Content

Problems in Mathematics
&
Experiments with Mathematica

9. Functions of several variables

9.1 Basic properties: definitions, domain, graph, contours

Theory

- **Definition**

The function

$$f : \mathbb{R}^2 \to \mathbb{R}$$

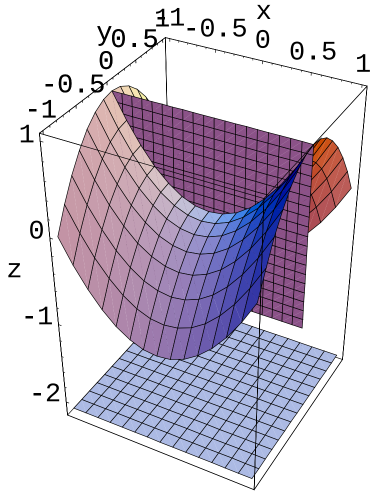
is called a scalar valued function of two variables. The domain is a subset of $\mathbb{R}^2$, and the range is a subset of $\mathbb{R}$. The independent variables are denoted by x and y. The value is $z = f(x, y)$.

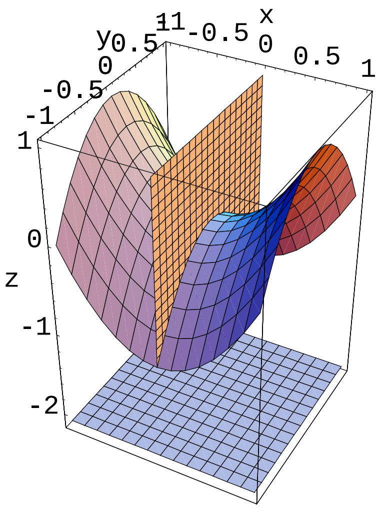
- **Graph**

The graph $G : \{(x, y, z) : (x, y, f(x, y))\}$ is a subset of $\mathbb{R}^3$.

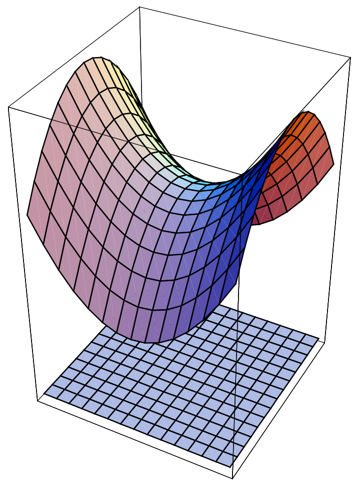
A set $G \subseteq \mathbb{R}^3$ represents the graph of a function $f : \mathbb{R}^2 \to \mathbb{R}$ if for every $(x, y) \in R^2$ there exists at most one z such that $f(x, y) = z$. Geometrically, every straight line parallel to the z-axis can have only one common point with G.

Practically, create a grid on the xy-plane (a net of lines parallel to the x-axis and the y-axis, given by the fixed values $x_1, x_2, \ldots$ and $y_1, y_2, \ldots$). Plot the graphs of the functions $f(x, y_i)$ of x and $f(x_i, y)$ of y in the corresponding planes.





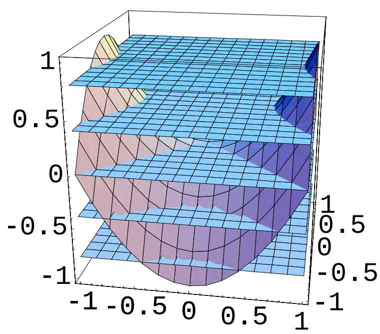
Then, the graph is

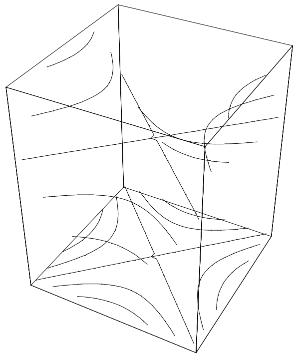


- **Contour lines**

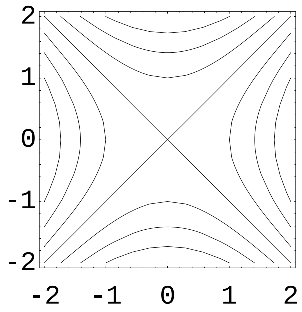
Let z_0 be given. The set $\{(x, y) : f(x, y) = z_0\} \subseteq \mathbb{R}^2$ is the contour line of the function at the value z_0. Geometrically, the contour line at z_0 is the intersection of the graph with the plane $z = z_0$ (the plane parallel to the xy-plane at $z = z_0$), projected to the xy-plane.

