Problems in Mathematics
&
Experiments with Mathematica

6. Definite integral

6.4 Applications

Area

Let *f(x)* and *g(x)* be integrable on the interval *[a,b*]. The area of the region bounded by the graphs and the lines $x=a$ and $x=b$ is the integral

$$\int_a^b |f(x) - g(x)|\, dx$$

Length of a curve

Let the planar curve be given parametrically by *(x(t), y(t))*, where the functions *x(t)* and *y(t)* are continuously differentiable. The length of the curve on the parameter interval $[t_1, t_2]$ is

$$\int_{t_1}^{t_2} \sqrt{x'^2(t) + y'^2(t)}\, dt$$

In particular, if $x(t)=t$ and $y(t)=f(t)$, we obtain

$$\int_{t_1}^{t_2} \sqrt{1 + f'^2(t)}\, dt$$

Volume of a cylindrical body

Let the body be given by revolving the function *f(x)* around the *x*-axis on the interval *[a,b]*. Then its volume is

$$\pi \int_a^b f^2(x)\, dx$$

You can examine this problem graphically in the Appendix.

Motion of a body

In particular, if the acceleration *a(t)* and the initial velocity $v(t_0) = v_0$ and initial position $s(t_0) = s_0$ are given, then

$$\boxed{v(t) = v_0 + \int_{t_0}^{t} a(u)\, du}$$

and

$$\boxed{s(t) = s_0 + \int_{t_0}^{t} v(u)\, du = s_0 + v_0 (t - t_0) + \int_{t_0}^{t} \int_{t_0}^{r} a(u)\, du\, dr.}$$

Average speed

Let *v(t)* the speed of a process. Using the mean value of a function, the average speed on $[t_1, t_2]$ can be obtained by the formula

$$\boxed{\overline{v}(t_1, t_2) := \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} f(u)\, du}$$

Exercises and problems

SOLVED PROBLEM 6.4.1 Area

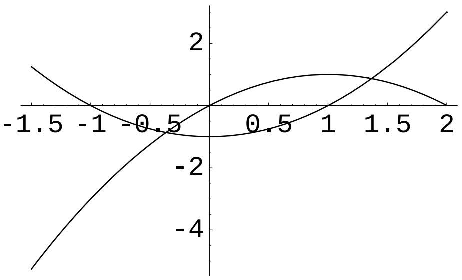
Calculate the closed area between the graps of the following functions.

$$f(x\_) = x^2 - 1; \quad g(x\_) := 2x - x^2;$$

- SOLUTION

Plot the functions:

```
Plot[{f[x], g[x]}, {x, -1.5, 2}];
```



Find the common points:

```
xbound = x /. Solve[f[x] == g[x], x]
```

$$\left\{\frac{1}{2}\left(1-\sqrt{3}\right),\ \frac{1}{2}\left(1+\sqrt{3}\right)\right\}$$

Calculate the area:

```
Simplify[∫_{xbound[[1]]}^{xbound[[2]]} Abs[f[x] - g[x]] dx]
```

$$\sqrt{3}$$

PROBLEM 6.4.2

Calculate the area of the closed region between the graps of the functions *f(x)* and *g(x)*.

