Content

Problems in Mathematics
&
Experiments with Mathematica

1. A short introduction to *MATHEMATICA*

1.1 Start Mathematica

- **Mathematica for Windows (PC-version)**

- start MS-Windows
- start Mathematica: double-click on the Mathematica icon

- **Mathematica for X-Windows (UNIX version)**

- Switch on the X-terminal, or start the X-windows program
- Start telnet:
  - Log in the UNIX-server (host)
  - Login name: login name <enter>
  - Password: password <enter>
- Start *Mathematica:* *Mathematica* <enter>

1.2 Basic concepts

- **Notations**

Notebook frontend	The graphical user interface
Kernel	The mathematical engine of *Mathematica*
Notebook	*Mathematica* document
Cell	The elementary unit of a notebook, a right bracket denotes it on the right side.
Group	The cells can be groupped, showed by right brackets.

- **Important cell types**

Title, Subtitle, Subsubtitle:	title cells
Section, Subsection, Subsubsection:	sectioning the notebook document
text, smalltext:	text descriptions
Input:	input for the *Mathematica* Kernel
Output:	result of calculations

- **File types**

$*$.m	*Mathematica* program packages
$*$.ma	*Mathematica* Pre – 3.0 notebook source file
$*$.mb	*Mathematica* Pre – 3.0 notebook binary file
$*$.nb	*Mathematica* 3.0 notebook file

1.3 The usage of *Mathematica*

- **Writing texts**

Mathematica has a text editor that is simple to use. The texts and commands are written into cells. The usual Windows "Edit" functions can be used.

- *Writing into a new cell*

Click at the end of the notebook or between two cells. A horizontal line shows the place of the new cell. The text now is written into it. The default cell style is Input. The cell type can be modified by celecting a type described in 1.2.2 from the menu Format→Style.

- *Writing into an existing cell*

Click with the mouse in the cell at the place of the new text or that one to be modified. The text can be written. There is a possibility for structured selection by multiple clicking in the versions 3.x.

- **Execute statements**

- Choose the cell(s) by the mouse
- Press Shift-Enter to perform the commands in the selected input cells

- **Some important menu functions**

Load a notebook	File menu : Open, Open Special
Close a notebook	File menu : Close
Save a notebook	File menu : Save, Save as
Windows editing	Edit menu
Control graphs	Graph menu (Pre – 3.0), Cell menu (3. x)
Kernel functions	Action menu (Pre – 3.0), Kernel menu (3. x)
Help	Help menu

The other functions are described in details in the documentations.

1.4 *Mathematica* as a calculator

• Numerical operations

+	addition
-	subtraction
* *or space*	multiplication
/	division
^	power
!	factorial
()	grouping of operations

• *Examples*

$5^2 + 2$

$3^{1/3}$

• Numerical approximation

$N[\sqrt{2}\ \sqrt{3}]$

2.44949

$N[\pi, 10]$

3.141592654

1.5 Functions, variables and symbolic expressions

• Some constants

E	e: the basis of the natural logarithm
Pi	π
Degree	one degree in radians: $\frac{\pi}{180}$
Infinity	infinity

• Some built-in functions in InputForm

a∗x + b	straight line
a∗x^2 + b∗x + c	parabola
x^a	power
Sqrt[x]	square root
Sin[x]	sine
Cos[x]	cosine
Tan[x]	tangent
Cot[x]	cotangent
ArcSin[x], ArcCos[x], ArcTan[x], ArcCot[x]	inverses of trigonometric functions
a^x	exponential function
Exp[x] = E^x	
Log[x]	natural logarithm
Log[a, x]	logarithm
Abs[x]	absolute value
Max[x, y]	maximum
Min[x, y]	minimum
N[x]	numerical approximation

• Operations with expressions

• *Variables and values*

```
a = Sin[x] + 1
```

```
1 + Sin[x]
```

```
b = 1
```

```
1
```

```
Clear[x];
x = 2.; Sin[x] Cos[x]
```

```
-0.378401
```

```
Clear[x];
x = 1.; 2 x^3 + 5 x + 1
```

```
8.
```

• *Composite functions*

```
Clear[x];
Sin[x^2 + x + 1]
```

```
Sin[1 + x + x^2]
```

```
Sin[x]^5
```

