Content

Problems in Mathematics
&
Experiments with Mathematica

9. Functions of several variables

9.2 Partial derivatives

Theory

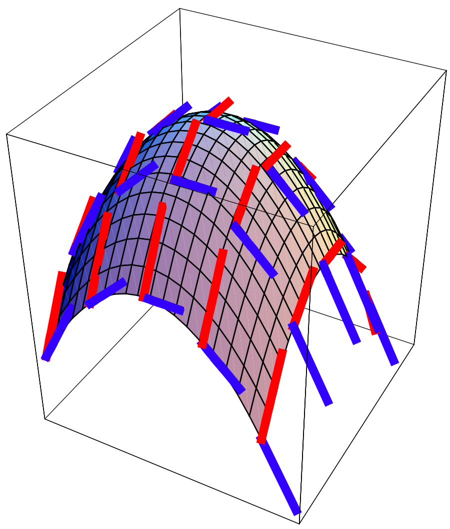
DEFINITION 9.2.1

Consider the function $f(x, y)$ in a neighborhood of the point (x_0, y_0). If the function $g(x) = f(x, y_0)$ is differentiable with respect to x (y_0 is fixed, see the definition in the first section on the derivative Definition, geometrical meaning, tangent line), then $g'(x_0)$ is called the partial derivative of $f(x, y)$ at (x_0, y_0) with respect to x.
If the function $h(y) = f(x_0, y)$ is differentiable with respect to y (x_0 is fixed), then $h'(y_0)$ is called the partial derivative of $f(x, y)$ at (x_0, y_0) with respect to y.

- **Notations**

$$f_x'(x, y) = \frac{\partial f(x,y)}{\partial x}; \qquad f_y'(x, y) = \frac{\partial f(x,y)}{\partial y}$$

- **Geometrical meaning**

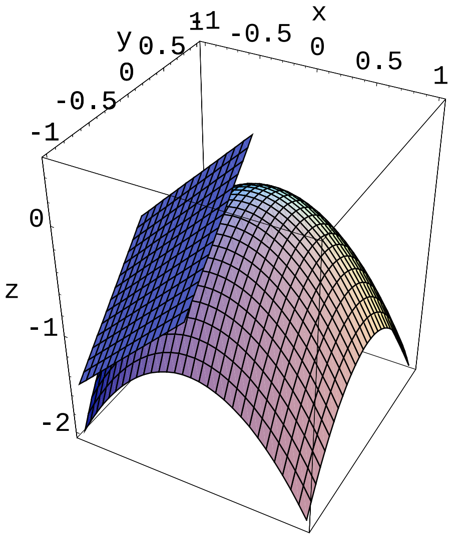


- *Tangent plane*

The tangent plane is determined at (x_0, y_0) by the direction vectors $(1, 0, f_x'(x_0, y_0))$ and $(0, 1, f_x'(x_0, y_0))$. The equation of the tangent plane at (x_0, y_0) is

$$z - z_0 = f_x'(x_0, y_0)(x - x_0) + f_y'(x_0, y_0)(y - y_0)$$

- *Example: $f(x,y)=-x^2- y^2$; tangent plane at (-0.5,-0.5).*



• Second order derivatives

The derivatives

$$\frac{\partial^2 f(x,y)}{\partial x^2} := \frac{\partial}{x}\left(\frac{\partial f(x,y)}{\partial x}\right)$$

$$\frac{\partial^2 f(x,y)}{\partial y^2} := \frac{\partial}{y}\left(\frac{\partial f(x,y)}{\partial y}\right)$$

are the pure, and

$$\frac{\partial^2 f(x,y)}{\partial y\,\partial x} := \frac{\partial}{y}\left(\frac{\partial f(x,y)}{\partial x}\right)$$

$$\frac{\partial^2 f(x,y)}{\partial x\,\partial y} := \frac{\partial}{x}\left(\frac{\partial f(x,y)}{\partial y}\right)$$

are the mixed second order partial derivatives.

THEOREM 9.2.2

If all the second order derivatives are continuous at (x_0, y_0), then

$$\frac{\partial^2 f(x,y)}{\partial y\,\partial x} = \frac{\partial^2 f(x,y)}{\partial x\,\partial y}.$$

Mathematica statements

See the differentiation statements in the section: A short introduction to *Mathematica*.

More details about the differentiation are in the Mathematica help: D[...].