- *The projection process*



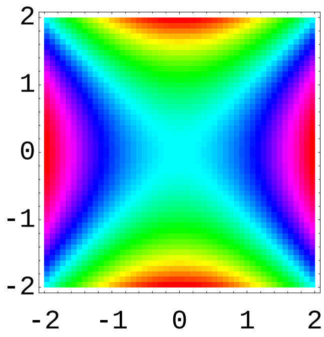


- *The contour lines in the xy-plane*



- **Density plot**

Make the same grid as for the `Plot`. In each little rectangle, take the average of $(f(x, y))$ (in some sense), and paint the rectangle according to some color mapping.



Mathematica statements

Plot3D[f, {x, x_0, x_1}, {y, y_0, y_1}]	Plots the graph of f
ContourPlot[f, {x, x_0, x_1}, {y, y_0, y_1}]	Plots the contourlines
DensityPlot[f, {x, x_0, x_1}, {y, y_0, y_1}]	Plots *a* densityplot

See some options and examples in the section: A short introduction to Mathematica

Exercises and problems

SOLVED PROBLEM 9.1.1

Find the domain, plot the contour lines for the *z*-values given in the variable `z0list`, plot the graph of the following function.

$$f(x\_, y\_) := \sqrt{x + \sin(y)}; \quad \text{z0list} = \{0, 0.2, 0.6, 1, 1.4\};$$

○ SOLUTION

- *Domain*

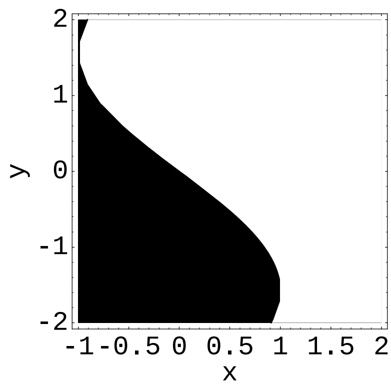
The function under the root has to be nonnegative.

```
Solve[f[x, y]^2 == 0, x]
```

```
{{x → -Sin[y]}}
```

The white area is the domain in the following figure.

```
pltdomain = ContourPlot[f[x, y]^2, {x, -1, 2}, {y, -2, 2},
    Contours → {0}, ContourShading → True, FrameLabel → {x, y}];
```

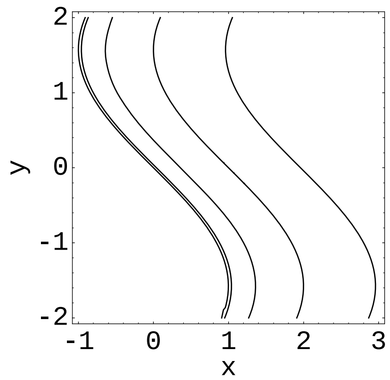


- *Contour lines*

```
Solve[f[x, y] == z0, x] /. z0 -> z0list
```

```
{{x → {-Sin[y], 0.04 - Sin[y],
    0.36 - Sin[y], 1 - Sin[y], 1.96 - Sin[y]}}}
```

```
pltcontours = ContourPlot[f[x, y]^2, {x, -1, 3}, {y, -2, 2},
    Contours → z0list^2, ContourShading → False, PlotPoints → 30,
    AspectRatio → Automatic, FrameLabel → {x, y}];
```

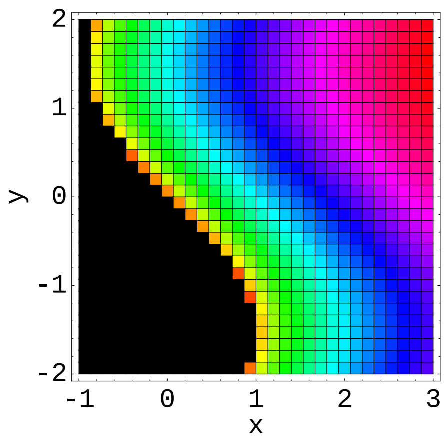


- *Remark.*

To avoid numerical problems, we took the second power of *f(x,y)* in the contourplots. Try these plots with only *f(x,y)*.

