Content

Problems in Mathematics
&
Experiments with Mathematica

3. Limit and continuity

3.1 Definitions and graphic meaning

The Intuitive definition of limit

DEFINITION 3.1.1 Intuitive definition

We say that the limit of *f(x)* at *a* is *L*, in notation

$$\lim_{x\to a} f(x) = L,$$

if the values of *f(x)* are as close to *L* as we desire for all $x \neq a$ sufficiently close to *a*.

- **Remark**

This definition does not assumes that *f(x)* is defined at *a*.
The place *a* and the value *L* can be either finite numbers or $\pm\infty$. For a number *x*, the meaning of being close to a finite number is obvious ($|x-a|$ is small), and to be close to ∞ means that the number is very big.

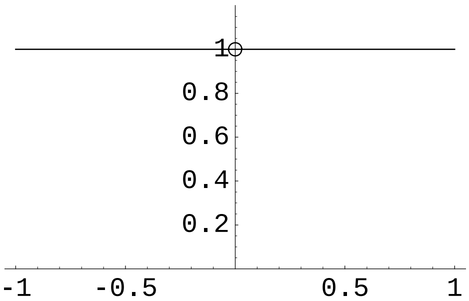
- **Remark**

We do not give hire the formal definitions of the special cases $\lim_{x\to a} f(x) = L$, $\lim_{x\to\infty} f(x) = L$, $\lim_{x\to a} f(x) = \infty$, $\lim_{x\to\infty} f(x) = \infty$. The reader find them in any Calculus textbook. Instead, let us consider some graphical examples.

- **Examples. Limit at a finite point**

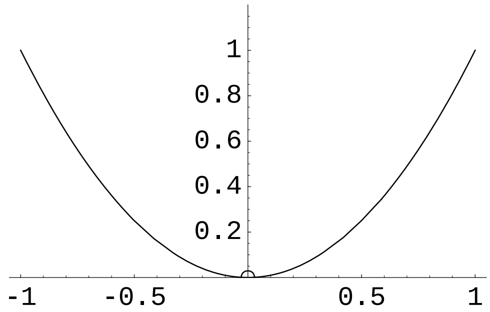
The following examples are almost obvious. Remember the graphs and the poperties of the power functions in the section Powers.

(1) $\lim_{x\to 0} \dfrac{x}{x} = 1.$



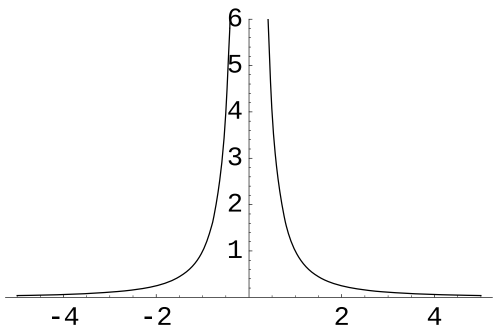
The function $\frac{x}{x}$ is not defined at zero, but $\frac{x}{x} \equiv 1$ for $x \neq 0$.

(2) $\lim_{x\to 0} \frac{x^3}{x}=0$.



The function $\frac{x^3}{x}$ is not defined at zero, but $\frac{x^3}{x} \equiv x^2$ for $x\neq 0$, and it obviously gets close to zero as the argument is sufficiently close to zero.

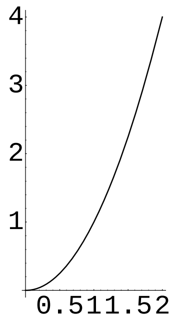
(3) $\lim_{x\to 0} \frac{1}{x^2} = \infty$.



The function $\frac{1}{x^2}$ is not defined at zero, and the values of the function are close to ∞ for both $x>0$ and $x<0$, provided x is close to zero.

