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The Use of Experimental Mathematics in the Classroom

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The use of computers is gaining importance in education today. Experimental Mathematics is particularly suitable for teaching (and learning) mathematics in a computer supported learning environment. In this manuscript, we show how Experimental Mathematics works in the classroom. We demonstrate that computers can be used in each phase of the whole learning process (formulating definitions, problems and proofs, detecting finite patterns, conjecturing, falsifying, and applying math knowledge).

1. Introduction

> *"... mathematics appears here as a close relative to the natural sciences, as a sort of "observational science" in which observation and analogy may lead to discoveries ..." And I hope that mathematical discovery, scientific method, and the inductive aspect of mathematics will not be so completely neglected in the high schools of the future as they are in the high schools of today."* (GYÖRGY PÓLYA [19])

1.1. Motivation

First we summarize our motivations citing Csermely [6].

"The quantity of easily accessible information has grown to a vast degree. However, often knowledge, which understands the relationships and interactions, is lost in the sea of information. Unexpected situations, which have become daily phenomena, require various problem-solving abilities. *More information is not more knowledge.* In the current sea of information, verifiable, systematized knowledge has only a real value."

The current process of Hungarian education is still dominated by an information-centric outlook. (Hungarian students do better than average in the Trends in International Mathematics and Science Study (TIMSS) mathematics survey which focuses on theoretical knowledge and worse than average in the Programme for International Student Assessment (PISA) survey which measures ability to use the knowledge in practice.) Because of the growing quantity of information (and its accessibility) the expectation of the society regarding to the tasks of the schools has changed. Workplaces and other members of the society expect the schools not only to present and systematize the information, but increasingly, to understand it and apply it. Two of the additional essential components of the pedagogical renewal needed to rebuild the education system is experimenting and info-communication methodology.

Now we give our point of view regarding the teaching of mathematics and emphasize the role of experiments, heuristic and intuition (Borwein [5] and Pólya [19]).

> *„Intuition comes to us much earlier and with much less outside influence than formal arguments which we cannot really understand unless we have reached a relatively high level of logical experience and sophistication."* (GYÖRGY PÓLYA [19])

In [10] the author claims that mathematical proofs have no longer a valid role in mathematical thinking. We think that one can discover and cultivate mathematics with the help of the computer.

"Experimental Mathematics connotes the use of the computer for some or all of:

(1) Gaining insight and intuition
(2) Discovering new patterns and relationships
(3) Using graphical displays to suggest underlying mathematical principles
(4) Testing and especially falsifying conjecture
(5) Exploring a possible result to see if it is worth a formal proof
(6) Suggesting approaches for formal proof
(7) Replacing lengthy hand derivations with computer based derivations
(8) Confirming analytically derived results [5]"

To show how mathematics really works, to clarify *why* we have our definitions and theorems in their form, and to explore the relationship between proof and experiment we should change our teaching methods:

(1) Use the computer to catch attention!
(2) Let the children act (play, ask, etc.)!
(3) Make experiments, use the computer as an experimental mathematician! - The use of computers can support the shift from frontal teaching to project based, self-paced or active small group learning.
(4) Use inquiry-based learning/teaching (IBL/T) techniques!

1.2. Outline

In Section 2 we show a computer-supported method for discovering identities and detecting patterns. We close this section with some further exercises. In Section 3 some examples are given regarding to the use of computer as a tool. In Section 4 we show some of the most important features of high speed computers. In Section 5 we show, how Experimental Mathematics works in teaching a complete new field.

2. Discovering identities behind complete induction

2.1. Historical motivation

> *„I have the result, but I do not yet know how to get it."* (GAUSS)

In the 1780's a provincial German schoolmaster gave his class the tedious assignment of summing the first 100 integers. The teacher's aim was to keep the kids quiet for half an hour, but one young pupil almost immediately produced an answer: $1+2+3+\ldots+98+99+100 = 5050$. Gauss had a deeper insight: If you "fold" the series of numbers in the middle and add them in pairs - $1+100$, $2+99$, $3+98$, and so on all the pairs sum to 101. There are 50 such pairs, and so the grand total is simply 50×101.[9] We can say, while the others only *counted*, Gauss *proved* the identity (or came up with a better algorithm).

2.2. Computers and *„deus ex machina"*

Exercise 1 *Let us prove that the Gauss' identity* $1+2+3+\ldots+n = \frac{n(n+1)}{2}$ *holds for all* $n \in \mathbb{N}$.

This kind of exercises do not support intuition and discovery. Let us try to *find* the identity!

Evaluate the first few terms of the series $1, 1+2, 1+2+3, \ldots$!

$$\begin{aligned} 1 &= 1 \\ 1+2 &= 3 \\ 1+2+3 &= 6 \\ 1+2+3+4 &= 10 \\ 1+2+3+4+5 &= 15 \end{aligned}$$

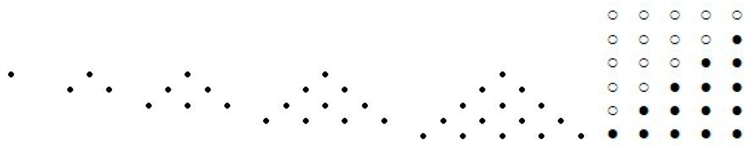
To recognize a pattern use The On-Line Encyclopedia of Integer Sequences (OEIS) [18]. Entering the sequence $1, 3, 6, 10, 15$, the site answers:

$$\text{Triangular numbers:} \quad a(n) = C(n+1, 2) = n(n+1)/2 = 0 + 1 + 2 + \ldots + n.$$

To prove the identity without more calculations try to answer the next question:

Exercise 2 Why do we call these numbers triangular numbers?