- *DensityPlot*

```
DensityPlot[f[x, y], {x, -1, 3},
  {y, -2, 2}, PlotPoints → 30, AspectRatio → Automatic,
  ColorFunction → Hue, FrameLabel → {x, y}];
```



— *DensityGraphics::zval : Non-numerical value 0. + 1.38177 I found at position {1, 1} in first argument.*

— *DensityGraphics::zval : Non-numerical value 0. + 1.33093 I found at position {1, 2} in first argument.*

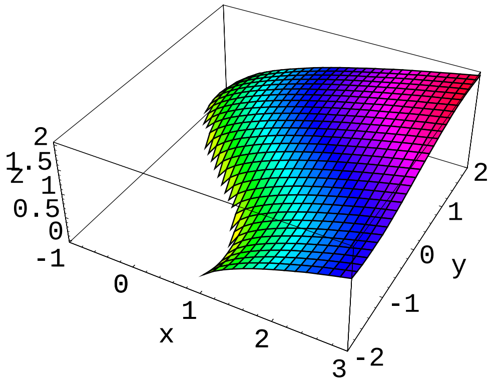
— *DensityGraphics::zval : Non-numerical value 0. + 1.27806 I found at position {1, 3} in first argument.*

- *Remark*

We got error messages because the function is not defined everywhere on the given set.

- *Plot the graph*

```
Plot3D[f[x, y], {x, -1, 3}, {y, -2, 2},
  PlotPoints → 30, AspectRatio → Automatic,
  ColorFunction → Hue, AxesLabel → {x, y, z}];
```



Solved Problem 9.1.2

Plot the graphs of the functions $f(x, y_0)$ of x and $f(x_0, y)$ for *f(x,y)* given in the previous problem if values of x_0 and y_0 are given in `x0list` and `y0list`, respectively. Find these curves on the graph of *f(x,y)*.

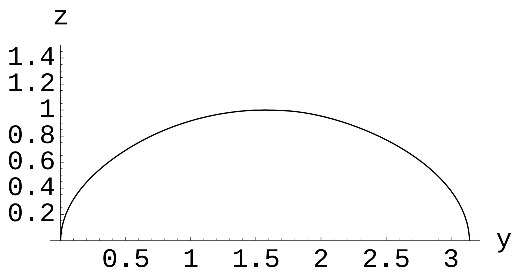
x0list = {0, 0.5, 1}; y0list = {0, 0.5, 1};

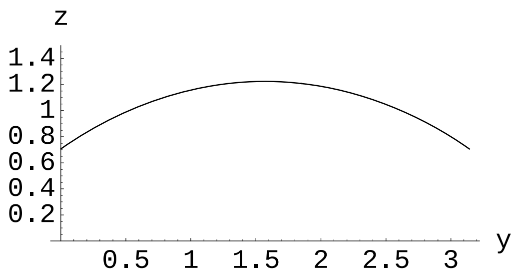
○ Solution

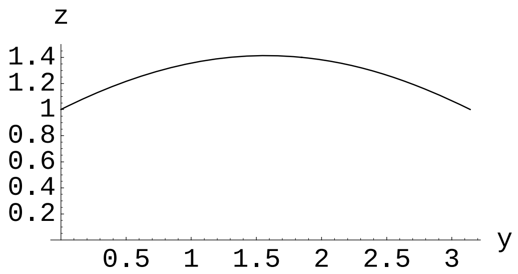
$$f(x\_, y\_) := \sqrt{x + \sin(y)};$$

- *The functions* $f(x_0, y)$

```
Table[Plot[Evaluate[f[x, y] /. x → x0list[[i]]], {y, 0, π},
   AxesLabel → {y, z}, PlotPoints → 30, PlotRange → {0, 1.5},
   AspectRatio → Automatic], {i, 1, Length[x0list]}];
```

