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Interesting Mathematical Problems in Sciences and Everyday Life - 2011

On the process of mathematical modeling

Đurđica Takači, Duška Pešic, Jelena Tatar

Department of Mathematics and Informatics,Faculty of Sciences,
High School '' Jovan Jovanović Zmaj'', Novi Sad
djtak@dmi.uns.ac.rs

Mathematical models and the process of mathematical modeling are often represented in mathematics and natural sciences. Today, the new technologies involve lot of changing in all segments of human activities, especially in the process of mathematical modeling receiving a new interactive dimension, which is an innovation in the teaching and learning. From a didactic point of view, interactive presentations are a great challenge and they are the subject of numerous studies that confirm their positive impact on the mathematical concepts and in particular on connecting mathematical concepts to real world.

This paper presents the mathematical modeling processes starting from various:

- mathematical formula, the binomial form, the degrees of 11, natural and polygonal numbers, geometric objects, arrays, graphics functions;
- problems appearing in natural sciences as division of living cells, disintegration of the atom;
- real situations, restaurants, coloring,

and always finishing with such a mathematical model that is based on Pascal's triangle. In this paper, special attention is devoted on the Fibonacci sequence and its relation to Pascal's triangle. The interactive presentations of mathematical formulas and real-world phenomena are presented. Interactive presentations are demonstrated by using the software package Power point, by using hyperlink, and they can be open by usual procedure.

1. Blaise Pascal

Blaise Pascal was born 1623, in Clermont, in France. His father, who was very educated person at that time, had decided to take care on his son's teaching and learning, because Pascal had shown, very early, exceptional abilities. Pascal's father did not force him learning mathematics before he was fifteen years old. But, Blaise decided to learn geometry, when he was years old and then he discovered that the sum of the interior angles of a triangle is equal to two right angles.

Figure 1.

Pascal has never got married because of his decision to devote himself to science. After a series of health problems, Pascal died when he was 39th years old, in 1662 year.

He discovered the first mechanical calculators and therefore a programming language, PASCAL, is named after Blaise Pascal in 1968 year.

Pascal's triangle is also named after Blaise Pascal, because he connected a lot of mathematical formulas with Pascal's triangle.

The first written records of Pascal's triangle came from the Indian mathematician Pingala, 460 years before the new ere where the Fibonacci sequence and the sum of the diagonals of Pascal triangle are appeared. Otherwise, Pascal's triangle was discovered by a Chinese mathematician, Chu Shu-Kie 1303rd year. Figure 2 shows the original triangle.

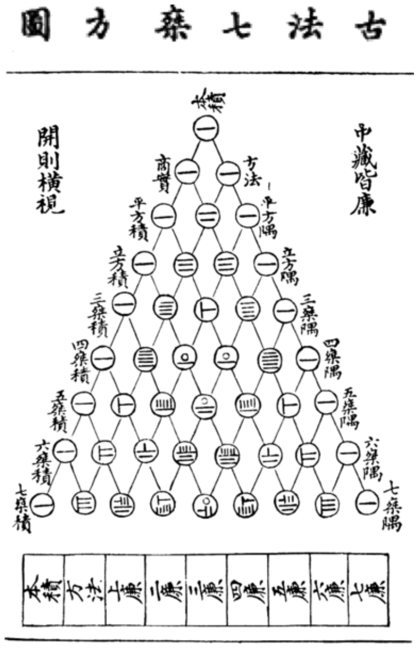


Figure 2.

2. On the construction of the Pascal triangle

At the beginning, at the top of the triangle, in “zero” row we put number 1.
In the first row we put two times number 1.
In each row we put the first number 1, and it can be considered as the sum of 1 +0.
Further on, the basic idea consists in the fact that we add two numbers from above (from above row), and the obtained sum we write in the box (in the middle) below (in the next row).
These steps are repeated until the row is fulfilled, then the process is repeated on a new row.

Construction of Pascal triangle

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
1 10 45 90 210 252 210 90 45 10 1
1 11 55 135 300 462 462 300 135 55 11 1

Figure 3.

