean algebra $BA = \{0,1\}^n$. We consider two cells neighbouring if their Hamming distance is 1. Our islands are Boolean algebras. We denote the maximum number of islands in $BA = \{0,1\}^n$ by $b(n)$. Then, $b(n) = 1 + 2^{n-1}$.

8. Further problems, partially investigated

If we put only finitely many heights into the square cells of a rectangle, we obtain a much more complicated problem for rectangular islands, which is solved only for one dimension in [8]; for the maximum number of rectangular islands $I(n,h)$ we have

$$I(n,h) \geq n - \left[\frac{n}{2^h}\right]$$

if we have n cells in one row and the number of possible heights is h.

If we have maximally many rectangular islands in a square grid of size $m \times n$, then it is interesting to investigate the number of possible levels, i.e. the heights of the digital hills, see [12].

For further interesting mathematical questions and answers in this topic we recommend [8], [11], [13], [14], [15], [16][17] [18], [19] or [20].

9. Acknowledgement

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