Content

Problems in Mathematics
&
Experiments with Mathematica

4. The derivative and its applications

4.6 Extremal values, monotonity, concavity and investigaton of functions

Theory: monotonity and extremal values

THEOREM 4.6.1

Let the function f be differentiable on the interval $[x_0, x_1]$. Then, f is nondecreasing (nonincreasing) on $[x_0, x_1]$ if and only if $f'(x)\geq 0$ $(f'(x)\leq 0)$ for every $x \in [x_0, x_1]$.

THEOREM 4.6.2

Let the function f be differentiable on the interval $[x_0, x_1]$. If $f(x)$ has local extremum at $\overline{x} \in (x_0, x_1)$, then $f'(\overline{x}) = 0$.
If $f'(\overline{x}) = 0$ and $f'(x)$ changes the sign at $\overline{x}$ from negative to positive (from positive to negative), then $f(x)$ has a local minimum (maximum) at $\overline{x}$.

Theory: convexity and inflection points

THEOREM 4.6.3

Let the function f be twice differentiable on the interval $[x_0, x_1]$. The function f is convex (concave) on $[x_0, x_1]$ if and only if $f'(x)$ is nondecreasing (nonincreasing) on $[x_0, x_1]$, i.e., $f''(x)\geq 0$ $(f''(x)\leq 0)$ for every $x \in [x_0, x_1]$. .

THEOREM 4.6.4

Let the function f be twice differentiable on the interval $[x_0, x_1]$. If f has an inflection point at $\overline{x} \in (x_0, x_1)$, then $f''(\overline{x}) = 0$.
If $f''(\overline{x}) = 0$ and $f''(x)$ changes the sign at $\overline{x}$ from negative to positive (from positive to negative), then $\overline{x}$ is an inflection point and $f(x)$ turns from concave to convex (convex to concave) point.

The classical way of the investigation of functions

Find the domain.
Find the zeros.
Find special properties (symmetry, etc).
Investigate continuity and differentiability.
Investigate monotonity, find local extremum points.
Investigate convexity, find inflection points.
Find limits at the boundary of the domain and at the points of discontinuity.
Draw the graph.

The computer-aided way of the investigation of functions

Find the domain.
Plot the graph.
Do experiments and draw more plots to discover details.
Investigate the plotted graph, have conjectures for the properties (monotonity, convexity, continuity, differentiability, limits, ...).
Estimate the zeros, points of discontinuity, extremal points.
Find special properties (symmetry, etc).
Investigate continuity, differentiability.
Find the specific places (zeros, local maxima, minima, etc) with computer.
If numeric method is needed use graphical estimations for initial estimates.
Calculate the limits at the boundary of the domain and at the points of discontinuity.

Mathematica statements

Plot[...]	Plots a functions.
Solve, NSolve	Finds the zeros formally.
FindRoot	Finds a zero numerically.
D[f[x], x], f'[x]	Calculates the first derivative.
D[f[x], {x, 2}], f''[x]	Calculates the second derivative
`Limit`	Theoretical limit
`NLimit`	Numerical limit in the package `NumericalMath`NLimit``

Exercises and problems

• Mathematica initialization

```
SetDirectory["FileName" /.
    NotebookInformation[EvaluationNotebook[]] /.
    Which[
            $System == "Microsoft Windows",
                FrontEnd`FileName[{_, d___}, nam_, ___] :>
     ToFileName[{d}] ,
             $System != "Microsoft Windows", FrontEnd`FileName[
                d_List, nam_, ___]     :> ToFileName[d]] ];
Abs'[x_] := Which[x < 0, -1, x > 0 , 1, True, 0]
Sign'[x_] := 0
<< package//npow.m;
```

SOLVED PROBLEM 4.6.1 Use the traditional method

Investigate the properties and sketch the graph of

$$f(x\_) = x^3 - 5x - 2x^2 + 6;$$

○ SOLUTION

• *Domain and continuity*

Since *f(x)* is a polynomial, it is defined for every real *x*, and it is continuous and differentiable everywhere.