$\text{Sin}[x]^5$

- *Remarks*

Clear[x]	Clears the variable
%	The result of the last evaluated statement; be careful when using it
x	To retrieve the content of the variable x

1.6 Algebraic operations

- **Expand**

```
Clear[x]
Expand[(x - 2) (x - 1) x (x + 1) (x + 2)]
```

$4\,x - 5\,x^3 + x^5$

```
Clear[x, y]
Expand[(-x + 2 y) (-3 x + y) (x + y)]
```

$3\,x^3 - 4\,x^2\,y - 5\,x\,y^2 + 2\,y^3$

- **Factorization**

```
Clear[x, a]
Factor[x^2 - 2 a x + a^2]
```

$(a - x)^2$

```
Clear[x]
Factor[1224 + 1084 x + 170 x^2 - 31 x^3 + x^4]
```

$(-18 + x)\,(-17 + x)\,(2 + x)^2$

- **Simplification**

```
Clear[a]
Simplify[(a^6 - 1)/(a^2 - 1)]
```

$1 + a^2 + a^4$

FullSimplify (version 3.x) usually does better job at the price of running more slowly.

```
Clear[a]
FullSimplify[(a^6 - 1)/(a^2 - 1)]
```

$1 + a^2 + a^4$

1.7 Lists, vectors and arrays

• Simple lists

A list is a collection of any kinds of Mathematica objects.

• *Defining a list*

```
{2, 4, 3}
```

```
{2, 4, 3}
```

• *Operations on lists*

```
{2, 3, 5} - 2
```

```
{0, 1, 3}
```

• *Operations with two lists*

```
{3, 2, 4} - {4, 5, 1}
```

```
{-1, -3, 3}
```

• *Mathematical functions on lists*

See `Listable` functions in the help.

```
Log[{1, 2, 3, 3}]
```

```
{0, Log[2], Log[3], Log[3]}
```

• *Elements*

The numbering of the elements starts at 1:

```
Clear[a];
a = {2, -1, 3};
```

```
{1, 2, 2, -3}[[4]]
```

```
-3
```

```
a[[1]]
```

```
2
```

• Composite lists, matrices, arrays

```
a = {{1, 2}, {-1, 3}, {5, 7}}
```

```
{{1, 2}, {-1, 3}, {5, 7}}
```

• *Elements*

```
a[[1]]
```

```
{1, 2}
```

```
a[[1, 1]]
```

1

```
a[[2, 1]]
```

-1

- **Generating lists**
  - *A simple sine table*

$$\text{Table}\left[\text{Sin}[x], \left\{x, 0, \frac{\pi}{2}, \frac{\pi}{12}\right\}\right]$$

$$\left\{0, \frac{-1+\sqrt{3}}{2\sqrt{2}}, \frac{1}{2}, \frac{1}{\sqrt{2}}, \frac{\sqrt{3}}{2}, \frac{1+\sqrt{3}}{2\sqrt{2}}, 1\right\}$$

  - *The same with the arguments*

$$\text{tbl} = \text{Table}\left[\{x, \text{Sin}[x]\}, \left\{x, 0, \frac{\pi}{2}, \frac{\pi}{12}\right\}\right]$$

$$\left\{\{0, 0\}, \left\{\frac{\pi}{12}, \frac{-1+\sqrt{3}}{2\sqrt{2}}\right\}, \left\{\frac{\pi}{6}, \frac{1}{2}\right\}, \left\{\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right\}, \left\{\frac{\pi}{3}, \frac{\sqrt{3}}{2}\right\}, \left\{\frac{5\pi}{12}, \frac{1+\sqrt{3}}{2\sqrt{2}}\right\}, \left\{\frac{\pi}{2}, 1\right\}\right\}$$

  - *Showing it in a nice form*

```
TableForm[Transpose[tbl], TableSpacing -> {1, 1},
    TableAlignments -> Center,
    TableHeadings -> {{"x", "Sin[x]"}, None}]
```

x	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{5\pi}{12}$	$\frac{\pi}{2}$
Sin[x]	0	$\frac{-1+\sqrt{3}}{2\sqrt{2}}$	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	$\frac{1+\sqrt{3}}{2\sqrt{2}}$	1

- **Summary: braces in Mathematica**

( )	grouping of mathematical operations
[ ]	argument of a function
{ }	list
[[ ]]	referring to elements of a list
(* note *)	comment in an input cell

