Problems in Mathematics & Experiments with Mathematica

3. Limit and continuity

3.2 Basic rules and techniques: polynomials and rational fractions

The arithmetic of limits

THEOREM 3.2.1 Finite limits

Let a be finite or $\pm\infty$, and $\lim_{x\to a} f(x) = L$, $\lim_{x\to a} g(x) = M$, where M and L are finite.

$$\begin{aligned}
\lim_{x\to a} f(x) \pm g(x) &= \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) = L \pm M \\
\lim_{x\to a} c\, f(x) &= c \lim_{x\to a} f(x) = c\, L \\
\lim_{x\to a} (f(x)\, g(x)) &= \lim_{x\to a} f(x) \lim_{x\to a} g(x) = L\, M \\
\lim_{x\to a} \frac{f(x)}{f(x)} &= \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{L}{M}, \text{ provided } M \neq 0
\end{aligned}$$

THEOREM 3.2.2 Infinite and zero limits

Here, we summarize the arithmetical rules for infinite and zero limits. Assume that $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ are nonzero numbers, zero, or infinity. Then

$\infty + \infty = \infty$		$\infty - \infty$,	undefined
$c \cdot \infty = \infty$,	$c > 0$	$0 \cdot \infty$,	undefined
$c \cdot \infty = -\infty$,	$c < 0$		
$\infty \cdot \infty = \infty$			
$\frac{c}{\infty} = 0$,	$\lvert c \rvert < \infty$	$\frac{\infty}{\infty}$,	undefined
$\frac{0}{c} = 0$,	$0 < \lvert c \rvert \leq \infty$	$\frac{0}{0}$	undefined
$\frac{c}{+0} = \infty$		$\frac{c}{0}$,	undefined

Here, $\lim_{x\to a} f(x) = +0$ means that $f(x) > 0$ for x, close to a.

THEOREM 3.2.3 Limit of a composite function

If $\lim_{x\to a} g(x) = A$ and $f(x)$ is continuous at A, then $\lim_{x\to a} f(g(x)) = f(A)$.

THEOREM 3.2.4 A simple criterion for monotone functions

Let $f(x)$ be monotone for $x<a$ and close to a. Then

$$\lim_{x\to a-0} f(x)$$

exists. If $f(x)$ is bounded, then the limit is finite. The same statements hold for the right-hand side limit.

Arithmetic of continuity

THEOREM 3.2.5 Basic continuous functions

The elementary functions (powers, trigonometric functions, exponential and logarithmic functions, absolute value) are continuous on their domain.

THEOREM 3.2.6

The sum and product of continuous functions are continuous. The fraction is continuous if the denominator is not zero at the given point. If $g(x)$ is continuous at a and $f(x)$ is continuous at $g(a)$, then the function $f(g(x))$ is continuous at a.

Exercises and problems

SOLVED PROBLEM 3.2.1 Limit of continuous functions

Find

$$\lim_{x \to 1} x^2 - x - 1$$

○ SOLUTION

The function is continuous at $a=1$. Consequently, the limit is equal to $f(1)= -1$.

○

PROBLEM 3.2.2

Find the following limits.

(1) $\lim_{x \to 1} x^2 - x - 1$

(2) $\lim_{x \to 0} x^3 - 2x - 3$

(3) $\lim_{x \to 0} \frac{x - 1}{x + 1}$

(4) $\lim_{x \to 0} \frac{\sin(x)}{x - 1}$

(5) $\lim_{x \to 1} \log(x)(2 - x^2)$

(6) $\lim_{x \to 1} \frac{\tan(x)}{e^x}$

(7) $\lim_{x \to -e} \frac{\log(-x)}{x}$

(8) $\lim_{x \to -1} \frac{x^4 - x^3 + x^2 - 1}{x - 1}$

(9) $\lim_{x\to -1} \frac{\sqrt{x^4+2x^3-1}}{x+2}$

(10) $\lim_{x\to 2} \sqrt[3]{\frac{x^3+x^2-3}{x-1}}$

SOLVED PROBLEM 3.2.3 Limit of a polynomial at ∞

Find

$$\lim_{x\to\infty} x^3-2x^2+1$$

○ SOLUTION

The expression in this form results in a limit of form ∞-∞. To get a better form, take the highest power out. Now

$$x^3-2x^2+1=x^3\left(-\frac{2}{x}+\frac{1}{x^3}+1\right).$$

The limit of the first factor is ∞, and $\lim_{x\to\infty}(-\frac{2}{x}+\frac{1}{x^3}+1)=1$. Consequently, $\lim_{x\to\infty} x^3-2x^2+1=\infty$.