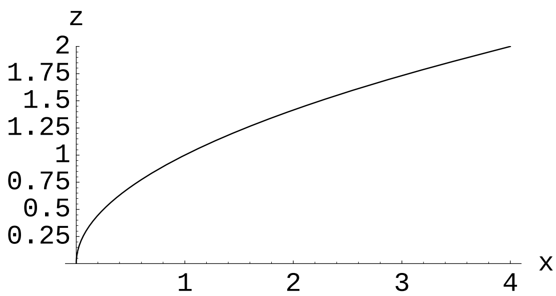


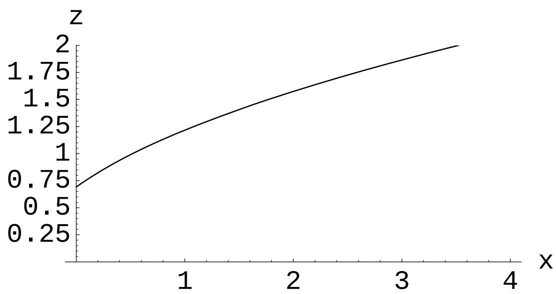


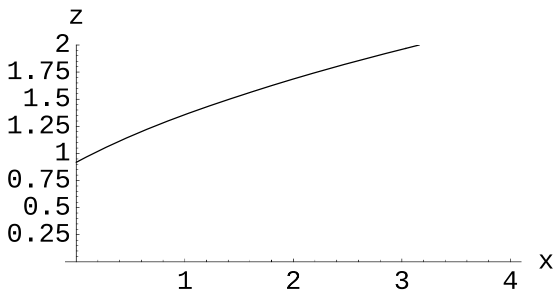


- *The functions* $f(x, y_0)$

```
Table[Plot[Evaluate[f[x, y] /. y → y0list[[i]]], {x, 0, 4},
   AxesLabel → {x, z}, PlotPoints → 30, PlotRange → {0, 2},
   AspectRatio → Automatic], {i, 1, Length[y0list]}];
```







PROBLEM 9.1.3

Find the domain, plot the contour lines for z-values given in the variable `z0list`, and plot the graph of the functions given below.

Plot the graphs of the functions $f(x, y_0)$ of x and $f(x_0, y)$ if the values of x_0 and y_0 are given in `x0list` and `y0list`, respectively.

(1) $f(x\_, y\_) = x^2 + 2y^2$;
z0list = {0, 1, 2}; x0list = {-1, 0, 1} ; y0list = {-1, 0, 1};

(2) $f(x\_, y\_) = 3x^2 - y^2$; z0list = {0, 1, 2}; x0list = {−1, 0, 1}; y0list = {−1, 0, 1};

(3) $f(x\_, y\_) = \log(y^2 + x)$;
z0list = {-2., -1, 1, 2}; x0list = {-1, 0, 2}; y0list = {-2, 0, 1};

(4) $f(x\_, y\_) = e^{x^2-y^2}$;
z0list = {0, 1, 2}; x0list = {-1, 0, 1}; y0list = {-1, 0, 1};

(5) $f(x\_, y\_) = \sin(x^2 + y^2)$;
z0list = {0, 1, 2}; x0list = {-1, 0, 1}; y0list = {-1, 0, 1};

(6) $f(x\_, y\_) = \sin(x + 3y)$;
z0list = {0, 1, 2}; x0list = {-1, 0, 1}; y0list = {-1, 0, 1};

PROBLEM 9.1.4

Find the domain, plot the contourlines for the z-values given in `z0list`, and plot the graph of the functions given below.

(1) $f(x\_, y\_) := \sqrt{\log x - y}$; z0list = {0, 1, 2};

(2) $f(x\_, y\_) := \sqrt{y^3 + x}$; z0list = $\left\{0, \frac{1}{2}, 1\right\}$;

(3) $f(x\_, y\_) := \tan(x^2 - y)$; z0list = {0, 1, −1};

(4) $f(x\_, y\_) := \sin(x^2 + y^2)$; z0list = {0, 1, −1};

(5) $f(x\_, y\_) := \log(x - y^2)$; z0list = {0, 1, −1};