• *Zeros of f(x)*

Use *Mathematica* to find the zeros, instead of manual calculations.

```
zeros = Solve[f[x] == 0, x]
```

```
{{x → -2}, {x → 1}, {x → 3}}
```

• *Monotonity and extremal points*

Find the zeros of the derivative:

```
f'[x]
```

$$-5 - 4x + 3x^2$$

```
{x1, x2} = x /. Solve[f'[x] == 0, x]
```

$$\left\{\frac{1}{3}\left(2-\sqrt{19}\right), \frac{1}{3}\left(2+\sqrt{19}\right)\right\}$$

The derivative is a parabola. Knowing its properties, the derivative changes sign at both roots. The function has a local maximum at the smaller place and local minimum at the larger one.

- *Convexity and inflection points*

Find the zeros of the second derivative

```
f''[x]
```

$-4+6x$

```
x3 = x /. Solve[f''[x] == 0, x]
```

$\left\{\frac{2}{3}\right\}$

The second derivative is a straight line. It changes sign at the root. The function has inflection at this place.

- *Limits*

It is obvious that

```
Limit[f[x], x -> ∞]
```

∞

and

```
Limit[f[x], x -> -∞]
```

$-\infty$

- *Summarize the results in a table.*

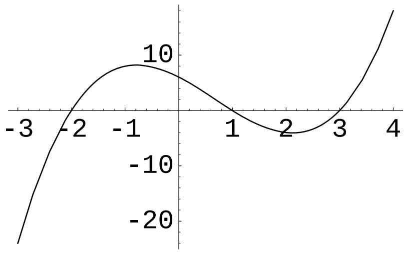
	$x<x_1$	$x=x_1$	$x_1<x<x_3$	$x=x_3$	$x_3<x<x_2$	$x_2=x$	$x_2<x$
$f(x)$	$-\infty\nearrow$ $\cap$	MAX.	$\searrow$ $\cap$	Infl.	$\searrow$ $\cup$	min.	$\nearrow\infty$ $\cup$
$f'(x)$	$+$ $\searrow$	0	$-$ $\searrow$	$-$	$-$ $\nearrow$	0	$+$ $\nearrow$
$f''(x)$	$-$	$-$	$-$	0	$+$	$+$	$+$

- *Meaning of signs*

$\nearrow$	increasing
a $\nearrow$ b	increasing from a to b
$\searrow$	decreasing
a $\searrow$ b	decreasing from a to b
$\cup$	convex
$\cap$	concave

- *Finally, plot the graph of the function.*

```
xmin = -3; xmax = 4;
Plot[f[x], {x, xmin, xmax}];
```



Solved Problem 4.6.2 Use the traditional method once more.

Investigate the properties, and sketch the graph of

$$f(x\_) := \frac{x^2 - x}{x^2 - 4x + 4};$$

○ Solution

- *Domain, continuity*

The function *f(x)* is not defined at the zeros of the denominator:

```
Excluded = {ex1, ex2} = x /. Solve[1/f[x] == 0, x]
```

{2, 2}

It is continuous and differentiable everywhere else.

- *The zeros of f(x)*

```
zeros = Solve[f[x] == 0, x]
```

{{x → 0}, {x → 1}}

- *Monotonity and extremal points*

Find the zeros of the derivative:

```
Factor[f'[x]]
```

$$-\frac{-2 + 3x}{(-2 + x)^3}$$

```
{x1} = x /. Solve[f'[x] == 0, x]
```

$$\left\{\frac{2}{3}\right\}$$

The numerator is a straight line. The derivative changes sign at the root, and it is a local minimum place.

- ### *Convexity and inflection points*

Find the zeros of the second derivative:

```
Factor[f''[x]]
```

$$\frac{6\,x}{(-2+x)^4}$$

```
x2 = x /. Solve[f''[x] == 0, x]
```

{0}

The second derivative changes sign at the root. The function has inflection point at this place.

- ### *Limits*

It is obvious that

```
Limit[f[x], x -> ∞]
```

1

and

```
Limit[f[x], x -> -∞]
```

1

The limits at the excluded point from the left-hand side

```
Limit[f[x], x -> ex1, Direction → 1]
```

∞

The limits at the excluded point from the righ-hand side

```
Limit[f[x], x -> ex1, Direction → -1]
```