Solution 3 First draw the subsequent figures for the terms 1, $1+2$, $1+2+3$, $\ldots$. Let the students make their own discoveries. If they are stuck, the role of the teacher is to help the experiments, and not to lecture anything new.



(a) Triangular numbers and a ... (b) ... rectangle

Figure 1: Discovery: The sum of consecutive triangular numbers

2.3. Further exercises

With similar methods the reader is urged to solve the following exercises. (A brief solution is given for the first exercise.)

Exercise 4 Try to find a closed form for $1^2 + 2^2 + 3^2 + \ldots$!

Solution 5 Typing the first few terms $(1, 5, 14, 30, 55)$, OEIS returns:

$$\text{Square pyramidal numbers :} \quad 0^2 + 1^2 + 2^2 + \ldots + n^2 = n(n+1)(2n+1)/6.$$

Exercise 6 Try to find a closed form for the sum of the first n cubic number!

Exercise 7 Prove for each $n \in \mathbb{N}$ that $1^2 + 3^2 + 5^2 + \ldots + (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$

Exercise 8 Prove for each $n \in \mathbb{N}$ that $1^3 + 3^3 + 5^3 + \ldots + (2n-1)^3 = n^2(2n^2-1)$

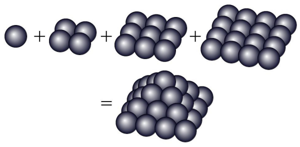


Figure 2: Square pyramidal numbers

Exercise 9 Prove for each $n \in \mathbb{N}$ that $(2+1)(2^2+1)\dots(2^{2^n}+1) = 2^{2^{n+1}} - 1$

Exercise 10 Prove for each $n \in \mathbb{N}$ that $1 \cdot 4 + 2 \cdot 7 + \dots n \cdot (3n+1) = n(n+1)^2$

Exercise 11 Prove for each $n \in \mathbb{N}$ that $1 \cdot 2 + 2 \cdot 3 + \dots + n \cdot (n+1) = \frac{n(n+1)(n+2)}{3}$

Exercise 12 Prove for each $n \in \mathbb{N}$ that $1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \dots + n(n+1)(n+2) = \frac{n(n+1)(n+2)(n+3)}{4}$

Exercise 13 Prove for each $n \in \mathbb{N}$ that $1 \cdot 1! + 2 \cdot 2! + \dots n \cdot n! = (n+1)! - 1$

3. Computer as a tool

„*Life is good for only two things, discovering mathematics and teaching mathematics.*" (SIMEON POISSON)

In this section we show some freeware softwares, animations and applets. In most of the cases it is very easy to use them, e.g. they do not need installation at all. The busy teacher does not need to waste his or her time to beg for technical support.

3.1. Discovering set theory identities

This program was developed by Zoltán Palincsár. It was available on Sulinet, a Hungarian educational portal for a long time.

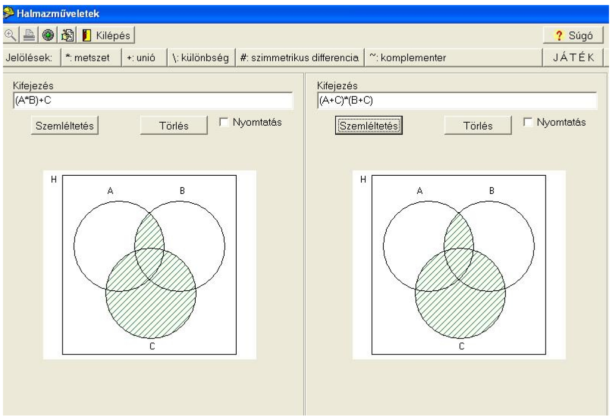


Figure 3: $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$

Starting the program the user gets a GUI with two windows. Via working in these windows (on Venn diagrams of three sets) it is easy the create and confirm set theory identities. The program has a very interesting function.

Choosing the function Game (Játék) one can play a game. The computer gives a Venn diagram and the user gives the related set operations or vica versa. This kind of approach supports *discovering* mathematics through *experiments*.

3.2. The Monty Hall problem and the Monte Carlo method

The Monty Hall dilemma was discussed in the popular ”Ask Marylin” column of the Parade magazine. The problem was originally posed by Steve Selvin in the American Statistician in 1975 [20].

Problem 14 ”Assume that a room is equipped with three doors. Behind two are goats, and behind the third is a shiny new car. You are asked to pick a door, and will win whatever is behind it. Let's say you pick door 1. Before the door is opened, however, someone who knows what's behind the doors (Monty Hall) opens one of the other two doors, revealing a goat, and asks you if you wish to change your selection to the third door (i.e., the door which neither you picked nor he opened).” [21] The Monty Hall problem is deciding whether you do [17].

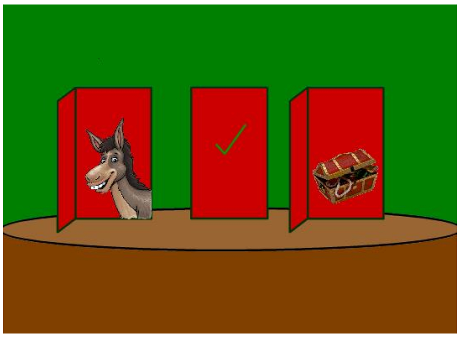


Figure 4: Monty Hall dilemma

To get closer to the problem, we propose use the Wolfram Demonstration Project page. After a few minutes of playing, discovering, learning (these are just the same!) students may come up with conjectures regarding to the decision in Monty Hall's brain teaser.

We can support these ideas with an another application, by which we can simulate thousands of games in a second.