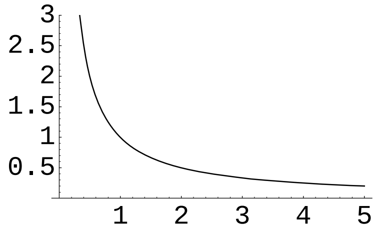
• Examples. Limits at infinity

(1) $\lim_{x\to\infty} x^2 = \infty$.



It is obvious that the values of the function x^2 are close to ∞ as x is close to ∞.

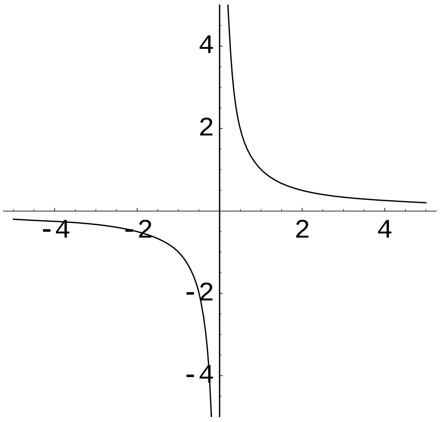
(2) $\lim_{x\to\infty} \frac{1}{x} = 0$.



It is clear that the closer is x to ∞, the closer is $\frac{1}{x}$ to 0.

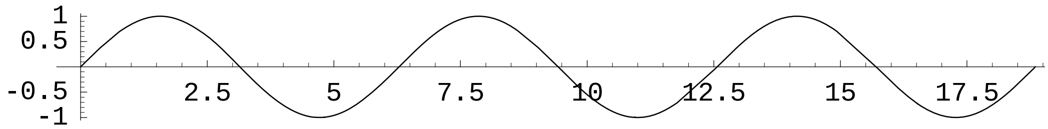
- **Counterexamples.**

(1) $\lim_{x\to 0} \frac{1}{x}$ does not exists.



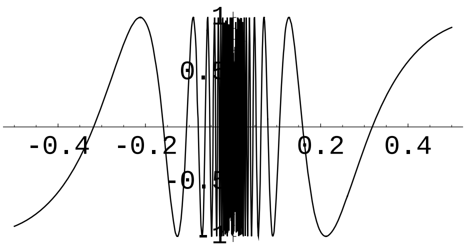
The function $\frac{1}{x}$ is not defined at zero, and the values of the function get close to ∞ for $x>0$, and close to $-\infty$ for $x<0$ as x is close to zero.

(2) $\lim_{x\to\infty} \sin(x)$ does not exists.



The function $\sin(x)$ can take every value from the interval [-1,1] arbitrarily large places on $[0, \infty)$. For example, $\sin(n\pi)=0$, but $\sin(n\pi+\frac{\pi}{2})=1$ for n=1,2,... Consequently, $\sin(x)$ cannot become definitely close to any number, as x is close to ∞.

(3) $\lim_{x\to 0} \sin(\frac{1}{x})$ does not exists.



The explanation is the same as at the previous example: $\sin(\frac{1}{n\pi})=0$, but $\sin(\frac{1}{n\pi+\frac{\pi}{2}})=1$ for n=1,2,...

Half-side limits

We saw that $\lim_{x\to 0} \frac{1}{x}$ does not exists, but if $x>0$, then the limit value can be well-defined.

DEFINITION 3.1.2 Left-hand side limit

We say that the left-hand side limit of $f(x)$ at a is L, in notation

$$\lim_{x\to a-0} f(x) = \lim_{x\to a^-} f(x) = L,$$

if the values of $f(x)$ are as close to L as we desire for all $x<a$ sufficiently close to a.

DEFINITION 3.1.3 Right-hand side limit

We say that the right-hand side limit of $f(x)$ at a is L, in notation

$$\lim_{x \to a+0} f(x) = \lim_{x \to a^+} f(x) = L,$$

if the values of *f(x)* are as close to L as we desire for all $a<x$ sufficiently close to a.