1.8 2D Graphics

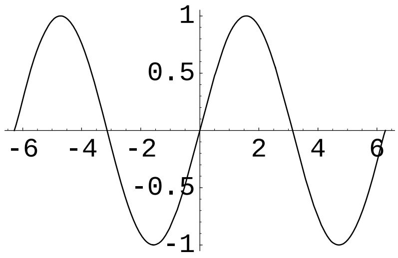
- **Preliminary remark**

Mathemoatica enables the user to read coordinates of the points in the plotted 2D Graphics object. Select the graph, press permanently the `Ctrl` button (or `Alt` in X-Windows systems). Then moving the mouse pointer, the coordinates of the actual point are printed in the left lower corner of the window.

• Plotting the graph of a function of one variable

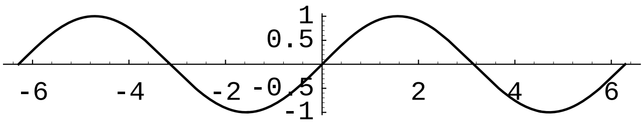
- *Simple plot*

```
Plot[Sin[x], {x, -2 π, 2 π}];
```



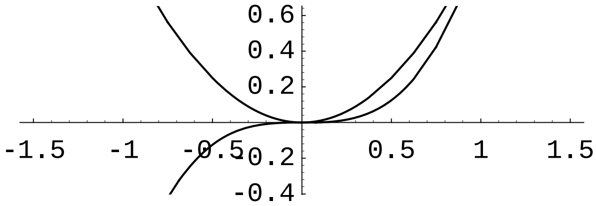
- *With equal scaling for both axes*

```
Plot[Sin[x], {x, -2 π, 2 π}, AspectRatio → Automatic];
```



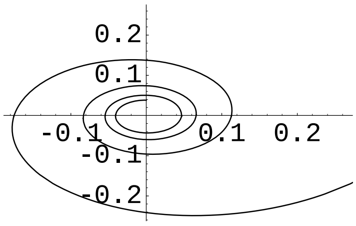
- *Two graphs together*

Plot[{x^2, x^3}, {x, -1.5, 1.5}, AspectRatio → Automatic];



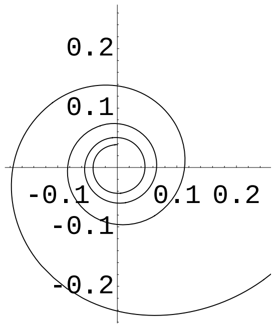
• Parametric curves

ParametricPlot$\left[\left\{\frac{\text{Sin[t]}}{1+\text{t}}, \frac{\text{Cos[t]}}{1+\text{t}}\right\}, \{\text{t}, 0, 8\,\pi\}\right]$;



- *With equal scaling*

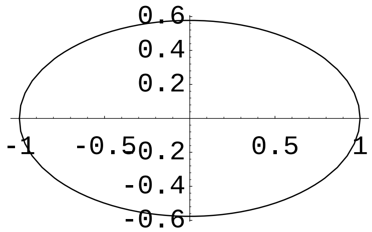
```
Show[%, AspectRatio → Automatic];
```



- **Plotting implicit functions**

```
<< "Graphics`ImplicitPlot`"
```

ImplicitPlot[$x^2 + 3\,y^2 == 1$, {x, -2, 2}];



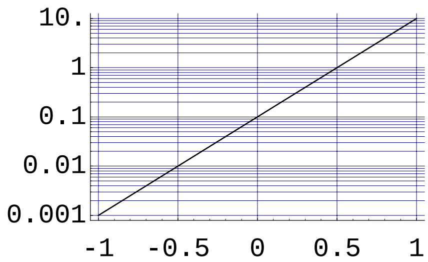
- **Logarithmic plots**

The statements `LogPlot` and `LogLogPlot` are included in the package `Graphics`.

```
<< "Graphics`Graphics`"
```

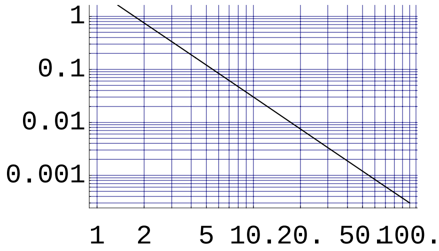
- *Logarithmic plot*

LogPlot[$10^{2\,x-1}$, {x, -1, 1}, GridLines → Automatic];