Not changing	Winning trials: 3340	Total: 10 000
Changing	Winning trials: 6660	Total: 10 000

Table 1: Monty Hall problem and Monte Carlo method

Remark 15 A very interesting question for teachers at different levels of teaching is: *Is it a proof or not?*

> „... *a proof is a psychological device for convincing somebody that something is true.”*[12]

> *„I do still believe that rigour is a relative notion, not an absolute one. It depends on the background readers have and are expected to use in their judgement.”* (RENÉ THOM)

Alternative sites for playing Monty's game:

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http://www.nytimes.com/2008/04/08/science/08monty.html
http://www.theproblemsite.com/games/monty_hall_game.asp
http://www.stat.sc.edu/~west/javahtml/LetsMakeaDeal.html
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4. Computer as a partner

4.1. Introduction

One of the most important features of high speed computers is that they enable us to see things that we could not visualize before. The computer allows us to see things more clearly and in more detail in new ways. Furthermore, the computer can try millions of combinations only in a few minutes, and there is a high chance of finding something new [12].

4.2. Thales' Theorem

Exercise 16 Prove that the diameter of a circle always subtends a right angle to any point on the circle!

Remark 17 Posing the problem in this old-fashioned style does not support experiments at all. What's more, if we discover something alone (not necessary new), it is easy to bring it back later. If we have to learn something without any personal connection, it can be very hopeless to recall it.

Handmade pictures and measurements are the most convincing ones. On the other hand, we should make thousands of measures to help the proof. To be able to do this, we can use the following applications: GeoGebra or Autograph.

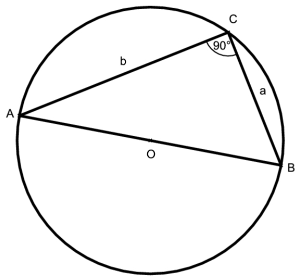


Figure 5: Thales' Theorem - Proof or certainty?

4.3. Discovering trigonometric identities

It is very difficult to learn (and to teach) trigonometric identities because we have several and they are very similar. An effective way of teaching this field is to make the students discover their own identity.

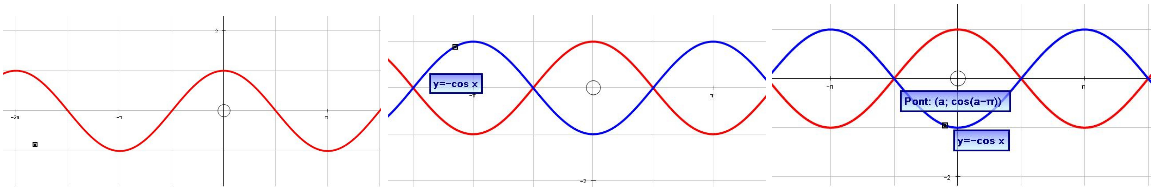
Let us consider the next animation. At the very beginning we can see the graph of $\cos x$ and a single point. (This point is the teacher's „partner". It means, that in this application the curve of this point is given by the teacher. This previously given orbit helps the discovery.) Let the point go on its own way! We see that this point moves along the curve $y = -\cos x$. Surprisingly the program gives that the curve is actually $y = cos(x - \pi)$. (This is the real curve given previously by the teacher.) Instead of Autograph we can use GeoGebra too.

4.4. Calculators and $\cos 30°$ - discovering Taylor polynomials

Problem 18 How does a calculator „know" the exact value of $\cos 30^o$?

Exercise 19 Using a scientific calculator let us fill in the following table!

Solution 20



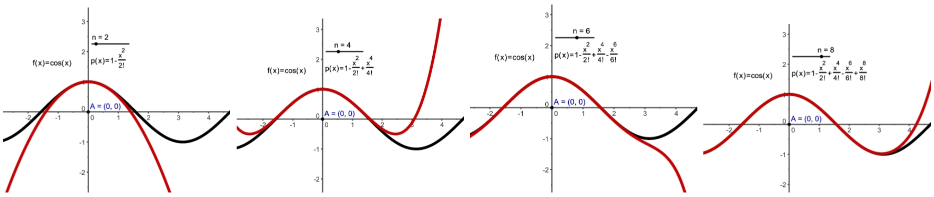
(a) (b) (c)

Figure 6: Discovering trigonometric identities: $-\cos x = cos(x - \pi)$

x	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	2	3	4	8
$T_2(x) = 1 - \frac{x^2}{2!}$								
$T_4(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$								
$T_6(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$								
$T_8(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!}$								
$\cos x$								

Table 2: Discovering Taylor polynomials I.

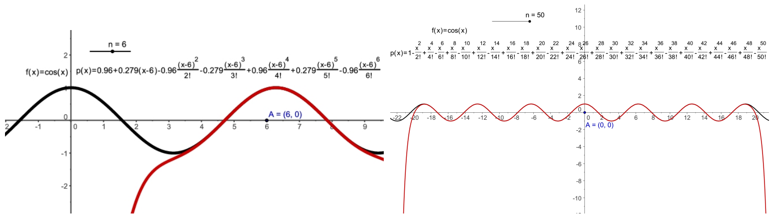
Exercise 21 To get closer to the problem above, use the next Mathematica worksheet.



(a) (b) (c) (d)

Figure 7: Taylor polynomials with different orders and the graph of $\cos x$

Changing the parameters we can discover that the error of the approximation depends on the *point* about which the polynomial is considered.



(a) Taylor polynomials around another point (b) Discovering Taylor series

Changing the parameters we can discover that the error of the approximation depends on the *degree* of the polynomial considered. Increasing the degree it is possible to discover the Taylor series.

4.5. Further possibilities

In this section we propose some programs supporting the learning process.