- **Remark**

The limit at ∞ is a left – hand side limit, and the limit at $-\infty$ is a right – hand side one.

- **Examples**

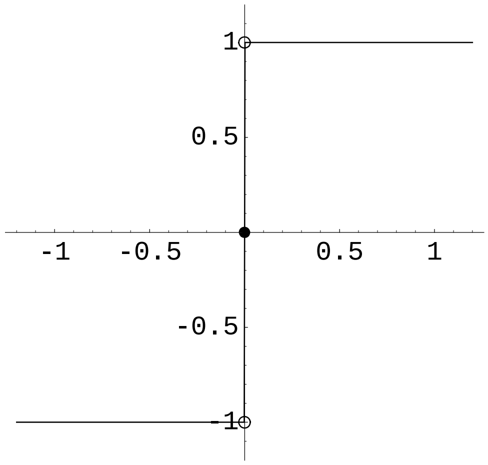
(1) It is obvious that $\lim\limits_{x \to 0-0} \frac{1}{x} = -\infty$ and $\lim\limits_{x \to 0+0} \frac{1}{x} = \infty$.

(2) The limit $\lim\limits_{x \to 0} \sin\left(\frac{1}{x}\right)$ does not exists from neither side.

Consider the function

$$f(x) = \text{sign}(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$$

It is clear that $\lim_{x \to 0-0} f(x) = -1$ and $\lim_{x \to 0-0} f(x) = 1$.



Basic properties

THEOREM 3.1.4 Uniqueness of the limit

If the limit exists at some point, then it is unique, that is, if

$$\lim_{x \to a} f(x) = L_1 \text{ and } \lim_{x :\to a} f(x) = L_2, \text{ then } L_1 = L_2.$$

Continuity

DEFINITION 3.1.5

Let the function *f(x)* be defined at and in a neighborhood of x_0. We say that is *f(x)* is continuous at x_0 if

$$\lim_{x \to x_0} f(x) = f(x_0).$$

The function is continuous from the left at x_0 if

$$\lim_{x \to x_0 - 0} f(x) = f(x_0).$$

The function is continuous from the right at x_0 if

$$\lim_{x \to x_0 + 0} f(x) = f(x_0).$$

The function *f(x)* is continuous on an interval (a, b) if it is continuous at every point $x_0 \in (a, b)$. The function is continuous on an interval $[a, b]$ if it is continuous at every point $x_0 \in (a, b)$, and it is continuous from the left and right at a and b, respectively.

- **"Graphic" meaning**

If the function is continuous on an interval, then it's graph is a "continuous" curve, that is, it can be plotted without lifting up the pencil of the paper.

Mathematica Summary

InputForm	Trad.Form	Meaning
`Limit[f[x], x -> a]`	$\lim_{x\to a} f(x)$	limit of $f(x)$ at a
`Limit[f[x], x -> a, Direction -> -1]`	$\lim_{x\to a^+} f(x)$	right – hand side limit
`Limit[f[x], x -> a, Direction -> 1]`	$\lim_{x\to a^-} f(x)$	left – hand side limit

- **Important Remark.**

Although the limit

```
Limit[f[x], x -> a]
```

may not exists, *Mathematica* gives a result. In this case, by default, the right-side limit, that is

```
Limit[f[x], x -> a, Direction -> -1]
```

