Content

Problems in Mathematics & Experiments with Mathematica

6. Definite integral

6.1 Definition, geometrical meaning, properties

Definitions, basic properties

- **The lower, upper and Riemann sums**

Let the nonnegative function *f(x)* be continuous on the interval *[a,b]*. Let the positive integer number n be given, let $h=\frac{b-a}{n}$, and let $x_0 = a$, $x_1 = a + h$, ..., $x_i = a + i\,h$, ..., $x_n = b$. Let $\overline{f_i} = \sup\{f(x) : x \in [x_i, x_{i+1}]\}$ and $\underline{f_i} = \inf\{f(x) : x \in [x_i, x_{i+1}]\}$.
The sums

$$\underline{S_n} = \sum_{i=1}^{n} \underline{f_i}\, h \quad \text{and} \quad \overline{S}_n = \sum_{i=1}^{n} \overline{f_i}\, h$$

are lower and upper approximation of the area between the graph of the function and the x-axis, respectively. It can be proved that

$$\lim_{n\to\infty} \underline{S_n} = \lim_{n\to\infty} \overline{S}_n.$$

This common limit is considered the area of the region between the graph of *f(x)* and the *x*-axis. A sum of form

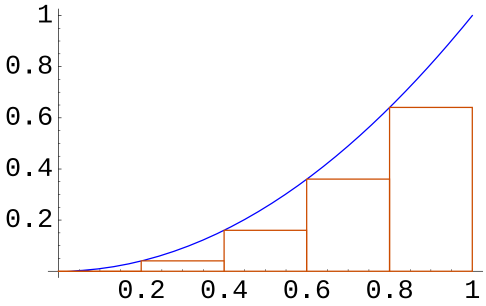
$$S_n = \sum_{i=1}^{n} f(\xi_i)\, h$$

where $x_i \le \xi_i \le x_{i+1}$, is called a Riemann sum. It is obvious that

$$\underline{S_n} \le S_n \le \overline{S}_n$$

- *Animation of the lower sums (electronic version only)*

```
Uniform partition into 5 subintervals
Left edge of subinterval used for height
Mesh = 0.2
Riemann Sum = 0.24
Definite Integral = 0.333333
```



- *Animation of the upper sums (electronic version only)*

```
Uniform partition into 5 subintervals
Right edge of subinterval used for height
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Riemann Sum = 0.44
Definite Integral = 0.333333
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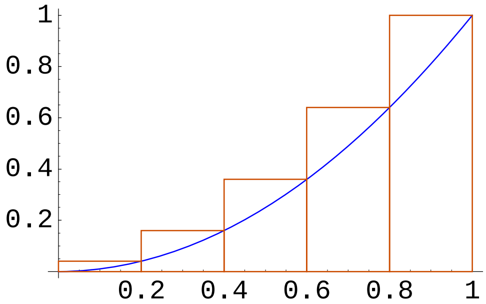


DEFINITION 6.1.1

In general, let the function $f(x)$ be defined on the interval $[a,b]$. The function $f(x)$ is Riemann integrable on $[a,b]$ if

$$\lim_{n\to\infty} \underline{S}_n = \lim_{n\to\infty} \overline{S}_n.$$

This common limit is called the definite integral or Riemann integral of $f(x)$ on [a,b], and it is denoted by

$$\int_a^b f(x)\,dx$$

For other integral approximations, see the section [Numerical integration: Integral approximation formulas](#).

Geometric meaning

If $f(x) \geq 0$, then $\int_a^b f(x)\,dx$ is nonnegative and gives the area between the graph of $f(x)$ and the x-axis. If $f(x) \leq 0$, then $\int_a^b f(x)\,dx$ is nonpositive. In general, $\int_a^b f(x)\,dx$ is the signed area.

• Basic properties

Let *f(x)* and *g(x)* be integrable on *[a,b]*.

$$\int_a^a f(x)\,dx = 0$$
$$f(x) \geq 0 \;\Rightarrow\; \int_a^b f(x)\,dx \geq 0$$
$$f(x) \geq g(x) \;\Rightarrow\; \int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$$
$$\int_b^a f(x)\,dx := -\int_a^b f(x)\,dx$$
$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$
$$\int_a^b k\,f(x)\,dx = k\int_a^b f(x)\,dx$$
$$\int_a^b f(x)\,dx + \int_a^b g(x)\,dx = \int_a^b (f(x)+g(x))\,dx$$

• Mean value of a function

The mean value of a function on an interval [a,b] is defined by the formula

$$M(f) := \frac{1}{b-a}\int_a^b f(u)\,du.$$

It is obvious that $\int_a^b M(f)\,du = M(f)\,(b-a) = \int_a^b f(u)\,du$. This is a natural generalization of the weighted aritmetical mean to integrable functions.

Area function, the Fundamental Theorem

DEFINITION 6.1.2 Area function

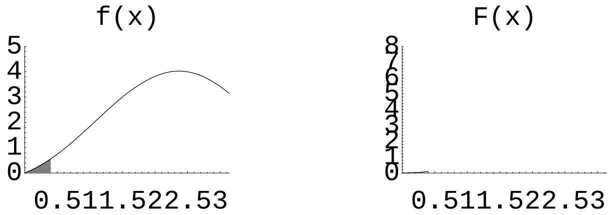
Let *f(x)* be integrable on the interval *[a,b]*. The function

$$\int_a^x f(x)\,dx$$

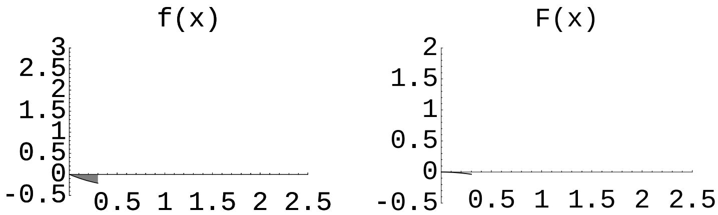
is called area function. For any other $a_1 \epsilon [a, b]$, the additivity of the integral provides the formula

$$\int_a^{a_1} f(u)\,du + \int_{a_1}^x f(u)\,du = \int_a^x f(u)\,du$$

- *Animation of the area function I. (electronic version only)*



- *Animation of the area function II. (electronic version only)*



THEOREM 6.1.3 Fundamental Theorem

Let *f(x)* be integrable on the interval *[a,b]*. If *f(x)* is continuous at $x_0 \in (a, b)$, then the area function $\int_a^x f(x)\,dx$ is differentiable at x_0 and

$$\boxed{\frac{d}{dx}\left(\int_a^x f(x)\,dx\right)\bigg|_{x=x_0} = f(x_0)}$$

Consequently, the area function is an antiderivative of *f(x)*.

• Application: variation of a function

The derivative *f(t)* (the speed of the change of a process) of a function *F(t)*, and the initial value $F(t_0)$ are given. Then *F(t)* is described by

$$\boxed{F(t) = F(t_0) + \int_{t_0}^{t} f(u)\,du.}$$

The variation of $F(t)$ between t_0 and t_1 is

$$F(t_1) - F(t_0) = \int_{t_0}^{t_1} f(u)\,du.$$

Exercises and problems

See the problems in the section Numerical integration: Integral approximation formulas.