3. Binomial Formula and Pascal's Triangle

Coefficients in binomial formula

$$(a+b)^n = \sum_{k=1}^{n} \binom{n}{k} a^{n-k} b^k = 1\cdot a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \cdots + 1\cdot b^n ,$$

where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} ,$$

form the nth row, $n = 1,2,3,...$ in Pascal triangle, because it holds:

$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}.$$

4. Power of number 11 and Pascal's Triangle

$$11^0 = 1$$
$$11^1 = 11$$
$$11^2 = (10+1)^2 = 100 + 2\cdot 10 + 1 = 121$$
$$11^3 = (10+1)^3 = 1000 + 3\cdot 100 + 3\cdot 10 + 1 = 1331$$
$$11^4 = (10+1)^4 = 10000 + 4\cdot 1000 + 6\cdot 100 + 4\cdot 10 + 1 = 14641$$
$$11^5 = (10+1)^5 = 100000 + 5\cdot 10000 + 10\cdot 1000 + 10\cdot 100 + 5\cdot 10 + 1 = 161051$$
$$11^6 = (10+1)^6 = 1000000 + 6\cdot 100000 + 15\cdot 10000 + 20\cdot 1000 + 15\cdot 100 + 6\cdot 10 + 1$$
$$= 1771561$$

It can be seen that digits in the numbers, representing powers of 11, are the corresponding rows in Pascal triangle, up to fourth row. In the fifth row of Pascal triangle two digit numbers appeared. See the interactive demonstration linked on the powers of 11.

Let us first consider:

$$11^5 = (10 + 1)^5 = 10^5 + 5\cdot 104 + 10\cdot 103 + 10\cdot 102 + 5\cdot 10 + 1 = 161051,$$

and then, the 5^{th} row of Pascal triangle:

$$1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1,$$

and compare them.

- 1 is the first number (from right side) of the corresponding row of Pascal's triangle and the last digits in the number 161051.
- 5 is the second number (from right side) of the corresponding row of Pascal's triangle and the second digit in the number (from right side) 161051.
- 10 is the third number (from right side) of the corresponding row of Pascal's triangle, but the last digit 0 is the third digit in the number (from right side) 161051.
- The fourth digit (from right side) 1=1+0 in the number 161051.
- The fifth digit (from right side) 6=1+5 in the number 161051
- 1 is the number in the corresponding row of Pascal's triangle and the first digit in the number 161051.

5. Natural Numbers and Pascal Triangle

By using Pascal triangle, one can express the sum of positive integers with a fixed sum of digits.

The total number of decimal numbers whose sum of digits is 1 makes the first column of Pascal triangle because there exist:

1 Single Digit 1
1 Two-digit number 10
1 Three digit number 100
1 Four-digit number 1000 ...

…….

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1

Figure 4.

The total number of decimal numbers whose sum digit is 2 corresponds the second column of Pascal triangle because there exist:

1 Single Digit 2
2 Two-digit number 20 11
3 three-digit number 200 110 101
4 four-digit number 2000 1100 1010 1001

……

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1

Figure 5.

The total number of decimal numbers whose sum of digits is 3 corresponds the second column of Pascal triangle because there exist:

1 Single Digit 3
3 Two-digit number 30 21 12
6 Three digit number 300 210 201 111 120 102
10 Four-digit numbers 3000 2100 2010 2001 1200 1020 1002 1110 1011 1101

......

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1

Figure 6.

Continuing the procedure we got:

The total number of decimal numbers whose sum of digits is k corresponds to the *k-th* column of Pascal triangle. See the interactive demonstration linked on the Natural Numbers

6. Polygonal numbers

Polygonal numbers are terms of the sequence which geometric form can be described as follows:
1. The first number in each group of polygonal numbers is 1, and it corresponds to one vertex of the polygon.
2. The second number equals the number of vertices, and types of polygons (for example, for triangular numbers this is 3, for the square number this is 4, for the pentagonal number is 5, etc.).
4. "Stretching" the two sides of the polygon, and adding points in order to obtain new polygon of the same type, and then by adding all points of the resulting figure we gets the third polygonal number.
5. The process continues as described.
The formula (the Winton formula) for determining $n-$th, $x-$gonal number is given by

$$f_{x,n} = \frac{(n^2-n)}{2}(x-2)+n\,.$$

Triangular numbers (linked on Triangular Numbers) are terms of the sequence:

$$f_{3,n} = \frac{(n^2-n)}{2}(3-2)+n = \frac{n(n+1)}{2}.$$

Square numbers (linked on Square numbers) are terms of the sequence:

$$f_{4,n} = \frac{(n^2-n)}{2}(4-2)+n = n^2.$$

Pentagonal numbers (linked on Pentagonal numbers) are terms of the sequence

$$f_{5,n} = \frac{(n^2-n)}{2}(5-2)+n = \frac{n(3n-1)}{2}.$$

The connection between polygonal numbers and Pascal triangle (linked on Polygonal numbers) can be displayed on the following:
The sequence of square numbers can be represent as:

- The sum of two sequence of triangular numbers:

$$f_{3,n} + f_{3,n-1} = \frac{n(n+1)}{2} + \frac{(n-1)n}{2} = n^2$$

- The difference between two sequence of Square numbers

$$\frac{n(n+1)(n+2)}{6} - \frac{(n-2)(n-1)n}{6} = n^2$$

The generalization of triangular numbers, tetraedralne and hipertetraedarne numbers, are respectively, terms of the sequences:

$$f_n = \frac{n(n+1)(n+2)}{6}, \quad f_n = \frac{n(n+1)(n+2)(n+3)}{24}.$$

7. Electronic configuration of atoms

An atom consists of protons, neutrons and electrons. Electrons are located in the energy layer of atoms, which surrounds the core, fill the shell. The first shell can contain up to 2 electrons, the other 8, and so on with a corresponding number of protons. The maximum number of electrons, $Z_{\max}$, which surround the nucleus of atoms are in *n*th shells can be determined based on formula:

$$Z_{\max} = 2 \cdot n^2 .$$

8. Sierpinski triangle

Fractals are geometric shapes which, in simple terms, give the same number of details, no matter which resolution we use. In other words, fractals can zoom in infinitely many times, and each time a new details appeared, which could not been visible before, and in addition the amount of detail is shown roughly remained the same. One of the simplest fractals is Sierpinski triangle, and it starts with an equilateral triangle, in which we write and paint an equilateral triangle, and then further in each of the obtained non-colored equilateral triangles we white and paint equilateral triangle, and so on.

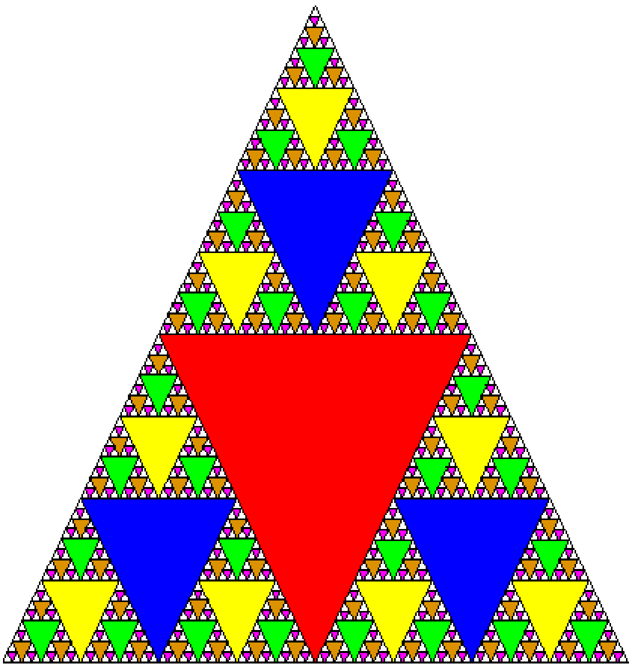
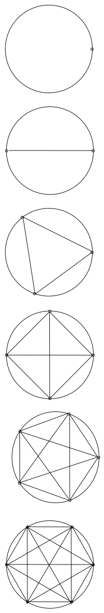


Figure 7. Sierpinski Triangle

9. The points on the circle

The number of points, thesegments and polygons on a circle can be connected with the Pascal's triangle as follows (Figure 8):



Number	points	segments	triangles	quadrilaterals	pentagons	hexagon
	1					
	2	1				
	3	3	1			
	4	6	4	1		
	5	10	10	5	1	
	6	15	20	15	6	1

Figure 8. Points on Circle

10. On the division of the cells

The cell is the basic unit of structure and function of every living organism. It is able to extend its own life due to its ability to replace or compensate for its substance. Processes that characterize the life cycle of cells take place in an orderly and controlled manner. They realize the change of two periods, and the interface is cell division. Cell division is the ability of cells to divide and thus form two daughter cells.

One way is mitosis and cell division, which means that the mother cell divides into two equal daughter cells, and she ceases to exist. It can be described as follows (Figure 9).