- *Double logarithmic plot*

```
LogLogPlot[3/x^2, {x, 0.1, 100}, GridLines → Automatic];
```



• Plotting {x,y} points and lists

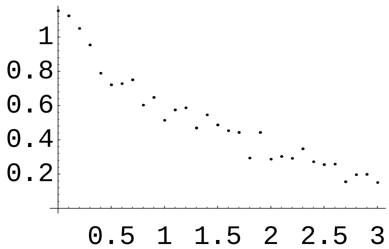
- *Simple plot*

Generate a list:

```
data = Table[{x, 2^-x + 0.2 Random[]}, {x, 0, 3, 0.1}];
```

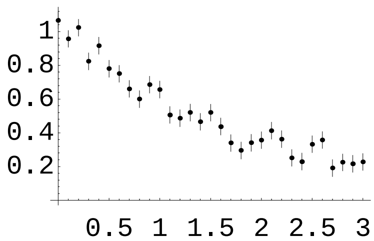
Then, plot it:

```
ListPlot[data];
```



- *Plotting with error*

```
data1 = Table[{x, 2^-x + 0.2 Random[], 0.05}, {x, 0, 3, 0.1}];
ErrorListPlot[data1];
```



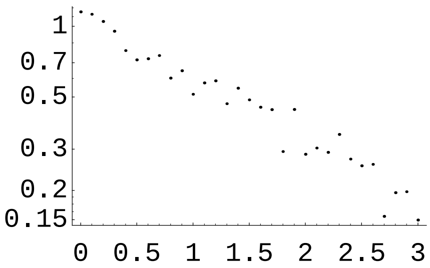
• Logarithmic plot of data

The statements `LogListPlot` and `LogLogListPlot` are included in the package `Graphics`.

```
<< "Graphics`Graphics`"
```

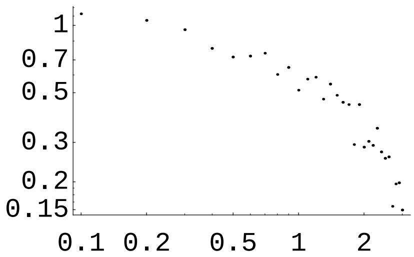
- *Logarithmic plot*

```
LogListPlot[data];
```



- *Double logarithmic plot*

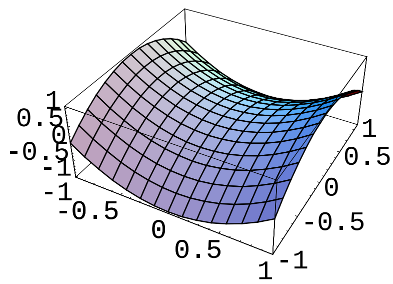
```
LogLogListPlot[data];
```



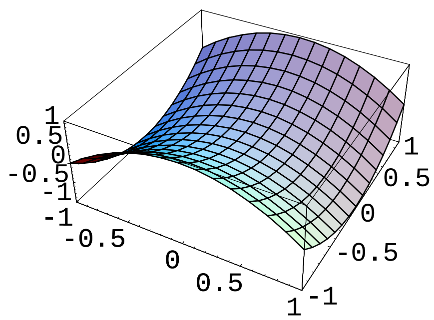
1.9 3D Graphics

- **Graphs of functions of two variables**
  - *Simple plot*

```
d3plt = Plot3D[x x - y y, {x, -1, 1}, {y, -1, 1}];
```

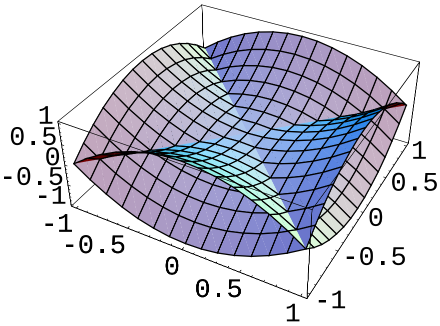



```
d3plt1 = Plot3D[-x x + y y, {x, -1, 1}, {y, -1, 1}];
```



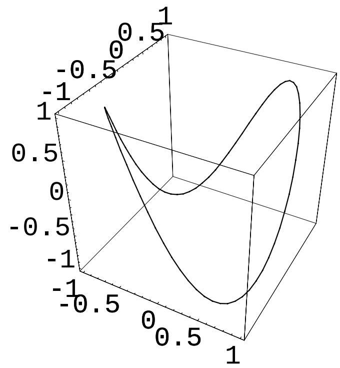
- *Two plots together*

```
Show[d3plt, d3plt1];
```