○

• Remark

The previous example shows the large scale behavior of polinomials. For large x, a polynomial behaves similarly to its dominant term, that is, the term of the highest power. This technique will be used in the following problems for functions containing polynomials.

• General method to find a limit of a rational fraction at a ∞

$$\boxed{\lim_{x\to\infty}\frac{a_n x^n+a_{n-1}x^{n-1}+\ldots+a_0}{b_m x^m+a_{m-1}x^{m-1}+\ldots+b_0}=\begin{cases}\frac{a_n}{b_m}, & n=m\\ 0, & n<m\\ \operatorname{sign}\left(\frac{a_n}{b_m}\right)\cdot\infty, & n>m\end{cases}}$$

SOLVED PROBLEM 3.2.4 Limit of a generalized polynomial at ∞.

Find

$$\lim_{x\to\infty}\sqrt{x^3-x}-x^2+1$$

○ SOLUTION

The expression results in a limit of form ∞-∞. To get a better form, take the highest power out. Now

$$\sqrt{x^3-x}-x^2+1=x^2\left(-1+\frac{1}{x^2}+\frac{1}{\sqrt{x}}\sqrt{1-\frac{1}{x^2}}\right).$$

It is obvious, that the limit is -∞.

Solved Problem 3.2.5 Limit of rational fractions at ∞.

Find the following limits.

(1) $\lim_{x\to\infty} \dfrac{x^3 - x^2 + x}{2x^2 - 3x + 1}$

Solution

The expression results in a limit of form $\frac{\infty}{\infty}$. Take the dominant power out in both the numerator and denominator:

$$\frac{x^3 - x^2 + x}{2x^2 - 3x + 1} = \frac{x^3(1 - \frac{1}{x} + \frac{1}{x^2})}{x^2(2 - \frac{3}{x} + \frac{1}{x^2})} = x \frac{1 - \frac{1}{x} + \frac{1}{x^2}}{2 - \frac{3}{x} + \frac{1}{x^2}}.$$

The limit of the fraction is $\frac{1}{2}$. Consequently, $\lim_{x\to\infty} \frac{x^3-x^2+x}{2x^2-3x+1} = \infty$.

(2) $\lim_{x\to\infty} \dfrac{2x^4 + x^2 + 1}{5x^5 - 3}$

Solution

The expression results in a limit of form $\frac{\infty}{\infty}$. Take the dominant power out in both the numerator and denominator:

$$\frac{2x^4 + x^2 + 1}{5x^5 - 3} = \frac{x^4(2 + \frac{1}{x^2} + \frac{1}{x^4})}{x^5(5 - \frac{3}{x^5})} = \frac{1}{x} \frac{2 + \frac{1}{x^2} + \frac{1}{x^4}}{5 - \frac{3}{x^5}}.$$

The limit of the second fraction is $\frac{2}{5}$. Since $\lim_{x\to\infty} \frac{1}{x} = 0$, we have $\lim_{x\to\infty} \frac{x^3-x^2+x}{2x^2-3x+1} = \infty$.

(3) $\lim_{x\to\infty} \dfrac{2x^4 + x^2 + 1}{5x^5 - 3}$

Solution

The expression results in a limit of form $\frac{\infty}{\infty}$. Take the dominant power out in both the numerator and denominator:

$$\frac{2x^3 + x^2 + 1}{x^3 - x} = \frac{x^3(2 + \frac{1}{x} + \frac{1}{x^3})}{x^3(1 - \frac{1}{x^3})} = \frac{2 + \frac{1}{x} + \frac{1}{x^3}}{1 - \frac{1}{x^3}} \quad \text{for } x \neq 0.$$

The limit is obviously equal to 2.

Problem 3.2.6

Find the following limits.

(1) $\lim_{x\to\infty} \dfrac{x^3 - 2x - 1}{x^2 - x}$

(2) $\lim_{x\to-\infty} \frac{x^5 - x^2 + 21}{2x^2 + x}$

(3) $\lim_{x\to\infty} \frac{-2x^2 + 2x + 1}{x^2 - x}$

(4) $\lim_{x\to\infty} \frac{x^3 + 3x^2 + 1}{x^2 - 3x^3}$

(5) $\lim_{x\to-\infty} \frac{5x^7 - 2x}{x^7 - x}$

(6) $\lim_{x\to\infty} \frac{x^4 - 5x + 1}{x^4 + x}$

SOLVED PROBLEM 3.2.7 Limit of rational fractions at finite places

Find the limits of the function

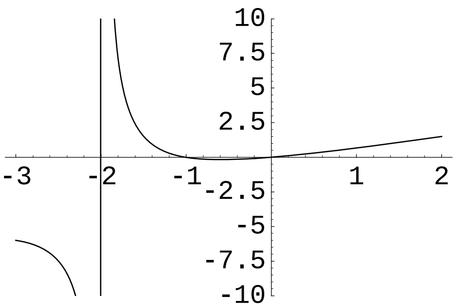
$$f(\text{x\_}) := \frac{x^3 - x}{x^2 + x - 2}$$

at the places, where it is not defined.