At the initial (zero) time we have only one cell (the mother), C_0 .

1. After the first cycle of division, it anhilira and gives two daughter cells, $2\,C_1$.
2. After the second cycle of division, each of these cells anhilira and get four daughter cells $4\,C_2$.
3. After the third cycle of cell division each anhilira and gives two daughter cells, so in total we have eight new cells $8\,C_3$.

After n cycles of cell division we shall have 2^n cells of the same age, which corresponds to a geometric series with quotient 2.

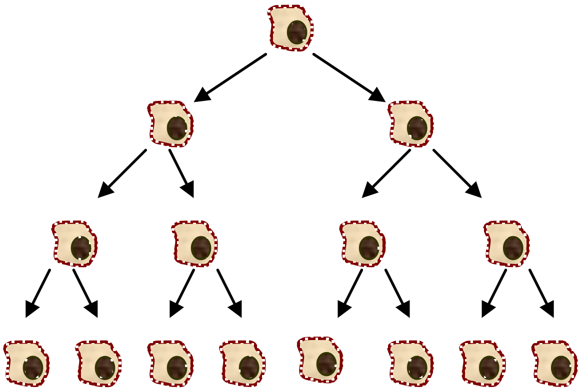


Figure 9.

Another approach involves cell division to cell division gives two mother cell, but one cell of his generation and one cell to the next generation (Fig. 9).

The initial (zero) time we have only one cell (the mother), C_0.

After the first cycle of division, it divides into two cells, one of its C_0 and a next-generation C_1. We have two cells of different generations $C_0 + C_1$.

After the second cycle of division, each of these cells is divided into a cell and his cell the next generation. The cell C_0 gives one cell generation C_0 and C_1, but the cell C_1 gives one cell generation C_1 and C_2, and finally we have one cell of generation C_0 and C_2 and 2 cells of generation C_1, i.e.,

$$C_0 + 2\,C_1 + C_2 .$$

After the third cycle of division, each of these cells gives one cell and one of its next generation, and we get $C_0 + 3\,C_1 + 3\,C_2 + C_3$. After the fourth cycle, the division again each of these cells gives one cell and one of its next generation, and we get

$$C_0 + 4\,C_1 + 6\,C_2 + 4\,C_3 + C_4 .$$

After the fifth cycle we have

$$C_0 + 5\,C_1 + 10\,C_2 + 10\,C_3 + 5\,C_4 + C_5 ,$$

And so on..

Note that here the number of cells after n cycle division also 2^n, which in this case will be all the same to each other.

$$
\begin{array}{cc}
1\,C_0 & 1 \\
1\,C_0 + 1\,C_1 & 2 \\
1\,C_0 + 2\,C_1 + 1\,C_2 & 4 \\
1\,C_0 + 3\,C_1 + 3\,C_2 + 1\,C_3 & 8 \\
1\,C_0 + 4\,C_1 + 6\,C_2 + 4\,C_3 + 1\,C_4 & 16 \\
1\,C_0 + 5\,C_1 + 10\,C_2 + 10\,C_3 + 5\,C_4 + 1\,C_5 & 32 \\
\vdots & \vdots
\end{array}
$$

Figure 10.

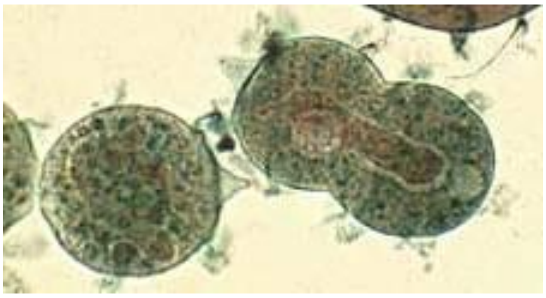


Figure 11.

11. Pascal's triangle in the restaurant

Using the Pascal triangle one can answer the question: How to order a meal? Depending on the type of restaurant, or the number of dishes in a restaurant can serve, the customer may want different combinations of dishes. If there is no food in the restaurant, then nothing can be ordered. If the restaurant has only one dish, then it can taken or not.

If the restaurant has two dishes, then the customer can take two meals, or each one separately, or may not take none. Continuing the process we can make the following simulation and connect with Pascal's triangle (Figure 12).

