Content

Problems in Mathematics
&
Experiments with Mathematica

Elementary functions

Sets, numbers, Cartesian systems

Sets

- **Definition of a set**

A set is a well-defined set of objects. The definition can be given by enrolling the elements or giving the algorithm that decides if the given object belongs to the set or not. The objects belonging to a given set are called elements. The sets are denoted by capital letters, $\{a_1, a_2, \ldots\}$ or $\{a\text{: rule}\}$.

$a \in$ A denotes that the object "a" is an element of the set "A".

Φ denotes the empty set that contains no element.

U denotes the universe or basic set, i.e., the set that contains every set taken into consideration.

$A \subset B$ means that A is a subset of B, i.e., $a \in A$ implies $a \in B$.

- **Operations on sets**

complement: $\neg A = \overline{A} := \{a\colon a \notin A\}$

divison: $A \setminus B := \{a\colon a \in A,\ a \notin B\}$

union: $A \bigcup B := \{a\colon a \in A \text{ or } a \in B\}$

intersection: $A \bigcap B := \{a\colon a \in A \text{ and } a \in B\}$

- **Basic properties**

$A \bigcap \Phi = \Phi$; $A \bigcap U = A$; $A \bigcup \Phi = A$; $A \bigcup U = U$

$A \subset B \Longrightarrow A \bigcap B = A$ and $A \bigcup B = B$.

$A \setminus B = A \setminus (A \bigcap B)$

$A \setminus B \bigcap (A \bigcap B) = \Phi$ and $A \setminus B \bigcup (A \bigcap B) = A$

$A \bigcap (B \bigcup C) = (A \bigcap B) \bigcup (A \bigcap C)$ and $A \bigcup (B \bigcap C) = (A \bigcup B) \bigcap (A \bigcup C)$

$\overline{\Phi} = U$, $\overline{U} = \Phi$, $\overline{A \bigcup B} = \overline{A} \bigcap \overline{B}$, $\overline{A \bigcap B} = \overline{A} \bigcup \overline{B}$.

Numbers

• Definitions

The set

$$N=\{1, 2, 3, 4, ...\}$$

is called the set of *natural numbers*.

The set of *integers* is

$$I=\{0, \pm 1, \pm 2, \pm 3, \pm 4, ...\}$$

The *rational numbers* are the fractions of form $\frac{p}{q}$,
where p is the numerator and $q \neq 0$ is the denominator:

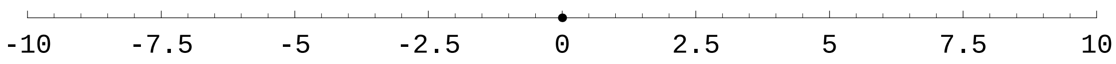
$$Q = \left\{\frac{p}{q} : p, q \in I, q \neq 0\right\}$$

The *irrational numbers* are the numbers that cannot be written in the form of fractions. Such numbers are $\sqrt{2}$, π, e.

The set of *real numbers* $\mathbb{R}$ is the union of rational and irrational numbers.

• The real line

There is a one-to-one correspondence between the real numbers and the points of a straight line. Fix a point on the line that is associated with zero. Fix another point on the right-hand side that is associated with one. The middle point of this obtained segment represents $\frac{1}{2}$. Then, divide the segment into three equal part to obtain the points $\frac{1}{3}$, $\frac{2}{3}$. Continuing this process, we obtain the following uniformly scaled *real line*:



• Inequalities

We say that $a<b$ $(a \leq b)$ if $b-a$ is positive (nonnegative). On the real line, b is to the right from a.

• *Properties*

$a<b$ and $b<c$ imply $a<c$ (transitivity).

If $a<b$, then $b<a$ is not true.

If $a \leq b$ and $b \leq a$, then $a=b$.

If $a<b$, then $a+c<b+c$ for any $c \in \mathbb{R}$.

If $a>0$ and $b>0$, then $a*b>0$. If $a<0$ and $b<0$, then $a*b>0$.

If $a>0$, then $-a<0$.

If $c>0$ and $a<b$, then $c*a<c*b$.

If $c<0$ and $a<b$, then $c*a>c*b$.

If $0 < a < b$, then $0 < \frac{1}{b} < \frac{1}{a}$.

• Intervals

$[a,b]:=\{x: a \le x \le b\}$ closed interval

$[a,b):=\{x: a \le x < b\}$ closed from left and open form right

$(a,b]:=\{x: a < x \le b\}$ open from left and closed from right

$(a,b):=\{x: a < x < b\}$ open interval

$<a,b>$ any of the above cases

$(-\infty, a]:=\{x: x \le a\}$, $(-\infty, a):=\{x: x < a\}$

$[a,\infty) :=\{x: x \ge a\}$, $(a,\infty) :=\{x: x > a\}$

• Neighborhoods

The neighborhoods of a real number a are open sets that contain this number. Closed neighborhoods are closed intervals containing a inside.
For example, if $a=1$, then the interval $(0.9, 1.1)$ is a neighborhood, $[0.8, 1.1]$ is a closed neigborhood, but $[1, 1.1]$ is not a neighborhood.