(a) Euler 3D
(b) Euklidesz

x	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	2	3	4	8
$T_2(x)$	1	0,8629221	0,4516886	-0,2337005	-1	-3,5	-7	-31
$T_4(x)$	1	0,8660538	0,5017962	0,0199689	-0,3333333	-0,125	3,6666667	139,666667
$T_6(x)$	1	0,8660252	0,4999645	-0,00008945	-0,4222222	-1,1375	-2,0222222	-224,42222
$T_8(x)$	1	0,8660254	0,50000004	0,0000247	-0,415873	-0,9747767	-0,3968254	191,67937
$\cos x$	1	0,8660254	0,5	0	-0,4161468	-0,9899925	-0,6536436	-0,1455

Table 3: Discovering Taylor polynomials II.

(c) Cabri
(d) The Geometer's Sketchpad
(e) GrafEq
(f) Poly
(g) Graph
(h) Derive
(i) WMI

5. Islands

In this section we show, how experimental mathematics works in an elaborate problem [15]. We use the computers to make appropriate definitions at different level of knowledge and abstraction, to make and test conjectures, and detecting finite patterns. With the help of the computers we will find out exercises and we will solve them with different methods. We show how to replace lengthy hand derivations with with computer based derivations.

5.1. Didactics

In most cases it is difficult to give real ”living” research topics and open problems to high-school students because understanding most research areas need university or even higher level knowledge [11]. The topic of islands and the methods for its investigation is suitable also for high school students, although some of the corresponding results are quite new. The arising questions need no advanced mathematical knowledge. Because most of the problems are of finitary type, experimental mathematics with computer support proves to be useful for the formulation of general conjectures related to the bounds of the number of islands in particular configurations [16].

5.2. Notion of Island

In geography class the question may pop up automatically: What can be called an island? What is an island?

Remark 22 It is important to show our students, that most of the problems are multi-disciplinary.

Definition 23 An island rises up out of the sea, each of its points is higher than any of the surrounding points. (See Figure 8(c).)

In the sequel, our simplified world will be a rectangular grid. A positive integer will be assigned to each cell (unit square): this number expresses the height of the cell. It can be assumed that any cell outside of the rectangular grid will have height zero (intuitively, zero can be thought as the sea level here). This means that the cells of an island are connected. Moreover, the islands cannot have inner holes. (See Figure 8(d).)

5.2.1. Searching for islands

Exercise 24 Find all islands in the pictures below (see Figure 9)!

(c) Artistic sketch of an island

4	4	4	3	4	4
2	4	2	1	5	4
1	5	1	1	2	2
3	4	3	1	3	3

(d) Our simplified world

Figure 8: Different point of views

3	3	2
3	2	2

(a)

4	3	1	2
5	2	1	3
4	4	1	4

(b)

4	4	4	3	4	4
2	4	2	1	5	4
1	5	1	1	2	2
3	4	3	1	3	3

(c)

Figure 9: Find the islands!

Solution 25 There are two islands in Figure 9(a): Besides the main island (the whole world) there is an L-shaped island on the upper left corner, all cells in that island have height 3. There are eight islands in Figure 9(b). Finally there are eight islands in Figure 9(c), too. It can be observed that if we allow islands with arbitrary shapes, then their cardinality can be big and therefore their investigation is difficult.

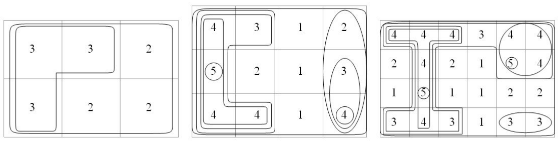


Figure 10: Solutions to Exercise 24

Remark 26 All figures were made in Mathematica 7.

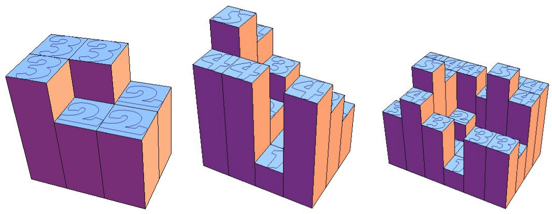
We have too many islands, it is too difficult to count them. (On the other hand it is not interesting at all, because it is easy to prove, that on a grid of size $m \times n$ we can have mn many islands.) We should consider islands with special shapes, for example rectangular islands.

Remark 27 Try to make the students discover the need and possibility of simplification.

Exercise 28 Give an exact mathematical definition of rectangular islands!

Solution 29 ***Definition 30*** Let a rectangular $m \times n$ board be given. We associate an integer number to each cell of the board. We can think of this number as a height above see level. A rectangular part of the board is called a rectangular island, if and only if there is a possible water level such that the rectangle is an island in the usual sense.

Definition 31 We say that some consecutive cells forming a rectangle constitute an island, if the integers in them are all greater then the integers in the neighbouring cells.



(a) (b) (c)

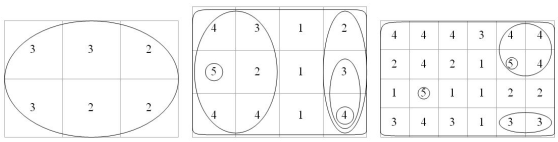
Figure 11: Solutions to Exercise 24

Definition 32 If $A = (a_{ij})_{m \times n}$ is an $m \times n$ matrix of real numbers and $\alpha, \beta, \gamma, \delta$ are integers with $1 \leq \alpha < \beta \leq m$ and $1 \leq \gamma < \delta \leq n$ then the elements a_{ij} with $\alpha \leq i \leq \beta$ and $\gamma \leq j \leq \delta$ form a submatrix R which we call a rectangle of A. Let r be the least element (or one of the least elements) of R. If for every element a_{ij} of A which is neighbouring with R we have $a_{ij} < r$ then R is called a rectangular island of A.

Exercise 33 Find all islands in the figures below (see Figure 12)!

(a) (b) (c)

Figure 12: Find all rectangular islands!

