Content

Problems in Mathematics
&
Experiments with Mathematica

7. 1D Ordinary differential equations

7.1 Definitions, properties, vectorfield, equilibria

Basic definitions

DEFINITION 7.1.1

Let $x(t)$ be a function of the independent variable t. The equation

$$x' = f(t, x)$$

is called first order explicit ordinary differential equation.

The problem is to find the solutions, and/or to investigate their properties without solving the equation.

The solutions of the equations are the differentiable functions that satisfy the equation. The collection of all solutions is called general solution.

DEFINITION 7.1.2

The problem

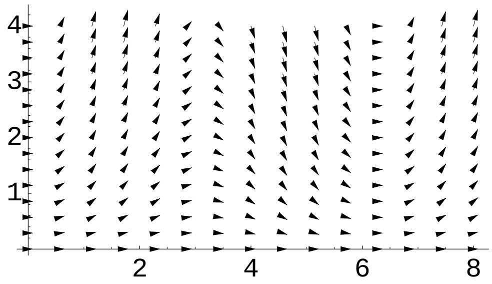
$$x' = f(t, x),$$
$$x(t_0) = x_0$$

is called initial value problem. We have to find the solution that satisfies the equation and the given initial condition (see the special case of finding a particular antiderivative). The solution of the initial value problem is a particular solution.

- **Geometrical meaning, direction field**

The geometrical meaning is a generalization of the geometrical meaning of the indefinite integral. Consider the tx-plane. At every point (t,x), draw a vector of slope $f(t,x)$ and length proportional with the slope. The obtained collection of vectors is called the direction field of the differential equation. If $x(t)$ is a solution, i.e., $x'=f(t,x)$, then it smoothly touches the elements of the direction field at every point $(t,x(t))$. The element of the direction field can be written in the form $\frac{1}{\sqrt{1+f^2(t,x)}}(1, f(t, x))$. The simpler vectors $(1, f(t, x))$ can also be drawn.

◦ *Example: direction field of the equation* $x'=x\sin(t)$



- *Importance*

The direction field can be plotted without solving the equation.

• Special cases

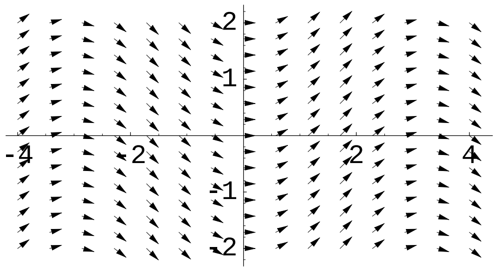
- *Indefinite integral*

The equation has the form

$$x' = f(t)$$

The elements of the direction field are parallel at every fixed t.

◦ *Example: direction field of the equation* $x'=\sin(t)$



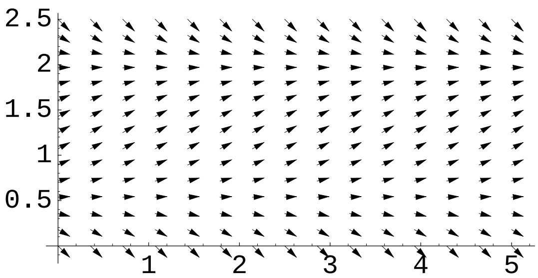
- *Autonomous equations*

The equation has the form

$$x' = f(x)$$

The derivative of $x(t)$ does not depend explicitly on the independent variable. The process can begin at any time, it has the same shape. Closed systems are described by autonomous equations. The vectors of the direction field are parallel at every fixed x.

◦ *Example: direction field of the equation* $x'=(x-0.5)(2-x)$



Equilibria

The constant solutions of the equation are called steady states or equilibria. If $x(t) = x_0$ is an equilibrium, then $f(t, x(t)) = 0$. On the other hand, the t-independent solutions of the equation $f(t, x) = 0$ are equilibria. If the differential equation is autonomous, then every solution of $f(t, x) = 0$) is t-independent.

For the techniques to solve the equation $f(t, x) = 0$), see the chapters Exercises on elementary functions and Approximation of zeros.