○ SOLUTION

First, plot the function to have a conjecture:

```
Plot[(x^3 - x)/(x^2 + x - 2), {x, -3, 2}, PlotRange -> {-10, 10}];
```



It seems that the only problematic point is $x_0 = -2$.

To be sure, factorize the denominator, and find its zeros:

```
Factor[x^2 + x - 2]
```

```
(-1 + x) (2 + x)
```

It gives that the function is not defined at $x_0 = -2$ and $x_1 = 1$. We have to find the limits

$$\lim_{x\to-2} \frac{x^3 - x}{x^2 + x - 2}$$

and

$$\lim_{x\to1} \frac{x^3 - x}{x^2 + x - 2}$$

Consider the case $x_0 = -2$. The limit can be written as

$$\lim_{x\to -2} \frac{x^3 - x}{x^2 + x - 2} = \lim_{x\to -2} \frac{x^3 - x}{x - 1} \lim_{x\to -2} \frac{1}{x + 2}$$

$\lim_{x\to -2} \frac{x^3 - x}{x - 1} = 2$, but for $\frac{1}{x + 2}$, only the half − side limits exist. Consequently, the limit

$$\lim_{x\to -2} \frac{x^3 - x}{x^2 + x - 2}$$

does not exist, but the half-side limits are

```
Limit[(x^3 - x)/(x^2 + x - 2), x → -2, Direction -> -1]
```

∞

```
Limit[(x^3 - x)/(x^2 + x - 2), x → -2, Direction -> 1]
```

$-\infty$

Now, consider the limit

$$\lim_{x\to 1} \frac{x^3 - x}{x^2 + x - 2}$$

The expression, in this form, results in a limit of form $\frac{0}{0}$. We also need to factorize the numerator:

$$x^3 - x = x(x - 1)(x + 1).$$

Consequently, if $x \neq 1$, then the fraction can be simplified:

```
Simplify[(x^3 - x)/(x^2 + x - 2)]
```

$\frac{x\,(1 + x)}{2 + x}$

Hence, $\lim_{x\to 1} \frac{x^3 - x}{x^2 + x - 2} = \frac{\lim_{x\to 1} x(1+x)}{\lim_{x\to 1}(2+x)} = \frac{2}{3}$.

With Mathematica:

```
Limit[(x^3 - x)/(x^2 + x - 2), x → 1]
```

$\frac{2}{3}$

• General method to find limits of rational fractions at finite places

1. The denominator is not zero at the given point x_0:
   Calculate the limit by substituting x_0 into the formula.
2. The denominator is zero, but the numerator is not, at x_0:
   Apply the rules for the limit $\frac{c}{0}$.
3. The fraction is $\frac{0}{0}$ at x_0:
   Simplify the fraction to relative prime numerator and denominator, and you can apply one of the above cases.

PROBLEM 3.2.8

Find the limits of the following functions at the places, where they are not defined. Plot the functions to have a conjecture, and also use *Mathematica* to simplify the fractions. Then compare your result with the limits obtained by *Mathematica*. Be aware, some points, where the function is not defined, cannot be found by the graph (explain, why!).

(1) $f(x\_) := \dfrac{1}{-x + x^2}$

(2) $f(x\_) := \dfrac{-3 + 2x + x^2}{-2 + x + x^2}$

(3) $f(x\_) := \dfrac{2 + 3x + x^2}{-4 + x^2}$

(4) $f(x\_) := \dfrac{2 + x}{2x^2 + x^3}$

(5) $f(x\_) := \dfrac{-2 + x + x^2}{2 - 3x + x^3}$

(6) $f(x\_) := \dfrac{1 - 2x}{-1 - x + 2x^2}$

(7) $f(x\_) := \dfrac{1 + x}{1 + 2x + 2x^2 + x^3}$

(8) $f(x\_) := \dfrac{1 + x + x^2}{1 + 2x + 2x^2 + x^3}$