Problems in Mathematics & Experiments with Mathematica

3. Limit and continuity

3.5 Comparison of growths

Compare the growth: investigate $\lim_{x\to a} f(x) - g(x)$

Let $f(x)$ and $g(x)$ be given around (but not necessarily at) a given point a. Assume that

$$\lim_{x\to a} g(x) = \lim_{x\to a} f(x) = \infty.$$

The problem is that which function converges faster to the given limit.

- **Method**

Take $f(x) - g(x)$ and find $\lim_{x\to a} f(x) - g(x)$.

If $\lim_{x\to a} f(x) - g(x) = A < \infty$, then $f(x) \approx g(x) + A$ around a.

If $\lim_{x\to a} f(x) - g(x) = \infty$, then $f(x) \gg g(x)$ around a.

If $\lim_{x\to a} f(x) - g(x)$ does not exist,

then $f(x)$ and $g(x)$ cannot be compared by this method around a.

The relative growth rate: investigate $\lim_{x\to a} \frac{f(x)}{g(x)}$

Let $f(x)$ and $g(x)$ be given around (but not necessarily at) a given point a. Assume that

$$\lim_{x\to a} g(x) = \lim_{x\to a} f(x) = 0$$

or

$$\lim_{x\to a} g(x) = \lim_{x\to a} f(x) = \infty.$$

The problem is that which function converges faster to the given limit.

• Method

Take $\frac{f(x)}{g(x)}$ and find $\lim_{x\to a}\frac{f(x)}{g(x)}$.

If $\lim_{x\to a}\frac{f(x)}{g(x)} = A > 0$, then $f(x) \approx A\, g(x)$ around a.

If $\lim_{x\to a}\frac{f(x)}{g(x)} = 0$, then $f(x) \ll g(x)$ around a.

If $\lim_{x\to a}\frac{f(x)}{g(x)} = \infty$, then $f(x) \gg g(x)$ around a.

If $\lim_{x\to a}\frac{f(x)}{g(x)}$ does not exist,

then $f(x)$ and $g(x)$ cannot be compared by this method around a.

Special case: Asymptotes at infinity

DEFINITION 3.5.1

Let $\lim_{x\to a} f(x) = \infty$. The linear function $y = a\,x + b$ is the asymptote of $f(x)$ at ∞ if $\lim_{x\to a} f(x) - (a\,x + b) = 0$.

• Method to find the asymptote

Step 1. $\lim_{x\to a}\frac{f(x)}{x} = a$

Step 2. $\lim_{x\to a} f(x) - a\,x = b$

Problems

PROBLEM 3.5.1

Consider the fractions of functions in the problems of the previous sections and in the section L'Hospital rule. Compare the growth rate of the numeration and the denominator.

PROBLEM 3.5.2

Find the asymptote for the following functions.

(1) $f(x\_) := \frac{x^2 - 2x + 1}{x - 1}$

(2) $f(x\_) := \frac{2x^2 + x + 1}{x - 2}$

(3) $f(x\_) := \frac{x^3 - x^2}{2x^2 - 1}$