DEFINITION 7.1.3 Stability of equilibria

An equilibrium $\overline{x}$ is *stable*, if for every $\epsilon > 0$ there exists $\delta > 0$ such that $|x(0)-\overline{x}| < \delta$ implies $|x(t)-\overline{x}| < \epsilon$ for all $t \geq 0$.
The equilibrium $\overline{x}$ is *asymptotically stable* if it is stable and there exists $\gamma > 0$ such that $|x(0)-\overline{x}| < \gamma$ implies $\lim_{t\to\infty} |x(t)| = \overline{x}$.
An equilibrium $\overline{x}$ is *unstable*, if it is not stable, i.e., there exists $\epsilon_0 > 0$ such that for every $\delta>0$ and $T>0$ there exists a solution x(t) such that $|x(0)-\overline{x}| < \delta$ and $|x(T)-\overline{x}| > \epsilon_0$.

THEOREM 7.1.4 A stability for autonomous equations

For an autonomous equation $x' = f(x)$, the equilibrium $\overline{x}$ is asymptotically stable if $(x - \overline{x})\, f(x) \leq 0$ and there is no other equilibrium in a neighborhood of $\overline{x}$.

If $(x - \overline{x})\, f(x) > 0$ in a neighborhood of $\overline{x}$, then $\overline{x}$ is unstable.

Mathematica statements

PlotVectorField[{1, xdot}, {t, t0, t1}, {x, x0, x1}, Axes –> True]	Graphics`PlotField` package : plots the directionfield in the square [t0, t1] x [x0, x1]
Solve[...]	Solves an equation formally
FindRoot[...]	Finds a root of an equation using the Newton method

Exercises and problems

Mathematica initialization

```
Needs["Graphics`PlotField`"]
```

SOLVED PROBLEM 7.1.1 A graphical problem

Without solving the equation, investigate the behavior of the solutions.

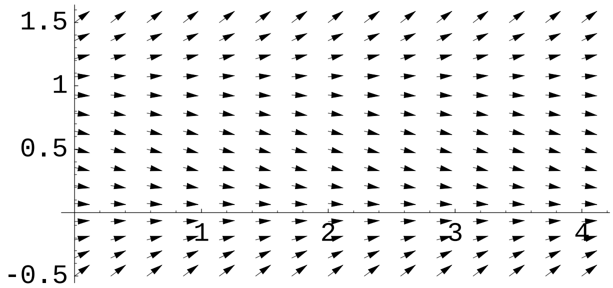
$$f(x\_) = x\,(x - 1); \quad \text{eqn} = x' == f(x);$$

Plot the direction field, find equilibria, and investigate their stability. Sketch some solutions in the plot of the direction field.

○ SOLUTION

- *Plot the direction field.*

```
x0 = -0.5; x1 = 1.5;
t0 = 0;  t1 = 4;
PlotVectorField[{1, f[x]}, {t, t0, t1}, {x, x0, x1}, Axes -> True];
```



- *Find the equilibria*

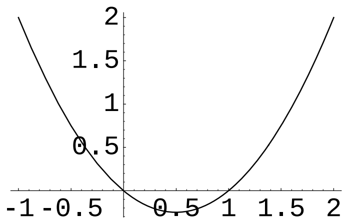
```
equilibium = Solve[f[x] == 0, x]
```

{{x → 0}, {x → 1}}

- *Stablility of equilibria*

The above plot suggests that the equilibrium $x=1$ is unstable and $x=0$ is stable. The plot of $f(x)$ confirms this conjecture.

```
Plot[f[x], {x, -1, 2}];
```



PROBLEM 7.1.2 Autonomous equations

Investigate the equations for which the right-hand sides are given below. Plot the direction fields, find equilibria, and investigate their stability. Sketch some solutions in the plot of the direction field.

(1) $f(x\_) := -3\,x$

(2) $f(x\_) := \frac{x}{2}$

(3) $f(x\_) := 2\,x - 3$

(4) $f(x\_) := 1 - 2\,x$

(5) $f(x\_) := (x - 1)\,(2\,x - 3)$

(6) $f(x\_) := x\,(2 - x) - 2$

(7) $f(x\_) := x\,(2 - x) - 0.5$

(8) $f(x\_) := x^2$

(9) $f(x\_) := x\,(x - 1)$

PROBLEM 7.1.3 Non-autonomous equations

Investigate the equations for which the right-hand sides are given below. Plot the direction fields, find equilibria -if there is any- and investigate their stability graphically. Sketch some solutions in the plot of the direction field.

(1) $f(x\_) := -\frac{x}{t+1}$

(2) $f(x\_) := t + \frac{x}{2}$

(3) $f(x\_) := x \log(t-2) - 3$

(4) $f(x\_) := 1 - 2x$

(5) $f(x\_) := \sin^2(t)\, x\, (1-x)$