Content

Problems in Mathematics
&
Experiments with Mathematica

7. 1D Ordinary differential equations

7.4 The equation $x' = k(A - x)(B - x) + C$

Summary of special cases

$x' = k\,x(A - x)$	Logistic population growth
$x' = k\,x(A - x) - H$	Logistic population growth with harvesting $(C < 0)$ or planting $(C > 0)$
$x' = k(A - x)(B - x)$	Chemical reaction of second order
$x' = k(A - x)$	Chemical reaction of first order
$x' = k(A - x)^2$	Selfexploding process in finite time if $x > A$ and $k > 0$

Logistic population growth

$$\mathbf{eqn} = x'(t) == k\,x(t)\,(A - x(t));$$

where

$k>0$ natural growth rate

$A>0$ capacity

- *General solution*

$$x(t) \to \frac{A\,e^{A k t}}{e^{A k t} - C}$$

- *The solving method*

```
DSolve[eqn, x[t], t] /. E^C[1] → C
```

Logistic population growth with external effect

$$\mathbf{eqn} = x'(t) == k\, x(t)\,(A - x(t)) - H;$$

where

k>0	natural growth rate
A >0	life space capacity
H	external influence rate:
	H<0 planting, immigration
	H>0 harvesting, emigration

- *Case of two equilibria :* $R = -4\,H\,k + A^2\,k^2 > 0$

$$x(t) \to \frac{A\,C\,k - A\,e^{R\,t}\,k + e^{R\,t}\,R + C\,R}{2\,C\,k - 2\,e^{R\,t}\,k}$$

- *The solving method*

Find the equilibria and factorize the right-hand side:

```
{x1, x2} =
  x /. Solve[k x (A - x) - H == 0, x] /. Sqrt[-4 H k + A^2 k^2] -> R
```

$$\left\{\frac{A\,k - R}{2\,k}, \frac{A\,k + R}{2\,k}\right\}$$

Then solve the equation:

```
Simplify[DSolve[
   x'[t] == k (x[t] - x1) (x[t] - x2), x[t], t] /. E^C[1] → C]
```

- *Case of one equilibrium :* $R = -4\,H\,k + A^2\,k^2 = 0$

$$x(t) \to \frac{A\,C - A\,k\,t + 2}{2\,C - 2\,k\,t}$$

- *The solving method*

Find the equilibria and factorize the right-hand side:

```
{x1, x2} =
  x /. Solve[k x (A - x) - H == 0, x] /. Sqrt[-4 H k + A^2 k^2] -> 0
```

$$\left\{\frac{A}{2}, \frac{A}{2}\right\}$$

Then solve the equation:

```
Simplify[DSolve[x'[t] == k (x[t] - x1)^2, x[t], t] /. C[1] -> C]
```

- *No real equilibria :* $R = -4\,H\,k + A^2\,k^2 < 0$

$$x(t) \to \frac{A\,k + R\,\tan\left(\frac{1}{2}\,(R\,t - C)\right)}{2\,k}$$

- *The solving method*

Find the equilibria and factorize the right-hand side:

```
{x1, x2} =
  x /. Solve[k x (A - x) - H == 0, x] /. Sqrt[-4 H k + A^2 k^2] -> ⅈ R
```

$$\left\{\frac{A\,k - i\,R}{2\,k}, \frac{A\,k + i\,R}{2\,k}\right\}$$

Then, solve the equation:

```
Simplify[DSolve[x'[t] == k (x[t] - x1) (x[t] - x2), x[t], t] /.
  C[1] -> C]
```

The constant C may be a complex number.

Chemical reaction of second order

$$\mathbf{eqn} = x'(t) == k\,(A - x(t))\,(B - x(t));$$

where

x(t)	amount of the resulted material
k>0	reaction speed constant
A >0	initial amount of reagent one
B >0	initial amount of reagent two

The speed of the reaction *x'(t)* at time *t* is proportional to the amount of the reagents.

- *General solution: x(0)≠0*

$$x(t) \to \frac{B\,e^{A\,k\,t} - A\,e^{C + B\,k\,t}}{e^{A\,k\,t} - e^{C + B\,k\,t}}$$

- *The solving method*

```
ComplexExpand[DSolve[eqn, x[t], t] /. C[1] -> C]
```

- *If x(0)=0, then*

$$x(t) \to \frac{A\,B\,(e^{A\,k\,t} - e^{B\,k\,t})}{A\,e^{A\,k\,t} - B\,e^{B\,k\,t}}$$

- *The solving method*

```
t0 = 0; x0 = 0;
Clear[x];
Simplify[ x[t] /. (gsol = DSolve[eqn, x[t], t] ) /.
(Solve[(x[t] /. gsol /. t -> t0) == {x0}, C[1]]) ]
```

Links

For the methods and tools, see the chapters:
Definitions, properties, vectorfield, equilibria,
Mathematica tools to solve a differential equation.

Exercises and problems

Mathematica initialization

```
Needs["Graphics`PlotField`"]
```

Use the *Mathematica* ODE Solving tools and ODE experiments to solving the following problems.

Problem 7.4.1

Solve the following initial value problems with each given initial values $(x_0, x_1, ...)$ at t_0. Draw the direction field and some solutions. Investigate the stability of the equilibria.

$$\mathbf{eqn} = x'(t) == f(x) \,/.\, x \to x(t);$$

(1) $f(\text{x\_}) = x\,(2 - x);\ t_0 = 0;\ x_0 = 1;\ x_1 = 0.1;$

(2) $f(\text{x\_}) = 2\,x\,(1 - x) - 1;\ t_0 = 0;\ x_0 = 3;\ x_1 = 1;$

(3) $f(\text{x\_}) = 2\,x\,(1 - x) - 0.5;\ t_0 = 0;\ x_0 = 3;\ x_1 = 1;$

(4) $f(\text{x\_}) = 2\,x\,(1 - x) - 0.2;\ t_0 = 0;\ x_0 = 3;\ x_1 = 1;$

(5) $f(\text{x\_}) = (1 - x)\,(3 - x);\ t_0 = 0;\ x_0 = 0.4;\ x_1 = 1.1;$

(6) $f(\text{x\_}) = x^2;\ \mathrm{t0} = 0;\ \mathrm{x0} = 1;$

Solved Problem 7.4.2

Find the set for the parameter C in the population model $x' = x(3 - x) - C$ to avoid overharvesting, that is, the population should not die out in finite time at large enough initial population.

○ Solution

The right-hand side of the equation is a parabola. If $C=0$, then the zeros are $x_1=0$ and $x_2=3$. For a negative C, it can happen that there are no zeros. The threshold value C_0 is that one at which the equation $x(3-x)-C=0$ has only one root. If $C>C_0$, then the system is overdamped, if $C<C_0$, then the population dies out, if the initial population is less than the smaller root of $x(3-x)-C=0$.

- *Find the equilibria, and the threshold C_0*

```
{x1, x2} = x /. Solve[x (3 - x) - C == 0, x]
```

$$\left\{\frac{1}{2}\left(3-\sqrt{9-4\,C}\right),\ \frac{1}{2}\left(3+\sqrt{9-4\,C}\right)\right\}$$

```
Solve[x1 == x2, C]
```

$$\left\{\left\{C\to\frac{9}{4}\right\}\right\}$$

○

EXPERIMENT 7.4.3

Investigate the equation in the previous problem at different values of C. Find the solutions formally.