Content

# Problems in Mathematics
# &
# Experiments with Mathematica

# 7. 1D Ordinary differential equations

## 7.3 Linear equations

### *Homogeneous linear equations*

**DEFINITION 7.3.1**

The equation

$$x' = a(t)\, x$$

is called first order linear homogeneous differential equation.

- **General solution**

$$x(t) = C\, e^{\int_0^t a(t)\, dt}$$

- **Properties**

Let $x_1(t)$ and $x_2(t)$ be solutions. Then $c\, x_1(t)$ (c is any constant), and $x_1(t) + x_2(t)$ are also solutions.

The solution $x(t)=0$ is the only equilibrium. It is stable if $a(t)\leq 0$. If $a(t)$ is a constant and $a>0$, then the solutions exponentially increase in magnitude, and exponentially decrease in magnitude if $a<0$.

- **Equation with constant coefficient**

$$x' = a\, x$$

The general solution is

$$x(t) = C\, e^{a\, t}$$

### *Non-homogeneous linear equations*

**DEFINITION 7.3.2**

The equation

$$x' = a(t)\, x + b(t)$$

is called first order nonhomogeneous linear differential equation.

- **General solution**

$$x(t) = e^{\int_0^t a(s)\,ds}\left(C + \int_0^t e^{-\int_0^s a(u)\,du}\, b(s)\, ds\right)$$

- **Properties**

Let $x_1(t)$ be a solution and $x(t)$ be a solution of the corresponding homogeneous equation. Then $x_1(t) + x(t)$ is also solution.

Let $x_1(t)$ and $x_2(t)$ be solutions. Then $x_1(t) - x_2(t)$ is a solution of the corresponding homogeneous equation $x' = a(t)\, x$.

If the coefficients are constants, the only equilibrium is $\dfrac{b}{a}$.

- **Equation with constant coefficient**

$$x' = a\,x + b$$

The general solution is

$$x(t) = C\, e^{a\,t} - \frac{b}{a}$$

## *Interpretations*

Let $a, b>0$.

| | |
|---|---|
| $x' = a\,x$ | Selfexciting<br>exponential growth : Malthus<br>population model |
| $x' = -a\,x$ | Exponential decay : drug dissimilation, dialysis, nuclear deplosion, dieing population |
| $x' = \mathrm{ax} + b$ | Increasing population with<br>rate $a$ and with immigration rate $b$ |
| $x' = \mathrm{ax} - b$ | Increasing population with rate<br>$a$ and with harvesting of rate $b$ |
| $x' = -\mathrm{ax} + b$ | Drug dissimilation with continuous<br>infusion; bounded population growth |

## *Connections, links*

For the methods and tools used, see the chapters:

[Definitions, properties, vectorfield, equilibria](),
[Mathematica tools to solve differential equations]().

# Exercises and problems

## Mathematica initialization

```
Needs["Graphics`PlotField`"]
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### PROBLEM 7.3.1

Solve the following initial value problems with each given initial values $(x_0, x_1, ....)$ at $t_0$. Draw the direction field, and some solutions. Investigate the stability of the equilibrium.

$$\text{eqn} = x'(t) == f(x(t));$$

(1) $f(\text{x\_}) = -3\,x;\ t_0 = 0;\ x_0 = 1;$

(2) $f(\text{x\_}) = \frac{x}{2};\ t_0 = 0;\ x_0 = 0.1;$

(3) $f(\text{x\_}) = 2\,x - 1;\ t_0 = 0;\ x_0 = 0.4;\ x_1 = 0.6;$

(4) $f(\text{x\_}) = 1 - 3\,x;\ t_0 = 0;\ x_0 = 1;\ x_1 = 0;$

(5) $f(\text{x\_}) = -x - 0.5;\ t_0 = 0;\ x_0 = 1;$

### PROBLEM 7.3.2

Find the set for the parameter $b$ in the equation $x'=0.2\ x\ \text{-}\ b$ such that the equilibrium is equal to $2$.

### PROBLEM 7.3.3 Non-autonomous equations

Solve the following initial value problems with each given initial values $(x_0, x_1, ....)$ at $t_0$. Draw the direction field, and some solutions. Investigate the stability of the equilibrium.

$$\text{eqn} = x'(t) == f(t, x(t));$$

(1) $f(\text{t\_}, \text{x\_}) := x\sin(t) + 1;\ \ t_0 = 0;\ x_0 = 0.2;\ \ x_1 = 0;$

(2) $f(\text{t\_}, \text{x\_}) := \log(t)\,x - 1;\ \ t_0 = 1;\ x_0 = 2;\ \ x_1 = 0;$

(3) $f(\text{t\_}, \text{x\_}) := t - t\,x;\ \ t_0 = 1;\ x_0 = 0;\ \ x_1 = 1.;$

(4) $f(\text{t\_}, \text{x\_}) := 1 - e^{-t}\,x;\ \ t_0 = 0;\ x_0 = 0;\ \ x_1 = 1;$