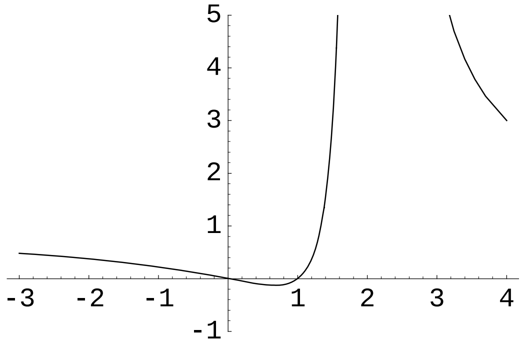
∞

- ### *Summarize the results in a table:*

	$x < x_2$	$x = x_2$	$x_2 < x < x_1$	x_1	$x_1 < x <$ ex1	$x =$ ex1	ex1 $< x$
$f(x)$	1 ↘ ∩	Infl.	↘ ∪	MIN	↗ ∞ ∪	× × × ×	∞ ↘ 1 ∪
$f'(x)$	− ↘	0	− ↗	0	+ ↗	× × × ×	− ↗
$f''(x)$	−	0	+	+	+	× × × ×	+

- *Plot the graph of the function*

```
xmin = -3; xmax = 4;
Plot[f[x], {x, xmin, xmax}, PlotRange -> {-1, 5}];
```



SOLVED PROBLEM 4.6.3 "Reversed" computer-aided investigation

Investigate the function

$$f(x\_) := x^6 - x^4 - 2x^5 - 4;$$

Find the extremal values, inflection points.

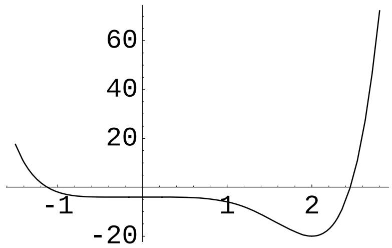
○ SOLUTION

- *Domain, continuity, limits*

Since *f(x)* is a polynomial, it is defined for every real number, and it is continuous and differentiable everywhere. No symmetry can be observed at the first sight. Since the highest power is even with positive coefficient, the limits at $\pm\infty$ are equal to ∞.

- *Plot the graph of f(x)*

```
xmin = -1.5; xmax = 2.8;
Plot[f[x], {x, xmin, xmax}, PlotRange -> All];
```



Note that we may have to do several plots to explore the properties of *f(x)*.

- *Zeros of f(x)*

Try to solve the equation $f(x)=0$ theoretically, although, there is no general rule to do it. For this case, *Mathematica* gives the following answer:

```
zeros = Solve[f[x] == 0, x]
```

```
{{x → Root[-4 - #1^4 - 2 #1^5 + #1^6 &, 1]},
 {x → Root[-4 - #1^4 - 2 #1^5 + #1^6 &, 2]},
 {x → Root[-4 - #1^4 - 2 #1^5 + #1^6 &, 3]},
 {x → Root[-4 - #1^4 - 2 #1^5 + #1^6 &, 4]},
 {x → Root[-4 - #1^4 - 2 #1^5 + #1^6 &, 5]},
 {x → Root[-4 - #1^4 - 2 #1^5 + #1^6 &, 6]}}
```

Estimate the zeros, and apply numerical approximations. It can be seen form the *Mathematica* plot that *f(x)* has two zeros, and their rough estimates are

```
root0 = {-1.1, 2.2};
```

Apply the Newton iteration (`FindRoot`) to obtain better approximate roots.

```
roots = Map[FindRoot[f[x] == 0, {x, #}] &, root0]
```

```
{{x → -1.1236}, {x → 2.45276}}
```

Check the approximation

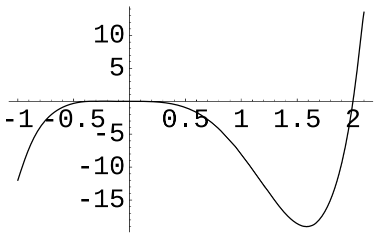
```
f[x] /. roots
```

$\{4.83683\times10^{-10},\ 2.98428\times10^{-12}\}$

- *Monotonity and extremal points*

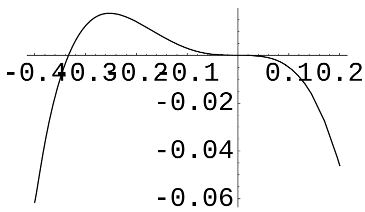
It is not clear from the plot if *f(x)* had two or three local extremal points. The plot of *f(x)* can be enlarged by the mouse. Plot and investigate the derivative:

```
xmin = -1; xmax = 2.1;
Plot[f'[x], {x, xmin, xmax}, PlotRange -> All];
```



Zoom in a neighborhood of the origin:

```
xmin = -0.4; xmax = 0.2;
Plot[f'[x], {x, xmin, xmax}, PlotRange -> All];
```



Find the zeros of the derivative.

```
Factor[f'[x]]
```

$2(-2+x)x^3(1+3x)$

The derivative can be factorized. It can be found from the graph (it can be ckecked also formally) that the derivative changes sign at each zeros.