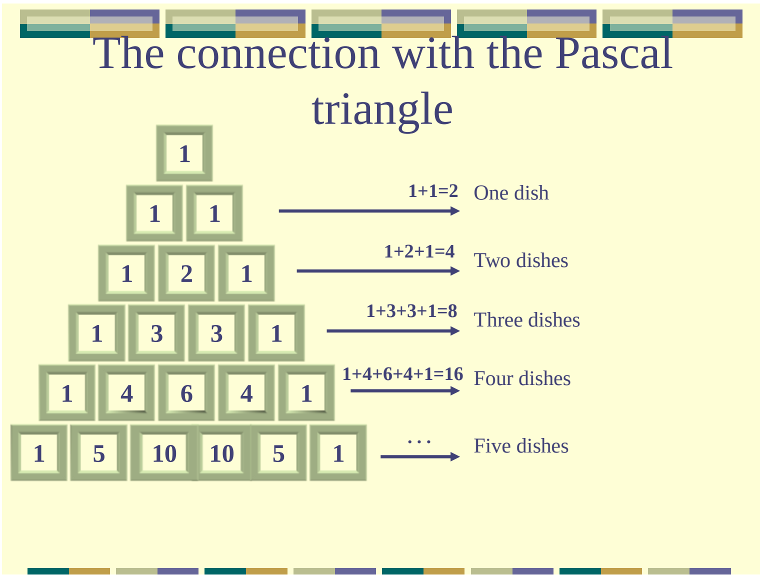


Figure 12. How to order meal

12. Coin tossing

If you throw a coin in the air then it landed on two ways, showing "heads" or "tails". When you throw two coins, at the same time, in the air then they can landed showing:
two heads, a letter to the head, or headto the letter, or two letters.
Throwing three, or more coins, at the same time, in the air then they can landed showing Pascal triangle:

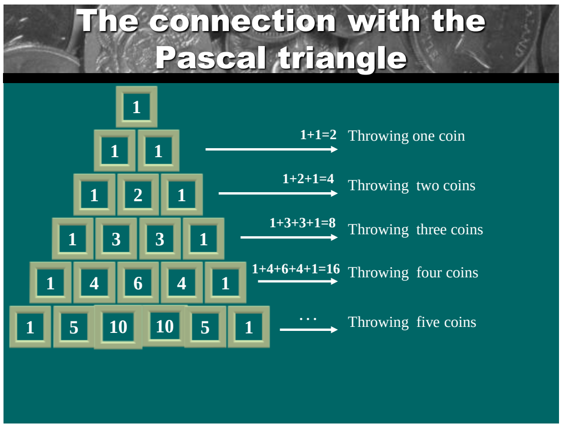


Figure 13. Toss-up

13. Coloring

Pascal's triangle can be used for a variety of interesting pictures. First we choose the number of colors, and then any number or its module is assigned by a color.
We shall show the painting with 5 colors using Pascal's triangle (Figures 14 and 15). Let:

- the number 1 corresponds to purple;
- the number 2 corresponds to yellow color;
- the number 3 corresponds to blue color;
- the number 4 corresponds to green color.
- Number 0 corresponds to the color red.

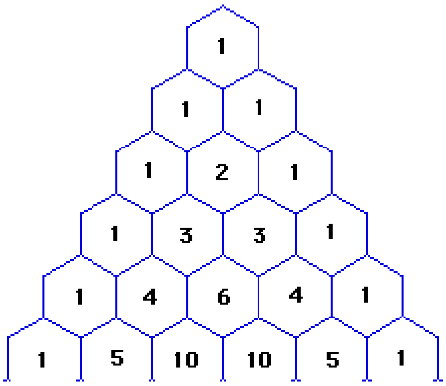


Figure 14.

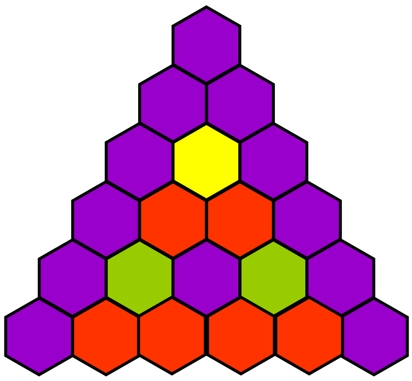


Figure 15.

Coloring

Coloring with three colors (as shown) receives the Figure 15, and by Figure 16.