• Absolute value

The absolute value of a real number a is the distance from zero. Formally,

$$|a| := \begin{cases} a & \text{if } a \ge 0 \\ -a & \text{if } a < 0 \end{cases}$$

• *Properties*

$|a| \ge 0$, $|a| = 0 \iff a=0$

$|a+b| \le |a|+|b|$

$||a|-|b|| \le |a-b|$

$|a-b| \le |a-c|+|c-b|$ *(triangle inequality)*

Cartesian product of sets
Cartesian systems

• Definition: Cartesian product of sets

Let the sets A and B be given. The Cartesian product $A \times B$ is the set of ordered pairs (a,b), where $a \in A$ and $b \in B$. The notation is

$$A \times B := \{(a,b): a \in A, b \in B\}.$$

Analogously,

$$A \times B \times C := \{(a,b,c): a \in A, b \in B, c \in C\},$$

and so on.

- *Example 1*

Let A=*{1,2}* and B=*{3,4}*. Then A×B=*{(1,3), (1,4), (2,3), (2,4)}*.

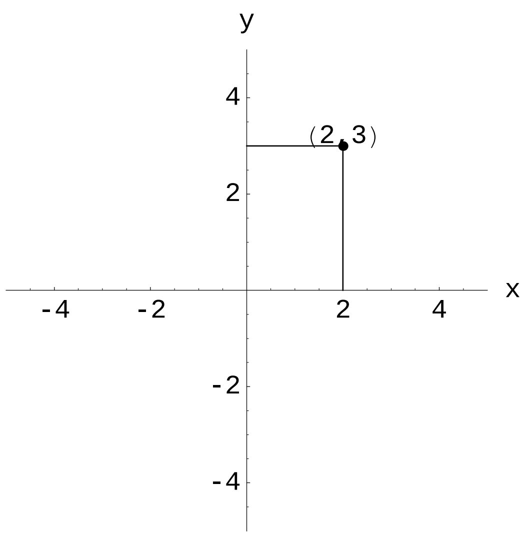
• Cartesian systems

By the definition

$$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} := \{(x, y) : x, y \in \mathbb{R}\}.$$

The planar Cartesian system consists of two perpendicular real lines with common origin. The first component (x-coordinate) of $(x,y) \in \mathbb{R}^2$ is measured on the first line (clockwise taken), and the second component (y-coordinate) is measured on the second line. It is obvious that every point on the plane can be described uniquely by an element of $\mathbb{R}^2$, and the plane itself can be considred as $\mathbb{R}^2$.

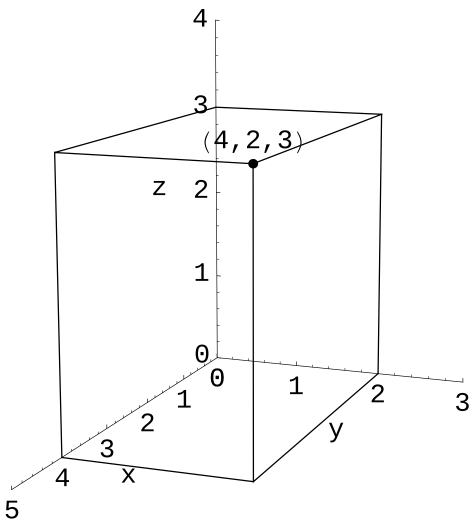
The point *(2,3)* in the *xy*-plane is plotted as follows:



Similarly,

$$\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R} := \{(x, y, z) : x, y, z \in \mathbb{R}\}.$$

The point *(4,2,3)* is plotted in the three dimensional *xyz*- space as follows:



For an arbitrary n

$$\mathbb{R}^n = \underbrace{\mathbb{R} \times \mathbb{R} \times \ldots \times \mathbb{R}}_{\text{n times}} := \{(x_1,\ x_2,\ \ldots,\ x_n) : x_i \in \mathbb{R}\},$$

Exercises

Here, the reader can find only some exercises. For more details, see any textbook on College Algebra.

PROBLEM 2.1.1

Find the union, intersection, difference and complement of A and B, if U is the universe set.

(1) $A = \{1, 2, 4, 7\};\ B = \{1, 3, 4, 5, 6\};\ U = \mathbb{N};$

(2) $A = [0, 1);\ B = [0.2, 3];\ U = \mathbb{R};$

(3) $A = (2, \infty);\ B = (-\infty, 1);\ U = \mathbb{R};$

PROBLEM 2.1.2

Solve the following inequalities.

(1) $x + 2 < 3 - 2x$

(2) $x^2 - 2x < x + 1$

(3) $\dfrac{x+3}{2x-1} > 1$

PROBLEM 2.1.3

Solve the following equation and inequalities.

(1) $|x - 2| + 1 = 3 - x$

(2) $|2x - 1| - |3 - x| = 1$

(3) $|x + 1| > x + 2$

(4) $|4 - x| < x - |x - 1|$

PROBLEM 2.1.4

Plot the following points in the planar Cartesian system:

(0,1), (1,2), (-3,1), (-2,2)

PROBLEM 2.1.5

Plot the following points in the 3D Cartesian system:

(2,0,1), (-1,1,1), (0,-1,2), (1,-2,-2)