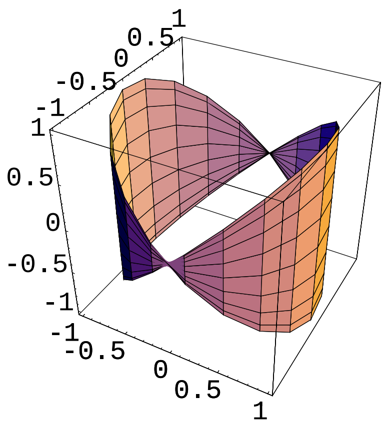
- **Parametric curves**

```
ParametricPlot3D[{Sin[t], Cos[t], Sin[2 t]}, {t, 0, 2 π}];
```



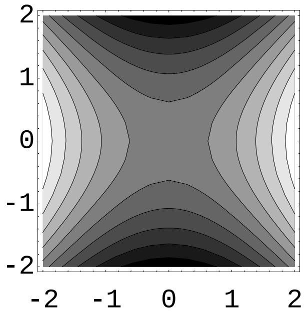
- **Parametric surfaces**

```
ParametricPlot3D[{Sin[u], Cos[u], Sin[u] Cos[v]},
```



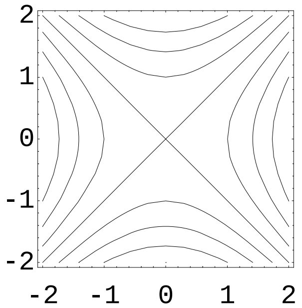
- **Contour plots**

```
ContourPlot[x x - y y, {x, -2, 2}, {y, -2, 2}];
```



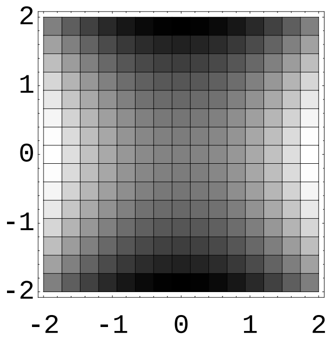
- *With given contour levels*

```
contplt = ContourPlot[x x - y y, {x, -2, 2}, {y, -2, 2},
        ContourShading → False,
        Contours → {-7, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, '
```



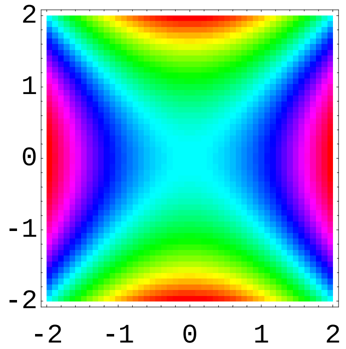
- **A map with color scale**

```
DensityPlot[x x - y y, {x, -2, 2}, {y, -2, 2}];
```



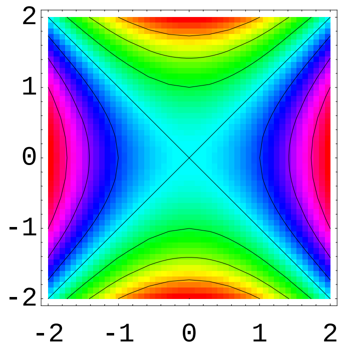
- *The same plot with some specific options*

```
densplt = DensityPlot[x x - y y, {x, -2, 2}, {y, -2, 2},
        PlotPoints → 50, Mesh → False, ColorFunction → Hue];
```



- *ContourPlot and DensityPlot together*

```
Show[densplt, contplt];
```



1.10 Calculus

1.10.1 Frequently used *Mathematica* functions for Calculus

Plot[f, {x, x_0, x_1}, options]	Plots a graph
Table[expr., {x, x_0, x_1, dx}]	Generates tables
Limit[expr., x −> x_0]	Limit
Solve[eqn., x]	Formal solution of (mainly
NSolve[eqn., x]	polynomial) equations
FindRoot[	N[Solve[...]]
eqn., {x, x_0}]	Numerical solution
	of equations
D[f, x]	First derivative by x
D[f, {x, n}]	n' th derivative by x
Integrate[f, x]	Indefinite integral
Integrate[	Definite integral
f, {x, x_0, x_1)]	
NIntegrate[f, {x, x_0, x_1}]	Numerical definite integral
Sum[expr., {x, x_0, x_1, dx}]	Sums
NSum[expr., {x, x_0, x_1, dx}]	Sums with
	numerical approximation
Series[f, {x, x_0, n}]	Taylor polynomial with error
Normal[Series[f, {x, x_0, n}]]	Taylor polynomial without error