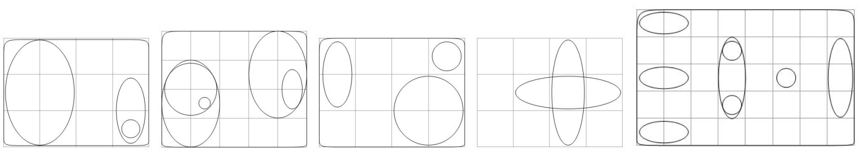


(a) (b) (c)

Figure 13: Solutions to Exercise 33

Solution 34

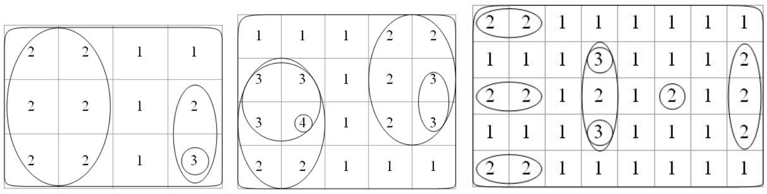
Exercise 35 Give height levels, such that exactly the circled/marked rectangular regions constitute islands, see Figure 14!



(a) (b) (c) (d) (e)

Figure 14: Give appropriate hight levels!

Solution 36 For Figure 14(c) and 14(d) there is no solution. Two islands cannot intersect or touch each other. For Figure 14(a), 14(b), 14(e) the pictures below give a possible solution, respectively(see Figure 15).



(a) Solution to 9(a) (b) Solution to 9(b) (c) Solution to 9(c)

Figure 15: Solutions to Figure 14

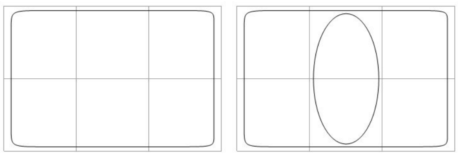
Remark 37 The nonexistence of solution for Figure 14(c) and 14(d) leads to discoveries regarding to the structure of the system of islands.

All exercises may serve as case studies for the general questions about the maximum number of islands on a grid. These can be partitioned into the following groups:

(a) Searching for islands on a rectangular grid, if the height function is given,
(b) Inverse problem: Searching for suitable height function, if a rectangle system is given,
(c) Inverse problem: Searching for suitable height function for a fixed number of islands on a grid,
(d) Inverse problem: Searching for suitable bounded height function, if a rectangle system is given,
(e) Inverse problem: Searching for suitable bounded height function for a fixed number of islands on a grid,
(f) Special: giving exact definitions, counterexamples, generalisations.

5.2.2. Further exercises

Exercise 38 Give height levels, such that exactly the marked rectangular regions constitute islands, if the maximum height h of an island is 1! (See Figure 16.)



(a) (b)

Figure 16: Give appropriate height levels!

Solution 39 It is obvious that in this case we can have only one island: the main island with 1-cells. Therefore, there is a unique solution for Figure 16(a) and there is no solution for Figure 16(b).

Exercise 40 Give height levels, such that exactly the marked rectangular regions constitute islands, if the maximum height h of an island is 2! (See Figure 16.)

Exercise 41 Give height levels, such that exactly the marked rectangular regions constitute islands, if the maximum height h of an island is 4! (See Figure 16.)

Exercise 42 Give height levels, such that there is exactly one island on the 3×4-grid!

Solution 43 Because the grid is always an island (main island), we cannot have an additional smaller island on the grid. This is obviously attainable if we let all cells have the same height. However, this is not the only possible solution type.

Exercise 44 Give height levels, such that there are exactly two islands on the 3×4-grid!

Exercise 45 Give height levels, such that there are exactly ten islands on the 3×4-grid!

Solution 46 There is no such configuration.

Exercise 47 Give height levels, such that there is exactly one island on the 3×4-grid, if the maximum height level is 1!

Exercise 48 Give height levels, such that there are exactly two islands on the 3×4-grid, if the maximum height level is 1!

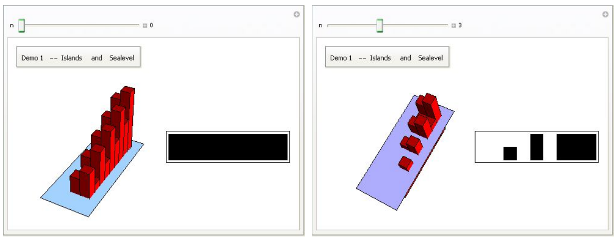
Exercise 49 Give height levels, such that there are exactly two islands on the 3×4-grid, if the maximum height level is 2!

Exercise 50 Change the parameters in the above exercises to make more and more exercises!

Remark 51 We think that it is important to give sufficient time for the aquisition of the above notions when working with younger students and we followed an inductive approach. Depending on time and other constraints, the instructor can choose a specialized case of the most general problem for further investigation.

5.2.3. Islands and conjectures

Use the next Mathematica demonstration to come up with conjectures regarding islands.



(a) (b)

Figure 17: Mathematica supporting the learning process

Conjecture 52 Two different islands can have a common cell or boundary only if one contains the other; i.e. two different islands either are far from each other (we can walk away (swim) between them) or the bigger one contains completely the cells of the smaller one in an island configuration.

Conjecture 53 All rectangle systems satisfying the above conditions can be made to an island system by a suitable choice of the height of the cells of the rectangles.

The above set theoretic conditions on rectangles completely characterize the island systems. Thus we obtained another definition of the island system which is useful for abstract mathematical investigations.

Conjecture 54 Let an $m \times n$ grid be given. We call a set R of rectangles of the grid an island system if for all $R_1 \neq R_2 \in R$, either $R_1 \subset R_2$, or $R_2 \subset R_1$, or no cell of R_1 is the neighbour of any cell of R_2.