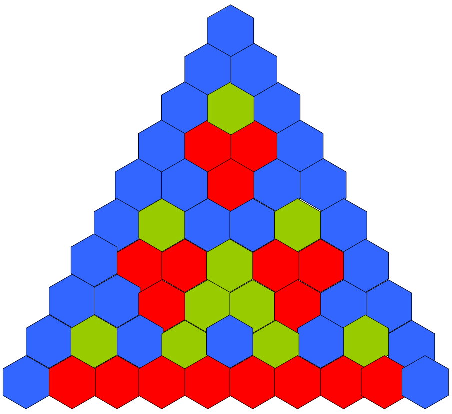


Figure 16.

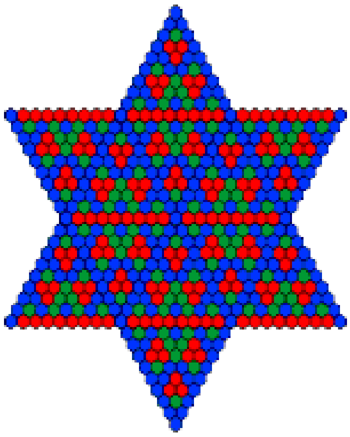


Figure 17.

Coloring with four colors and merging six of these triangles it is obtained:

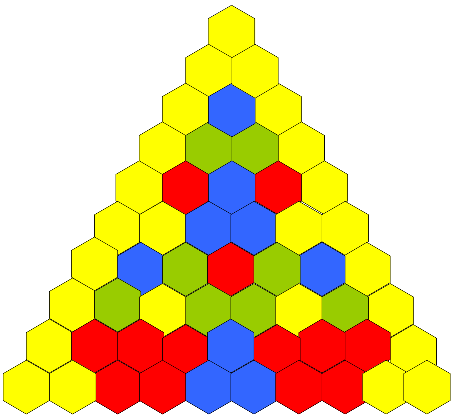


Figure 18.

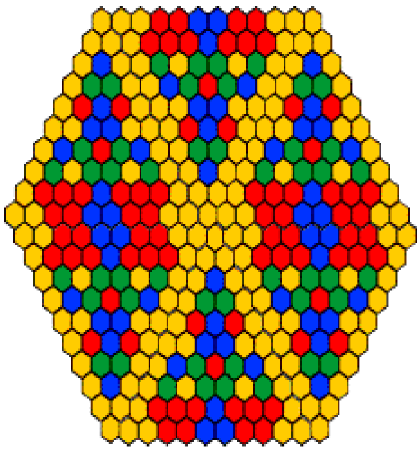


Figure19.

14. Fibonacci sequence

Fibonacci sequence is given by recurrent formula $f_n = f_{n-1} + f_{n-2}$, and its terms are given by:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, ...$$

Fibonacci sequence can be visualized by using package *GeoGebra* as: (open Fibonacci).
In the following we shall explain the connection between the raising of rabbits and Fibonacci sequence.
A man buys a couple of rabbits and raised their offspring in the following order (Figure 19): Parental pair of rabbits is marked with number 1.

After the first month a couple of young rabbits (first generation) is born, which is also marked by 1.

After the second month the parental pair get another pair and then the parents breaks on reproduction, but a couple of rabbits from the first generation also gets some new rabbits. The number of pairs is now 2.

This mode of reproduction is true for all the descendants of the parental couple. Each new pair of rabbits will be given in two consecutive months a pair of young rabbits, and after that its propagation will be terminated.

In the fourth month the pair of new born obtain: a pair of first-generation and second generation of two pairs, so the number of pairs of third

In the fifth month will be 5 pairs of rabbits, in the sixth month will be eight pairs of rabbits, and so on. The number of pairs of rabbits after each month corresponds to Fibonacci series. We can say that after half a year we have 21 pair of rabbits, and after one year 233 pairs of rabbits.