1.10.2 Defining functions

- *A function of one variable*

```
f[x_] := x^2 - 3 x + 1
```

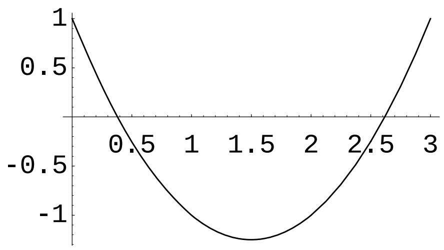
- *Function of several variables*

```
g[x_, y_] := Sin[2 π y x]
```

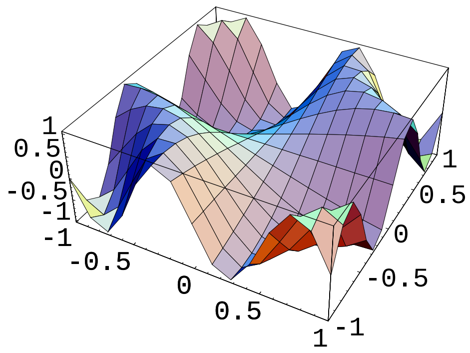
- *Usage of functions*

The evaluation of expressions is very simple by using functions. The user-defined functions can be used as the built-in's.

```
Plot[f[x], {x, 0, 3}];
```



```
Plot3D[g[x, y], {x, -1, 1}, {y, -1, 1}];
```



1.10.3 Solving equations

- *Solve*

`Solve` yields the formal solutions of an equation (if it can do).

```
tsol = Solve[f[x] == 0, x]
```

```
         3 - Sqrt[5]          3 + Sqrt[5]
{{x -> -----------}, {x -> -----------}}
             2                    2
```

The numeric approximation of the solutions is

```
N[tsol]
```

```
{{x -> 0.381966}, {x -> 2.61803}}
```

- *NSolve*

`NSolve` is equivalent to N[Solve[...]] (see the previous point).

```
nsol1 = NSolve[f[x] == 0, {x}]
```

```
{{x -> 0.381966}, {x -> 2.61803}}
```

- *FindRoot*

The equation is solved by Newton iteration or the secant method.

$$\texttt{nsol2 = FindRoot}\left[\texttt{Log[2, x]} == \frac{1}{x}, \{x, 2\}\right]$$

```
{x -> 1.55961}
```

$$\texttt{nsol3 = FindRoot}\left[\texttt{Log[2, x]} == \frac{1}{x}, \{x, \{1, 2\}\}\right]$$

$\{x \to 1.55961\}$

1.10.4 Limits

```
f[x_] := (x + 1)/(x - 1)
Limit[f[x], x → 0]
```

-1

Left-hand side limit at 1:

```
Limit[f[x], x → 1, Direction -> 1]
```

$-\infty$

Right-hand side limit at 1:

```
Limit[f[x], x → 1, Direction -> -1]
```

∞

The limit at ∞:

```
Limit[f[x], x → ∞]
```

1

1.10.5 Formal differentiation

```
f[x_] := Sin[x]/x
```

- *First derivative*

```
D[f[x], x]; ∂x f[x]; f′[x]
```

$$\frac{\mathrm{Cos}[x]}{x} - \frac{\mathrm{Sin}[x]}{x^2}$$

- *Second order derivative*

```
D[f[x], {x, 2}]; ∂{x,2} f[x]; f″[x]
```

$$\frac{-2\ \mathrm{Cos}[x]}{x^2} + \frac{2\ \mathrm{Sin}[x]}{x^3} - \frac{\mathrm{Sin}[x]}{x}$$