is taken. For example:

```
Limit[1/x, x -> 0]
```

∞

On the other hand,

```
Limit[1/x, x -> 0, Direction -> 1]
```

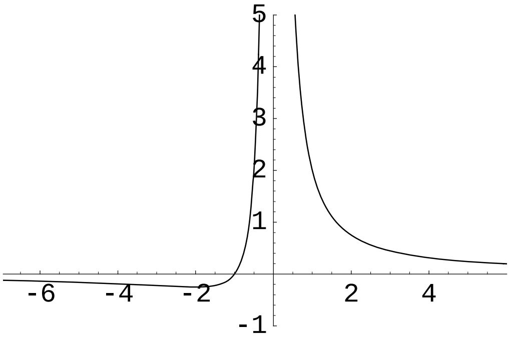
$-\infty$

Exercises and problems

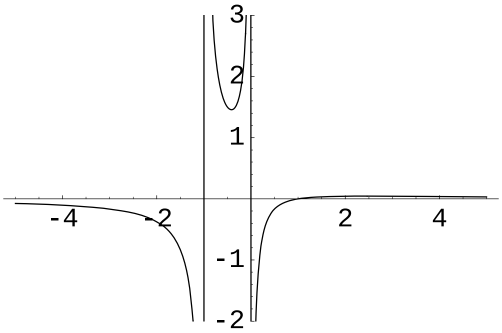
PROBLEM 3.1.1

Knowing the graph of the following functions, intvestigate their continuity. Estimate (graphically) the limits at the break points and at $\pm\infty$.

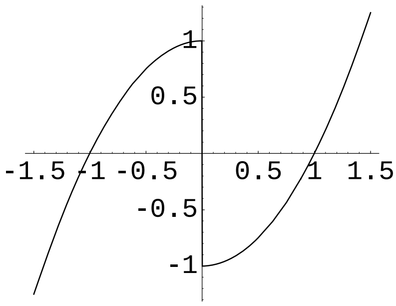
(1)



(2)



(3)



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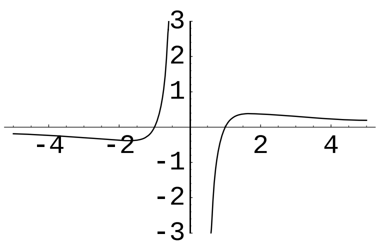
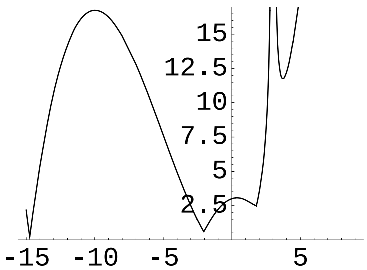
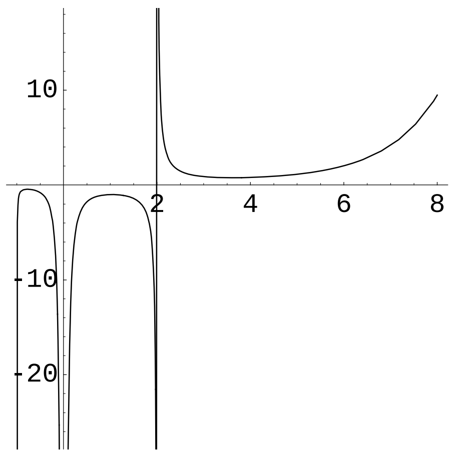


(5)



(6)



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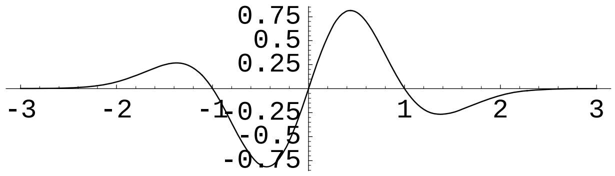
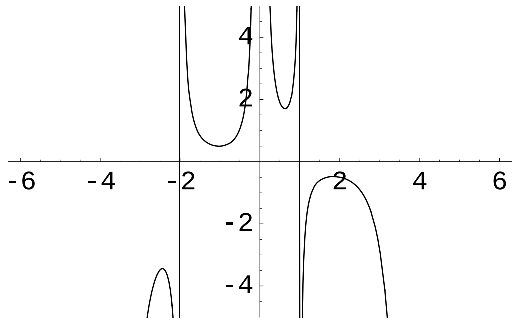


(8)