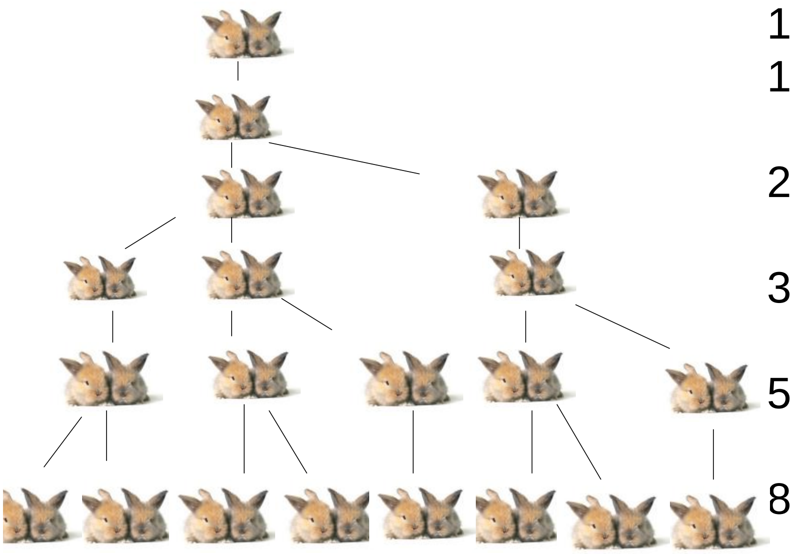


Figure 20. [Fibonacci Rabbits]

If we sum up the numbers on the diagonals of Pascal triangle then we get a series Fibonacci numbers.

If the term of Fibonacci sequence is divided with the following term of Fibonacci sequence, then the following sequence is obtained:

1/1 1/2 2/3 3/5 5/8 8/13 13/21 21/34 34/55 55/89 ……..

Obviously, the Fibonacci sequence diverges, but the sequence of Fabonacci quotients of terms, converges to the golden section.

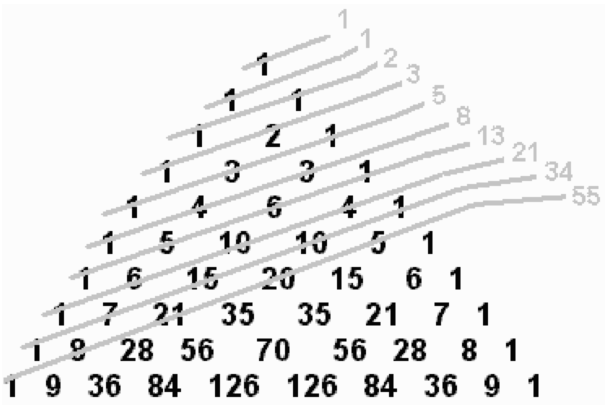


Figure 21.

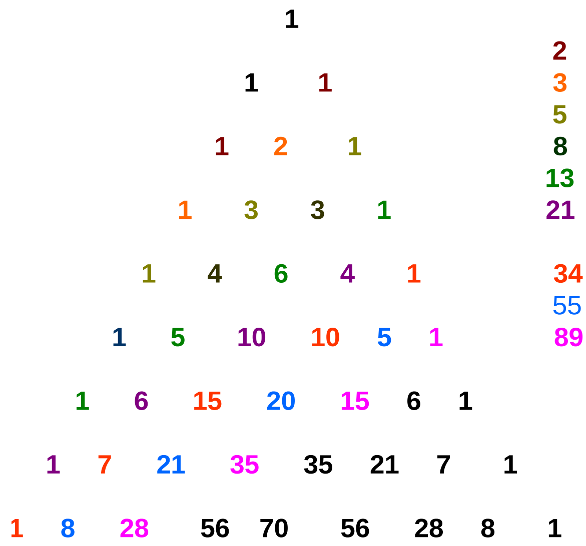


Figure 22.

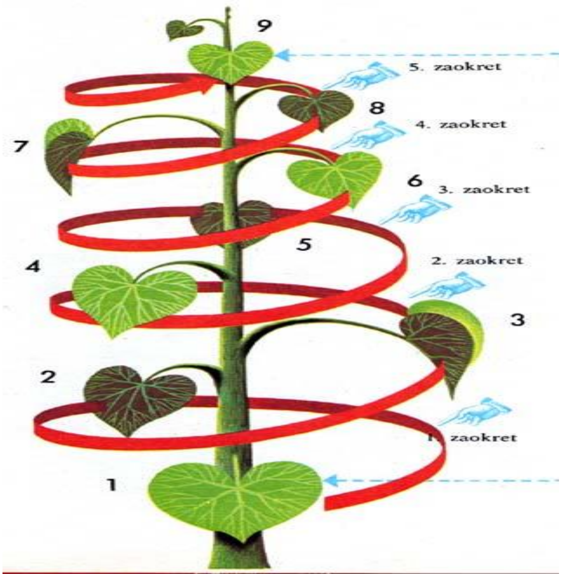


Figure 23.