- *Higher order derivatives*

```
D[f[x], {x, 3}]; ∂{x,3} f[x]
```

$$\frac{6\,\mathrm{Cos}[x]}{x^3} - \frac{\mathrm{Cos}[x]}{x} - \frac{6\,\mathrm{Sin}[x]}{x^4} + \frac{3\,\mathrm{Sin}[x]}{x^2}$$

```
D[f[x], {x, 5}]; ∂{x,5} f[x]
```

$$\frac{120\,\text{Cos}[x]}{x^5}-\frac{20\,\text{Cos}[x]}{x^3}+\frac{\text{Cos}[x]}{x}-\frac{120\,\text{Sin}[x]}{x^6}+\frac{60\,\text{Sin}[x]}{x^4}-\frac{5\,\text{Sin}[x]}{x^2}$$

1.10.6 Integration

- *Antiderivative*

```
Integrate[Sin[x] * Cos[x], x]; ∫Sin[x] Cos[x] ⅆx
```

$$-\frac{1}{2}\,\text{Cos}[x]^2$$

```
Integrate[1 / Sqrt[1 - x^4], x]; ∫ 1/Sqrt[1 - x^4] ⅆx
```

$$\int \frac{1}{\sqrt{1-x^4}}\,dx$$

```
EllipticF[ArcSin[x], -1]
```

- *Definite integral*

```
Integrate[Sin[x] * Cos[x], {x, 0, Pi / 2}]; ∫_0^(π/2) Sin[x] Cos[x] ⅆx
```

$$\int_0^{\frac{\pi}{2}} \text{Sin}[x]\,\text{Cos}[x]\,dx$$

$$\frac{1}{2}$$

- *Numerical integration*

```
NIntegrate[Sin[x] Cos[x], {x, 0, π/2}]
```

```
0.5
```

```
NIntegrate[1/Sqrt[1 - x^3], {x, 0, 1}]
```

```
1.40218
```

```
NIntegrate[1/(x x), {x, 1, ∞}]
```

```
1.
```

1.11 Series

1.11.1 Taylor series around a given point

```
f[x_] := Sin[x]
```

- *With error*

```
Series[f[x], {x, 0, 10}]
```

$$1 - \frac{x^2}{6} + \frac{x^4}{120} - \frac{x^6}{5040} + \frac{x^8}{362880} - \frac{x^{10}}{39916800} + O[x]^{11}$$

- *Without error*

```
Normal[Series[f[x], {x, 0, 10}]]
```

$$1 - \frac{x^2}{6} + \frac{x^4}{120} - \frac{x^6}{5040} + \frac{x^8}{362880} - \frac{x^{10}}{39916800}$$

1.12 Fitting by the method of least squares

1.12.1 Fitting with linear combination of functions

```
data = Table[{x, Sin[x] + Random[Real, {-0.2, 0.2}]},
{x, 0, 3, 0.1}];
sqfit = Fit[data, {1, x, x^2}, {x}]
```

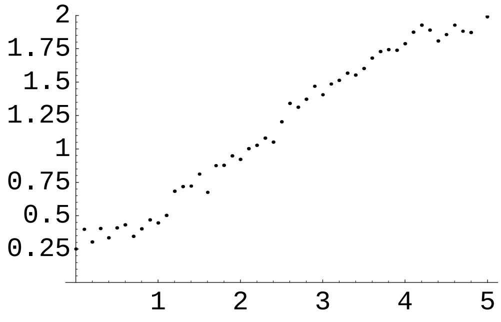
$$-0.102853 + 1.3404\ x - 0.424273\ x^2$$

1.12.2 Nonlinear fitting

```
<< Statistics`NonlinearFit`
```

```
data = Table[
  {t, (2 Exp[t])/(10 + Exp[t]) + Random[Real, {0, 0.2}, 3]}, {t, 0, 5, 0.1}];
```

```
pltdata = ListPlot[data, PlotRange -> {0, 2}];
```



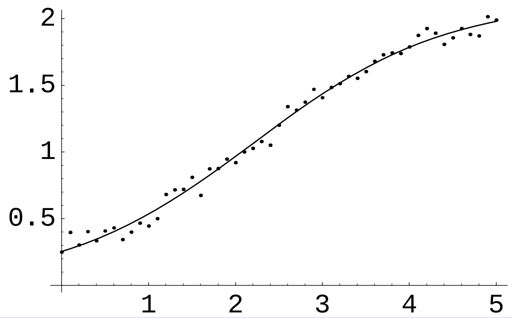

```
model = (A Exp[k A t])/((A-X0)/X0 + Exp[k A t])
```

$$\frac{A\, E^{A\,k\,t}}{E^{A\,k\,t} + \frac{A - x_0}{x_0}}$$

```
fitfv = NonlinearFit[data, model, {t},
  {{k, 1.2}, {A, 0.5}, {x0, 0.1}}, AccuracyGoal → 15]
```

$$\frac{2.13731\,E^{0.906252\,t}}{7.41672 + E^{0.906252\,t}}$$

```
pltf = Plot[fitfv, {t, 0, 5}, DisplayFunction -> Identity];
Show[pltf, pltdata, DisplayFunction -> $DisplayFunction];
```



