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Problems in Mathematics
&
Experiments with Mathematica

4. The derivative and its applications

4.7 Taylor polynomials, Taylor series

Theory

Let *f(x)* be defined and be n-times differentiable at and around the place x_0. Then the n-th order Taylor polynomial $T_n(x)$ for $f(x)$ and x_0 is the best approximating n'th order polynomial at x_0. Formally,

$$f^{(i)}(x_0) = T_n^{(i)}(x_0) \qquad (i = 0, \ \ldots, n)$$

$$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(x_0)}{i!} (x - x_0)^i$$

If *f(x)* is (n+1) times differentiable on some $[x_0-\epsilon,\ x_0+\epsilon]$, then the error can be estimated by

$$| T_n(x) - f(x) | \le \underset{u \in [x_0-\epsilon, x_0+\epsilon]}{\mathrm{Max}} \left(\frac{|f^{(n+1)}(u)|}{((n+1)!)} \right) | x - x_0 |^{n+1}$$

Mathematica summary

The used *Mathematica* statements:

`Series[f[x], {x, x0, n}]`	Taylor polynomial with error term
`Normal[Series[f[x], {x, x0, n}]]`	Taylor polynomial without the error term

Exercises and problems

Mathematica initialization

```
Taylor[x_, x0_, n_] :=
 Normal[Series[f[u], {u, x0, n}]] /. u → x
FMax[ff_, {x_, x0_, x1_, dx_: 0.1}] :=
  Max[Table[N[ff], {x, x0, x1, dx}]];
```

$$\text{Error}[\text{ff\_}, \text{x0\_}, \text{x1\_}] := \frac{1}{(n+1)!}\left(\text{FMax}[\text{Evaluate}[\text{Abs}[\partial_{\{x,n+1\}}\text{ff}]], \{x, x0, x1\}]\,(x1 - x0)^{n+1}\right)$$

SOLVED PROBLEM 4.7.1

Find the n-th order Taylor polynomial of the function *f(x)* at x_0. Plot the graphs and calculate the error of the approximation at x_1 .

$$f(\text{x\_}) := \sin(x);\quad x_0 = 0;\quad x_1 = 1;\quad n = 5;$$

○ SOLUTION

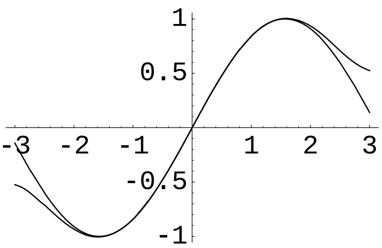
- *The Taylor polynomial at x_0*

```
Taylor[x, x0, n]
```

$$\frac{x^5}{120} - \frac{x^3}{6} + x$$

- *Graphs*

```
Plot[Evaluate[{f[x], Taylor[x, x0, n]}], {x, -3, 3}];
```



- *Estimating the error*

```
Error[f[x], x0, x1]
```

0.00116871

- *The theoretical error*

```
N[Abs[f[x1] - Taylor[x1, x0, n]]]
```

0.000195682

PROBLEM 4.7.2

Find the n-th order Taylor polynomial of the function *f(x)* at x_0. Plot the graph of the functions and calculate the error of the approximation at x_1 .

(1) $f(x\_) := \sin(x);\ x_0 = \frac{\pi}{2};\ x_1 = 1;\ n = 5;$

(2) $f(x\_) := \sin(x);\ x_0 = \frac{\pi}{4};\ x_1 = 1;\ n = 5;$

(3) $f(x\_) := \cos(x);\ x_0 = 0;\ x_1 = 1;\ n = 5;$

(4) $f(x\_) := e^x;\ x_0 = 0;\ x_1 = 1;\ n = 5;$

(5) $f(x\_) := \log(x+1);\ x_0 = 0;\ x_1 = 1;\ n = 5;$

(6) $f(x\_) := \log^2(x+1);\ x_0 = 0;\ x_1 = 1;\ n = 5;$

(7) $f(x\_) := x\log(x+1);\ x_0 = 0;\ x_1 = 1;\ n = 5;$

(8) $f(x\_) := \frac{1}{x+1};\ x_0 = 0;\ x_1 = 1;\ n = 5;$

(9) $f(x\_) := x\sqrt{x+1};\ x_0 = 0;\ x_1 = 1;\ n = 5;$

(10) $f(x\_) := e^{-x^2};\ x_0 = 0;\ x_1 = 1;\ n = 3;$

(11) $f(x\_) := \sqrt{x^2+1};\ x_0 = 1;\ x_1 = 2;\ n = 4;$

(12) $f(x\_) := e^{\cos(x)};\ x_0 = \frac{\pi}{2};\ x_1 = 1.5;\ n = 3;$

(13) $f(x\_) := \log(\log(x+e));\ x_0 = 0;\ x_1 = 1.;\ n = 3;$