Conjecture 55 Islands on a given grid with inclusion form a partially order set.

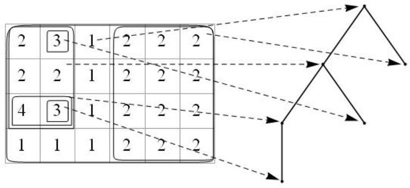
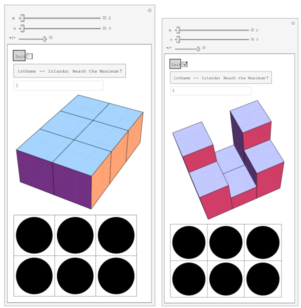


Figure 18: Islands and graphs

5.3. The maximum number of islands

We illustrate a game where a rectangular grid of fixed size is given. The final goal of the game is to construct a configuration with as many islands as possible.



(a) (b)

Figure 19: Discovering (the maximum number of) islands

Remark 56 We think that in a computer lab the students can experiment with well-designed games and might come up with conjectures regarding the lower or upper bounds on the number of islands. The teacher can motivate the activity by giving extra awards to those students who find at first an optimal configuration or finds all the possible optimal configuration in a small grid of fixed size. Another alternative is to use the interactive games together with a smartboard.

5.3.1. The number of islands on an $1 \times n$ grid

Exercise 57 How many islands can be at maximum on a

(a) 1×2,
(b) 1×3,
(c) 1×4,
(d) 1×5 grid?

Solution 58 The maximum number of islands are 2, 3, 4 and 5. A realization of such configurations are given by height levels in the next figure (see Figure 20).

1	2

(a)

1	2	3

(b)

1	2	3	4

(c)

1	2	3	4	5

(d)

Figure 20: Grinds with 2, 3, 4, 5 islands

Exercise 59 Formulate a general conjecture based on the solution of the previous exercise!

Conjecture 60 The maximum number of rectangular islands on an $1 \times n$ grid is n.

Proof. The proof is by induction on n. For $n = 1, 2$, it can be easily checked that the theorem holds. Assume now, that the theorem holds for all n $(n > 2)$ and we have a grid with length $n + 1$. We can assume that the grid contains a 1-cell (otherwise we can decrease each cell-height one by one, until we achieve a 1-cell, without changing the number of islands on the grid). Consider one of the 1-cells: It divides the grid into three parts: there is a part with length k, a part with length $n - k$ and the 1-cell itself (see Figure 21).

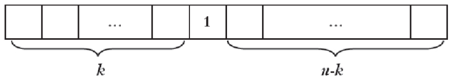


Figure 21: A grid with $n + 1$ length containing a 1-cell

According to the induction hypothesis, the number of islands $I \leq k + (n - k) + 1 = n + 1$, because on the first part there are at most k islands , on the second part there are at most $(n - k)$ islands and the grid is an additional island (we cannot have a new hybrid island which contain cells from part 1, from part 2, and part three, except for the whole grid). We saw that we can always have an island system with cardinality n on the $1 \times n$ grid. Therefore, the upper estimate cannot be improved (see Figure 22).

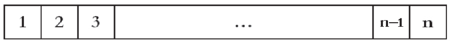


Figure 22: The $1 \times n$ grind with n islands

■

5.3.2. The number of islands on an $2 \times n$ grid

Exercise 61 Give a good lower estimate for the maximal number of islands on the $2 \times n$ grid! (Here is a simple universal construction required that produces ”many” islands, which does not necessarily need to reach the exact maximum.)

Islands and OEIS

Exercise 62 How many islands can be at maximum on a

(a) 2×1
(b) 2×2
(c) 2×3
(d) 2×4
(e) 2×5 grid?

Solution 63 $2, 3, 5, 6, 8$.

Exercise 64 What kind of pattern can be recognized? How can the above sequence be continued?

Solution 65 Let us use the The On-Line Encyclopedia of Integer Sequences! Entering the sequence the site answers: **`2, 3, 5, 6, 8, 9, 11, 12, 14,`** `15, 17, 18, 20, 21, 23, 24, 26, 27, 29, 30, 32, 33, 35, 36, 38, 39, 41, 42, 44, 45, 47, 48, 50, 51, 53, 54, 56, 57, 59, 60, 62, 63, 65, 66, 68, 69, 71, 72, 74, 75, 77, 78, 80, 81, 83, 84, 86, 87, 89, 90, 92, 93, 95, 96, 98, 99,` `The map n -> a(n) (where a(n) = 3n/2 if n even or (3n+1)/2 if n odd)`. Now we can formulate a conjecture.

Conjecture 66 On a board of size $2 \times n$ we can have at least a_n islands where

$$a_n = \begin{cases} \frac{3n}{2} & \text{if } n \text{ is even} \\ \frac{3n+1}{2} & \text{if } n \text{ is odd.} \end{cases}$$

Islands and cuts

Solution 67 We saw earlier that there are at most 2 islands on a 1×2 grid (see picture 20(a)), and it is also easy to see that there are at most 3 islands on a 2×2 grid. Denote by a_n the number of islands obtained in the $2 \times n$ grid by the the following, recursive cutting-technique starting from $a_1 = 2, a_2 = 3$. If $n > 2$, assume that we have all the $2n$ cells unfilled (i.e., no heights are assigned to any of the cells yet). Cut a 2×2 part off the grid and assign ones to the cells in the last but one column and assign 2 and 3 to the cells in the last column. We split the grid by the last but one-column into two smaller parts. If n¿5, then we continue the cutting on the bigger part. Since there are infinitely many positive natural numbers greater than 1, if we assign height levels which are greater than 1 to the cells in the bigger part, each island of it will be an island with respect to the initial whole grid, too.