We will show how the Fibonacci quotions can be found among the leaves that have grown on the stem. It can be said that the growth of new leaves takes place in a spiral, so that the ratio of the turns of spirals and an appropriate number of spaces between the leaves is a Fibonacci fraction.

On the stem in the Figure 23, the leaves and the spiral following the growth of leaves can be seen. For example, Fibonacci fraction 2 / 3 can be considered as the ratio of the number 2, of full turns of spiral, and number 3, the numbers of spaces between leaves 1 and 4. Analogously, the Fibonacci fraction is 5 / 8 ratio of the number 5, the number of full turn of the spiral and 8, the numbers of spaces between leaves 1 one 9.

Let us consider the quotients of Fibonacci sequence given by: $x_n = \dfrac{f_{n+1}}{f_n}, n \in N,$ where $(f_n)_{n\in N}$, are the terms of Fibonacci sequence. The sequence $(x_n)_{n\in N}$, can be written as

$$x_n = \frac{f_n + f_{n-1}}{f_n} = 1 + \frac{f_{n-1}}{f_n} = 1 + \frac{1}{x_{n-1}}.$$

The means that the terms of sequences $(x_n)_{n\in N}$, are given by:

$$2, \quad \frac{3}{2} = 1.5, \quad \frac{5}{3} = 1.667, \quad \frac{8}{5} = 1.6, \quad \frac{13}{8} = 1.625, \quad \frac{21}{13} = 1.615.$$

In order to prove the convergence of the sequence $(x_n)_{n\in N}$, we shall prove that the subsequences $(x_{2k-1})_{k\in N}$, and $(x_{2k})_{k\in N}$, converge to the same limit.

By using mathematical induction it can be shown that the subsequence $(x_{2k-1})_{k\in N}$ decrease, and subsequence $(x_{2k})_{k\in N}$ increase. All terms of the sequence are in the range $(1,2)$, meaning that both subsequence converge. Let us denote by

$$\lim_{k\to\infty} x_{2k-1} = p, \quad \lim_{k\to\infty} x_{2k} = l.$$

Then from

$$x_{2k-1} = 1 + \frac{1}{x_{2k-2}}, \ k \in N, \qquad x_{2k} = 1 + \frac{1}{x_{2k-1}}, \ k \in N,$$

it follows

$$\lim_{k\to\infty} x_{2k-1} = 1 + \frac{1}{\lim\limits_{k\to\infty} x_{2k-2}}, \quad \lim_{k\to\infty} x_{2k} = 1 + \frac{1}{\lim\limits_{k\to\infty} x_{2k-1}},$$

$$p = 1 + \frac{1}{l}, \quad l = 1 + \frac{1}{p}.$$

Simply, we obtain the equation $p^2 - p - 1 = 0,$ with the solution $p = \frac{1+\sqrt{5}}{2}$, which is the limit of the subsequence x_{2k-1} of sequence $x_n, n \in N.$ Analogously, one get $l = \dfrac{1+\sqrt{5}}{2}$, meaning that $\lim\limits_{n\to\infty} x_n = \dfrac{1+\sqrt{5}}{2}$, i.e., golden ration..

The terms of Fibonacci sequence, f_n, satisfies the Binet formula:

$$f_n = \frac{\varphi^n - (1-\varphi)^n}{\sqrt{5}} = \frac{\varphi^n - (-1/\varphi)^n}{\sqrt{5}},$$

where $\varphi = \frac{1+\sqrt{5}}{2} \approx 1.6180339887$, is the golden ratio.

Using the last two relations we can write:

$$f_n = \frac{\varphi^n - (1-\varphi)^n}{\sqrt{5}} = \frac{(\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n}{\sqrt{5}}$$
$$= \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}}, \quad n \in N.$$

Fibonacci sequence in nature can be seen on the website:

http://britton.disted.camosun.bc.ca/fibslide/jbfibslide.htm.

In her presentation Irina Boyadjiev, as can be seen on the website showed the Fibonacci spirals in nature:

http://www.lima.ohio-state.edu/people/iboyadzhiev/GeoGebra/spiralInNature.html .

Analogously the authors made:

FibonacijevS.ggb FibonacijevSK1.ggb FibonacijevSL.ggb.

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