1.13 Differential equations

1.13.1 Command summary

PlotVectorField[...]	Plots 2 D vectorfields : package Graphics`PlotField`
PlotVectorField3D[..]	Plots 3 D vectorfields : package Graphics`PlotField3D`
DSolve[...]	Solves ODE's formally
NDSolve[...]	Solves ODE's numerically

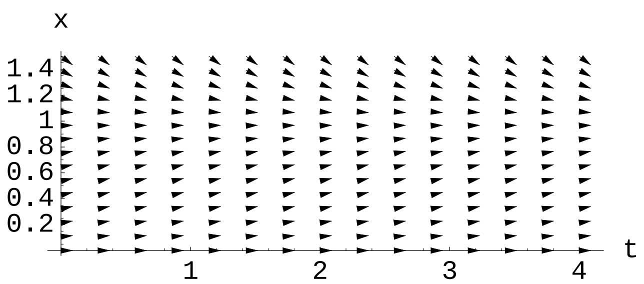
1.13.2 Plot direction field and vectorfield for an ODE

- *Direction field of 1D equations*

```
eqn1 = x'[t] == x[t] (1 - x[t]);
field = {1, x (1 - x)};
```

```
Needs["Graphics`PlotField`"];
```

```
PlotVectorField[field, {t, 0, 4},
  {x, 0, 1.5}, Axes -> True, AxesLabel -> {t, x}];
```

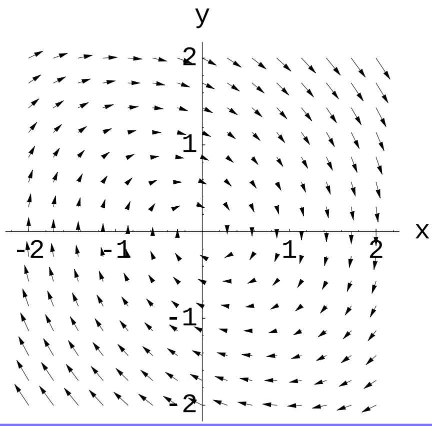


- *Vectorfield of 2D equations*

```
eqn2 = {x'[t] == y[t], y'[t] == -x[t] - y[t] / 2};
field2 = {y, -x - y / 2};
```

```
Needs["Graphics`PlotField`"];
```

```
PlotVectorField[field2, {x, -2, 2},
  {y, -2, 2}, Axes -> True, AxesLabel -> {x, y}];
```



1.13.3 Formal solution

- *Finding the general solution*

Give the equation:

```
eqn = x'[t] == a x[t] (1 - x[t]);
```

Solve it:

```
sol = DSolve[eqn, x[t], t]
```

$$\left\{\left\{x[t] \to \frac{E^{a\,t}}{E^{a\,t} - E^{C[1]}}\right\}\right\}$$

The general solution is given as a substitution rule. The initial condition will determine the integration constant C[1].

- *Solving an initial value problem*

```
t0 = 0;
x0 = 0.2;
```

```
sol = DSolve[{eqn, x[t0] == x0}, x[t], t]
```

$$\{\{x[t] \to 2.71828182845905^{a\,t} / (4.00000000000000 + 2.71828182845905^{a\,t})\,\text{^}\,1.00000000000000\}\}$$

1.13.4 Numerical solution

Give the equation and the initial conditions:

```
t0 = 0; T = 4;
x0 = 0.2;
eqn = x'[t] == x[t] (1 - x[t]);
```

Solve the initial value problem numerically:

```
NDSolve[{eqn, x[t0] == x0}, x[t], {t, t0, T}]
```

```
{{x[t] → InterpolatingFunction[{{0., 4.}}, <>][t]}}
```

The solution is given by an `InterpolatingFunction` object in a substitution rule.