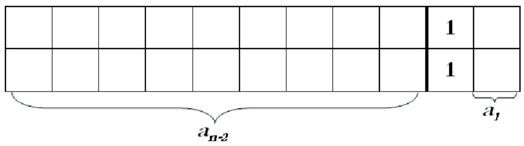


Figure 23: Cutting

We stop cutting if the remaining part has no more than two columns. (See Figure 23) Assume now that on the first part with length $(n-2)$ there are a_{n-2} islands. Then in the last column we have an additional two islands and the main islands.

With this procedure we are certainly be able to construct $a_n = a_{n-2} + 3$ $(n > 2)$ islands. Thus, we reduced the problem of giving a lower bound to finding a closed form solution to a second order linear inhomogeneous recursion:

$$a_1 = 2, a_2 = 3, a_n = a_{n-2} + 3, \quad (n > 2).$$

Solution 1 For a quick introduction on the general theory of higher order linear recursions with constant coefficients, we refer to [**?**]. First the inhomogeneous equation is reduced to a homogeneous second order recursion by considering the differences of the consecutive elements, $d_n := a_{n+1} - a_n$. Then $d_1 = 1$, $d_2 = 2$, moreover $d_n = (a_{n-1} + 3) - (a_{n-2} + 3) = a_{n-1} - a_{n-2} = d_{n-2}$

$$d_n = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 2 & \text{if } n \text{ is even} \end{cases}$$

The corresponding characteristic equation is $\lambda^2 = 0 \cdot \lambda + 1$. Its roots are $\lambda_{1,2} = \pm 1$. We are searching for the closed form of d_n as the following linear combination: $d_n = c_1 \cdot 1^n + c_2 \cdot (-1)^n$. Using the initial conditions

$$1 = c_1 - c_2$$
$$2 = c_1 + c_2,$$

we get: $c_1 = \frac{3}{2}$, $c_2 = \frac{1}{2}$. Hence $d_n = \frac{3}{2} \cdot 1^n + \frac{1}{2} \cdot (-1)^n$. Now we switch back to a_n by summation. Consider the following system of equations:

$$\begin{aligned} d_1 &= a_2 - a_1 \\ d_2 &= a_3 - a_2 \\ d_3 &= a_4 - a_3 \\ &\vdots \\ d_{n-1} &= a_n - a_{n-1} \end{aligned}$$

Summing it up, we obtain: $d_1 + d_2 + \ldots + d_{n-1} = a_2 - a_1 + a_3 - a_2 + a_4 - a_3 + \ldots + a_n - a_{n-1} = a_n - a_1$, or in a more compact form:

$$a_n = \sum_{i=1}^{n-1} d_i + a_1.$$

Hence $a_n = 2 + (n-1) \cdot \frac{3}{2} + b$, where

$$b = \begin{cases} 0 & \text{if } n \text{ is odd} \\ -1/2 & \text{if } n \text{ is even} \end{cases}$$

If fact, a nicer simplified form can be obtained as:

$$a_n = \left\lfloor \frac{3n+1}{2} \right\rfloor.$$

Solution 2 We can resolve the recursion without introducing the intermediate sequence of d_n by simply reducing it to a third order homogeneous equation as follows: Subtract two consecutive elements of a to get $a_n - a_{n-1} = a_{n-2} - a_{n-3}$, hence $a_n = a_{n-1} + a_{n-2} - a_{n-3}$. The characteristic equation for a_n is: $\lambda^3 - \lambda^2 - \lambda + 1 = 0$. Factoring the left hand side we get: $(\lambda+1)(\lambda-1)^2 = 0$. The roots are: $\lambda_{1,2} = 1$, $\lambda_3 = -1$. We search for a_n in the following form: $a_n = c_1 \cdot 1^n + c_2 \cdot n \cdot 1^n + c_3 \cdot (-1)^n$. For the unknown coefficients c_1, c_2 and c_3 we write down the initial conditions determined by the first three entries of a:

$$\begin{aligned} 2 &= c_1 + c_2 - c_3 \\ 3 &= c_1 + 2c_2 + c_3 \\ 5 &= c_1 + 3c_2 - c_3. \end{aligned}$$

By solving the system of equations we obtain that $c_1 = \frac{1}{4}$, $c_2 = \frac{3}{2}$, $c_3 = -\frac{1}{4}$. As a consequence,

$$a_n = \frac{1}{4} \cdot 1^n + \frac{3}{2} \cdot n \cdot 1^n + \left(-\frac{1}{4}\right) \cdot (-1)^n = \frac{6n + 1 + (-1)^{n+1}}{4}.$$

Solution 3 If we want keep the process of finding a closed form as black box [**?**], we may call for computer-algebra support; E.g. the next MATHEMATICA command helps:

```
RSolve[{a[n] == a[n-2]+3, a[1] == 2, a[2] == 3}, a[n], n]
```

It returns:

`a[n]` $\to \frac{1}{4}\left(4 - (-1)^n - 3(-1)^{2n} + 6n\right)$

This expression can be simplified further by the command

```
FullSimplify[a[n]/.RSolve[{a[n]==a[n-2]+3,a[1]==2,a[2]==3},a[n],n][[1]],
Assumptions→ {n∈ Integers}]
```

So finally we get the answer

$$\frac{1}{4}\left(1-(-1)^n+6n\right),$$

hence

$$a_n = \frac{6n+1+(-1)^{n+1}}{4},$$

which is the same expression that we already had before.

5.3.3. The number of islands on an $3 \times n$ grid

Exercise 68 Give a lower estimate for the maximal number of islands on the $3 \times n$ grid!

Solution 69 Denote by a_n the number of islands obtained in the $3 \times n$ grid by the above cutting-technique starting from $a_1 = 3, a_2 = 5$. If $n > 2$, the following recursion can be obtained: $a_n = a_{n-2} + 4$ $(n > 2)$.