(1) $f(\text{x\_}) := x^2 + 2x;\quad g(\text{x\_}) := 2x^2 + 7x + 4;$

(2) $f(\text{x\_}) := 2x^2 - 2x;\quad g(\text{x\_}) := x^2 - x - 1;$

(3) $f(\text{x\_}) := x^2 - 1;\quad g(\text{x\_}) := 2x^2 - 3x - 1;$

(4) $f(\text{x\_}) := x - 1;\quad g(\text{x\_}) := x^2 - x - 1;$

(5) $f(\text{x\_}) := x^2 - 2x - 1;\quad g(\text{x\_}) := 2x^2 - x - 1;$

(6) $f(\text{x\_}) := x^2 - 4x - 4;\quad g(\text{x\_}) := 3x^2 - 4x - 12;$

SOLVED PROBLEM 6.4.3 Volume

Calculate the volume of the jug obtained by revolving the function

$$f(\text{x\_}) = \sin(x) + e^{-\frac{x}{10}} + 1;$$

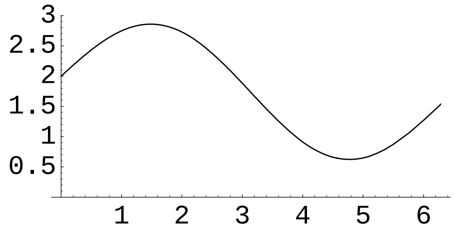
between the places

$$a = 0;\quad b = 2\pi;$$

○ SOLUTION

Plot the function:

```
Plot[{f[x]}, {x, a, b},
  PlotRange -> {0, 3}, AspectRatio -> Automatic];
```



See more details in the Appendix.

Calculate the volume:

$$\text{vol} = \pi \int_a^b f[x]^2 \, dx$$

$$\pi \left(\frac{2725}{101} - 5\, E^{-2\pi/5} - \frac{2220\, E^{-\pi/5}}{101} + 3\pi \right)$$

```
N[vol]
```

73.0601

SOLVED PROBLEM 6.4.4 Variation of a process

The variation speed of concentration of a solute is

$$c(t\_) = 2\, e^{-2t};$$

The initial concentration is zero. Calculate the concentration at

$$t_1 = 5;$$

○ SOLUTION

The concentration at t=5 is obtained by the formula

$$\int_0^{t_1} c[t] \, dt$$

$$1 - \frac{1}{E^{10}}$$

PROBLEM 6.4.5

The speed of a process is described by *v(t).* Find the function describing the process itself, and calculate the average speed between t_0 and t_1.

(1) $v(t\_) := 10\, e^{t-1}\, t;\ t_0 = 0;\ t_1 = 2;$

(2) $v(t\_) := 10\, e^{-t^2}\, t;\ t_0 = 0;\ t_1 = 1;$

SOLVED PROBLEM 6.4.6   Motion

The acceleration of a moving body at the t'th second is $a(t) = 10 \cos \pi t \ \frac{m}{s^2}$. Find the velocity and the displacement function, if $v(0)=0$ and $s(0) = 1$. Calculate mean value of the velocity between $t_0 = 0.25$ and $t_1 = 0.5$.

○ SOLUTION

The data:

$$aa = 10\,\text{Cos}[\pi\, t];\ t_0 = 0.25;\ t_1 = 0.5;$$

Velocity:

$$vv = \int_0^t aa\, dt$$

$$\frac{10\,\mathrm{Sin}[\pi\, t]}{\pi}$$

$$\mathrm{Meanofvv} = \frac{\int_{t_0}^{t_1} vv\, dt}{t_1 - t_0}$$

2.8658

Displacement:

$$ss = 1 + \int_0^t vv\, dt$$

$$1 + \frac{10}{\pi^2} - \frac{10\,\mathrm{Cos}[\pi\, t]}{\pi^2}$$

Problem 6.4.7

The acceleration of a moving body at the t'th second is $a(t)\ \frac{m}{s^2}$. Find the velocity and the displacement function, if $v(t_0) = v_0$ and $s(t_0) = s_0$. Calculate the mean value of the velocity and the acceleration between t_1 and t_2.

(1) $a(t\_) := 10\, e^{1-2t};\ t_0 = 0;\ v_0 = 0;\ s_0 = 1;\ t_1 = 1;\ t_2 = 3;$

(2) $a(t\_) := (3\,t - 2)^2;\ t_0 = 1;\ v_0 = 1;\ s_0 = 0;\ t_1 = 0;\ t_2 = 2;$

(3) $a(t\_) := 3\,t + 1;\ t_0 = 0;\ v_0 = 0;\ s_0 = 2;\ t_1 = 0.5;\ t_2 = 1.5;$

(4) $a(t\_) := 2^t - 1;\ t_0 = 0;\ v_0 = 1;\ s_0 = 1;\ t_1 = 1;\ t_2 = 1.5;$

(5) $a(t\_) := \sin(\pi\, t) + 1;\ t_0 = 0;\ v_0 = 0;\ s_0 = 2;\ t_1 = 0.5;\ t_2 = 1.5;$