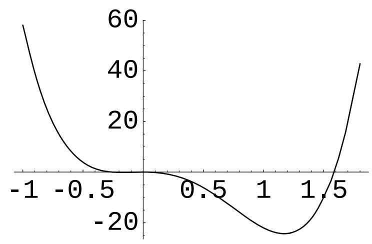
```
droots = x /. Solve[f'[x] == 0, x]
```

$\left\{-\frac{1}{3}, 0, 0, 0, 2\right\}$

- *Convexity and inflection points*

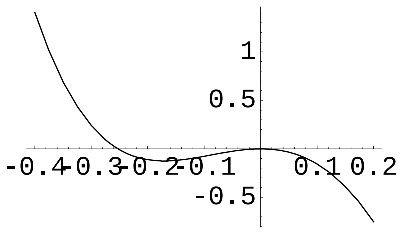
Plot and investigate the second derivative:

```
xmin = -1; xmax = 1.8;
Plot[f''[x], {x, xmin, xmax}, PlotRange -> All];
```



Zoom in a neighborhood of the origin:

```
xmin = -0.4; xmax = 0.2;
Plot[f''[x], {x, xmin, xmax}, PlotRange -> All];
```



This plot is quite good. Find the zeros of the second derivative formally or numerically.

```
Factor[f''[x]]
```

$2x^2(-6-20x+15x^2)$

```
ddrots = x /. Solve[f''[x] == 0, x]
```

$\left\{0, 0, \frac{1}{15}\left(10-\sqrt{190}\right), \frac{1}{15}\left(10+\sqrt{190}\right)\right\}$

The second derivative changes sign at the roots except the origin.

- *Summarize the results in a table.*

Let

$$x_1 = -\frac{1}{3}; \quad x_2 = \frac{1}{15}\left(10 - \sqrt{190}\right); \quad x_3 = 0;$$
$$x_4 = \frac{1}{15}\left(10 + \sqrt{190}\right); \quad x_5 = 2;$$

	$x <$	x_1	$< x <$	x_2	$< x <$	x_3	$< x <$	x_4	$< x <$	x_5	$< x$
$f(x)$	↘ ∪	min.	↗ ∪	infl.	↗ ∩	MAX	↘ ∩	infl.	↘ ∪	min.	↗ ∪
$f'(x)$	− ↗	0	+ ↗	+	+ ↘	0	− ↘	−	− ↗	0	+ ↗
$f''(x)$	+	+	+	0	−	0	−	0	+	+	+

Problem 4.6.4

Investigate the following functions. Use both the classical and the computer-aided methods.

(1) $f(\text{x\_}) := \frac{1}{12} x^2 (3x^2 - 4x - 12);$

(2) $f(\text{x\_}) := \frac{4x^3}{3} - \frac{x^4}{4} - 2x^2 + \pi;$

(3) $f(\text{x\_}) := \frac{x^4}{4} + \frac{x^2}{2} - \frac{2x^3}{3} + 1;$

(4) $f(\text{x\_}) := -(x-2)^2 x^2;$

(5) $f(\text{x\_}) := x\, e^{-x^2};$

(6) $f(\text{x\_}) := x^2 e^{-x};$

(7) $f(\text{x\_}) := e^{-x^2}$

(8) $f(\text{x\_}) := x \log(x);$

(9) $f(\text{x\_}) := x^2 \log(x)$

(10) $f(\text{x\_}) := x^2 \log^2(x)$

(11) $f(\text{x\_}) := \frac{x^2}{x+2};$

(12) $f(\text{x\_}) := \frac{x^2}{x^2-1};$

(13) $f(\text{x\_}) := \frac{1}{12} x^2 (x^2 - 2x - 2);$

(14) $f(\text{x\_}) := -\frac{1}{12} x^2 (x^2 - 4x - 18);$