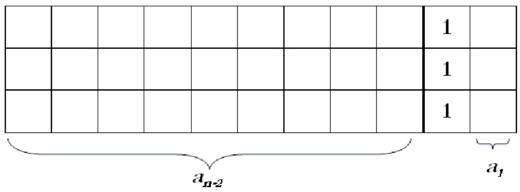


Figure 24: Cutting

We leave the details here to the reader (see Figure 24). The same solving methods can be applied as before. For the sake of brevity, let us just give the third, computer supported solution.

```
RSolve[{a[n] == a[n-2]+4, a[1] == 3, a[2] == 5}, a[n], n]
```

Mathematica returns:

$a \to 2-(-1)^{2n}+2n$

or after simplification

$$a_n = 2n+1.$$

5.3.4. The number of islands on an $m \times n$ grid - lower estimate

Exercise 70 Give a lower estimate for the maximal number of islands on the $m \times n$ grid by the same cutting technique!

Solution 71 Denote by a_n the number of islands obtained in the $m \times n$ grid by the cutting-technique (see Figure 25). First note that $a_1 = m$ (Exercise 59) and $a_2 = \frac{6m+1+(-1)^{m+1}}{4}$ (Exercise **??**).

Generalizing the above recursions, we can write:

$$a_n = a_{n-2} + m + 1 \ (n > 2)$$

This is a second order parametric recursion. Solving it by MATHEMATICA

```
RSolve[{a[n] == a[n-2]+m+1, a[1] == m, a[2] == (6m+1+(-1)^(m+1))/4}, a[n], n]
```

we get

$$\frac{1}{8}\left((-1)^n-6(-1)^{2n}-3(-1)^{3n}-6(-1)^n m-2(-1)^{2n}m+10n+6(-1)^n n+4mn\right)$$

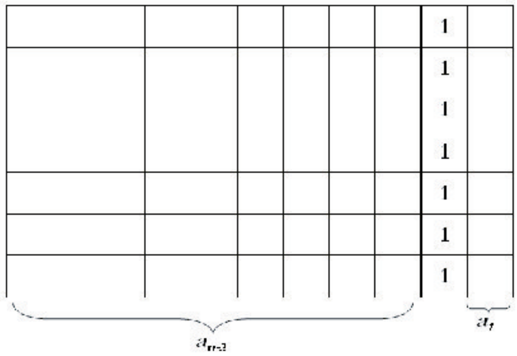


Figure 25: Cutting

An equivalent form of the result is:

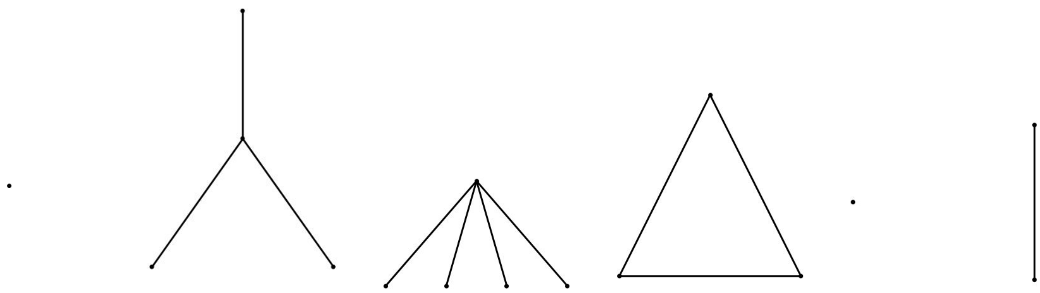
$$a_n = \left\lfloor \frac{mn + m + n - 1}{2} \right\rfloor .$$

5.3.5. The number of islands on an $m \times n$ grid - upper estimate

Exercise 72 Give an upper estimate for the number of islands on the $m \times n$ grid using the graph representation!

5.3.6. Further exercises

Exercise 73 Give island systems to whom the following graphs can be assigned (see Figure 26)!



(a) (b) (c) (d) (e)

Figure 26: Give island systems!

Solution 74 *There is no solution for d) and e).*

Exercise 75 Give an island system on the 3×4 grid such that the associated graph has exactly

(a) 1,
(b) 2,
(c) 4,
(d) 8,
(e) 9,
(f) 10 vertices.

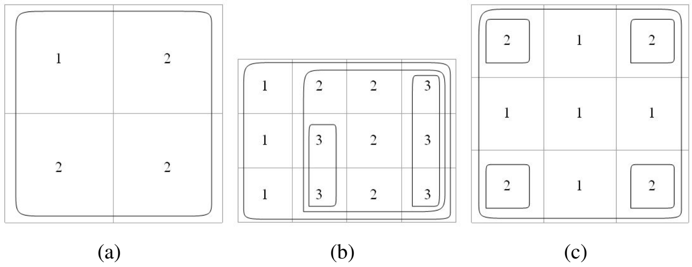


Figure 27: Solution to Exercise 73

Exercise 76 Give an island system on the 3×4 grid such that in the graph model each nonleaf vertex has exactly two sons!

Exercise 77 Give an island system on the 3×4 grid such that in the graph model there is a vertex with five sons!

Exercise 78 Prove the formula on the upper bound (which is exactly the same as the lower bound) of the number of rectangular islands on the $m \times n$ grid by induction!

Exercise 79 Prove that on the $m \times n$ grid there is an island system with cardinality k, where $k \in \mathbb{Z}$, $m + n - 1 \leq k \leq \lfloor \frac{mn+m+n-1}{2} \rfloor$!

„Theory and experiment feed on each other, and the mathematical community stands to benefit from a more complete exposure to the experimental process."[5]

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