EXPERIMENT 3.1.2

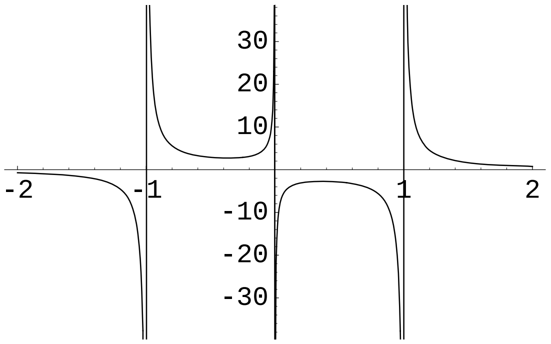
Take different sequences, approximating a, and show that the following limits do or do not exist. Such approximation helps to estimate the speed at which the function approximates the limit value. Using the basic theorems or the rules in the next sections (after working them out), prove formally your numerically obtained conjectures.

(1) $\lim\limits_{x\to 0} \dfrac{1}{x\log(|x|)}$; $\lim\limits_{x\to 1} \dfrac{1}{x\log(|x|)}$;

○ SOLUTION

First, plot the function

$$\mathrm{Plot}\left[\frac{1}{x\log(|x|)}, \{x, -2, 2\}\right];$$



Consider the case at zero. Take the points $x_n = \frac{1}{10^n}$, and $y_n = -\frac{1}{10^n}$.

The values of the function at x_n are

```
TableForm[Table[{1/10^n, N[1/(1/10^n Log[1/10^n])]}, {n, 2, 10}],
 TableHeadings -> {None, {"x_n", "f(x_n)"}}]
```

x_n	$f(x_n)$
$\frac{1}{100}$	-21.7147
$\frac{1}{1000}$	-144.765
$\frac{1}{10000}$	-1085.74
$\frac{1}{100000}$	-8685.89
$\frac{1}{1000000}$	-72382.4
$\frac{1}{10000000}$	-620421.
$\frac{1}{100000000}$	-5.42868×10^6
$\frac{1}{1000000000}$	-4.82549×10^7
$\frac{1}{10000000000}$	-4.34294×10^8

It suggests that the right-side limit is -∞, what can be prove formally by using the L'Hospital rule. Since $y_n = -x_n$, we have that left-side limit is ∞.

Consider the case at $a = 1$. Take the points $x_n = \frac{n+1}{n}$ from the right and $y_n = \frac{n}{1+n}$ from the left.

The values of the function at x_n are

```
TableForm[
 Table[{(n + 1)/n, N[1/((n + 1)/n Log[Abs[(n + 1)/n]])]}, {n, 2, 10}],
 TableHeadings -> {None, {"x_n", "f(x_n)"}}]
```

x_n	$f(x_n)$
$\frac{3}{2}$	1.6442
$\frac{4}{3}$	2.60704
$\frac{5}{4}$	3.58514
$\frac{6}{5}$	4.57068
$\frac{7}{6}$	5.56042
$\frac{8}{7}$	6.55277
$\frac{9}{8}$	7.54683
$\frac{10}{9}$	8.5421
$\frac{11}{10}$	9.53824

It confirms that the right-side limit is -∞. Since $y_n = \frac{1}{x_n}$ and $\log\left(\frac{1}{x_n}\right) = -\log(x_n)$, we have that left-side limit is -∞.

(2) $\lim_{x\to 0} x \log(|x|)$;

(3) $\lim_{x\to\infty} \frac{\log(|x|)}{x}$;

(4) $\lim_{x\to 0} \frac{\sin(x)}{x}$;

(5) $\lim_{x\to\infty} \frac{\sin(x)}{x}$

(6) $\lim_{x\to -\frac{\pi}{2}} \frac{\mathrm{Cot}(x)}{x}$

(7) $\lim_{x\to 0} \frac{e^x}{x}$

(8) $\lim_{x\to\infty} \frac{e^x}{x}$

(9) $\lim_{x\to -\infty} \frac{e^x}{x}$