Exercises and problems

SOLVED PROBLEM 9.2.1

Calculate the first and second order derivatives of the following functions. What is the tangent plane at the given (x_0, y_0). Plot the graph of the function and the tangent plane.

$$f(\mathrm{x\_}, \mathrm{y\_}) = \cos^2(y) + \sin^2(x); \quad \{x_0, y_0\} = \left\{-\frac{7\pi}{12}, 0\right\};$$

○ SOLUTION

- *The derivatives*

```
dx = ∂x f[x, y]
```

```
2 Cos[x] Sin[x]
```

```
dy = ∂y f[x, y]
```

```
-2 Cos[y] Sin[y]
```

```
dxx = ∂x,x f[x, y]
```

$2\,\mathrm{Cos}[x]^2 - 2\,\mathrm{Sin}[x]^2$

```
dyy = ∂y,y f[x, y]
```

$-2\,\mathrm{Cos}[y]^2 + 2\,\mathrm{Sin}[y]^2$

```
dxy = ∂x,y f[x, y]
```

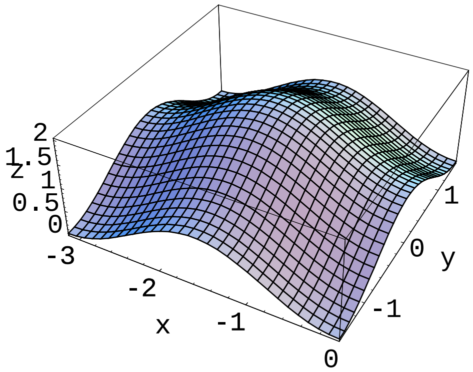
```
0
```

- *The tangent plane*

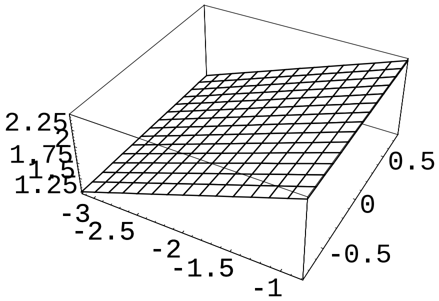
```
Plane[x_, y_] = f[x0, y0] +
  (dx /. {x → x0, y → y0}) (x - x0) + (dy /. {x → x0, y → y0}) (y - y0)
```

$$1 + \frac{1}{8}\left(1 + \sqrt{3}\right)^2 + \frac{1}{4}\left(-1 + \sqrt{3}\right)\left(1 + \sqrt{3}\right)\left(\frac{7\pi}{12} + x\right)$$

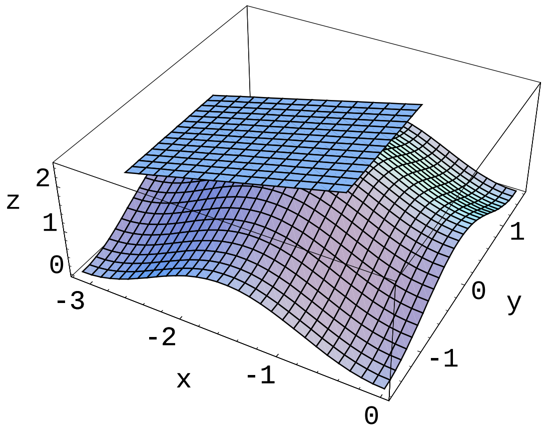
```
pltf = Plot3D[f[x, y], {x, -π, 0},
   {y, -π/2, π/2}, PlotPoints → 30, AxesLabel → {x, y, z}];
```




```
pltpl = Plot3D[Plane[x, y],
   {x, -π, -π/4}, {y, -π/4, π/4}, Shading → False];
```




```
sh = Show[pltf, pltpl];
```



PROBLEM 9.2.2

Calculate the first and second order partial derivatives of the following functions. Find the equation of the tangent plane at the given $\{x_0, y_0\}$. Plot the graph of the function and the tangent plane. Use the solution of the previous problem.

(1) $f(\text{x\_}, \text{y\_}) = x^2 - y^3;\ \{x_0, y_0\} = \{-1, 1\};$

(2) $f(\text{x\_}, \text{y\_}) = e^{\{-x^2-y^2\}};\ \{x_0, y_0\} = \left\{\frac{1}{e^{0.5}}, 0\right\};$

(3) $f(\text{x\_}, \text{y\_}) = \cos(x - y) + \sin(x\, y);\ \{x_0, y_0\} = \left\{0, \frac{\pi}{3}\right\};$

Problem 9.2.3

Calculate the first and second order partial derivatives of the following functions.

(1) $f(\text{x\_}, \text{y\_}) := y^3 + \log(y\, x^2) + \sin(x);$

(2) $f(\text{x\_}, \text{y\_}) := x^{\sin(y)} + \log(y^2 + x);$

(3) $f(\text{x\_}, \text{y\_}) := \cos^y(x) + \sqrt{x};$

(4) $f(\text{x\_}, \text{y\_}) := e^{x^2+y} + \tan(x\, y^2);$

(5) $f(\text{x\_}, \text{y\_}) := e^x\, y^3 + \log(y\, x^2)$