Content

Problems in Mathematics
&
Experiments with Mathematica

3. Limit and continuity

3.3 Inequalities, applications of $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$

Theory

THEOREM 3.3.1

Let $f(x) \le g(x)$ in a neighborhood of the place a (finite or infinite) except possibly $x=a$. Then

$$\lim_{x\to a} f(x) \le \lim_{x\to a} g(x),$$

provided the limits exist. The theorem also holds for half-side limits.

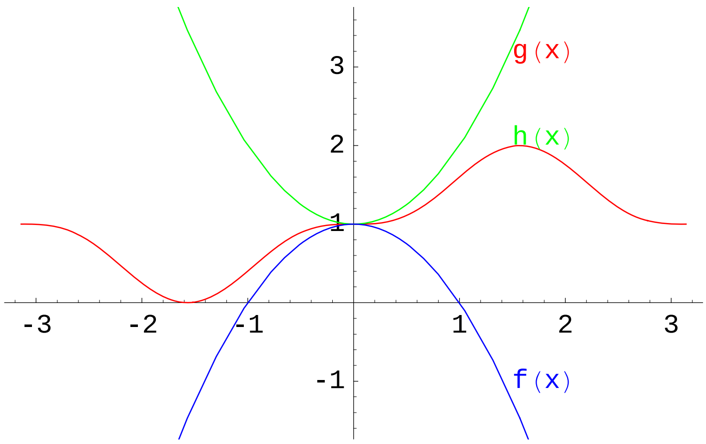
THEOREM 3.3.2 The ``Pinching Theorem''

Let $f(x) \le h(x) \le g(x)$ in a neighborhood of the place a (finite or infinite) except possibly $x=a$. If

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = L,$$

then

$$\lim_{x\to a} h(x) = L.$$



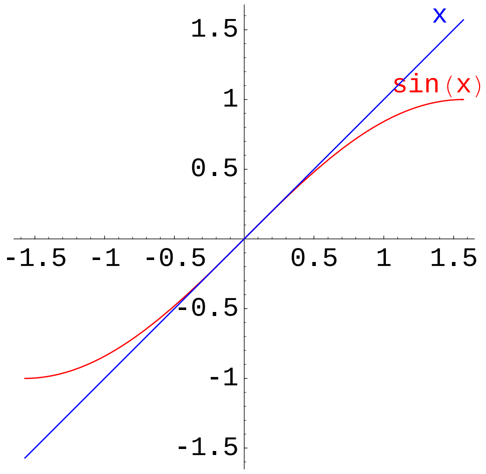
THEOREM 3.3.3

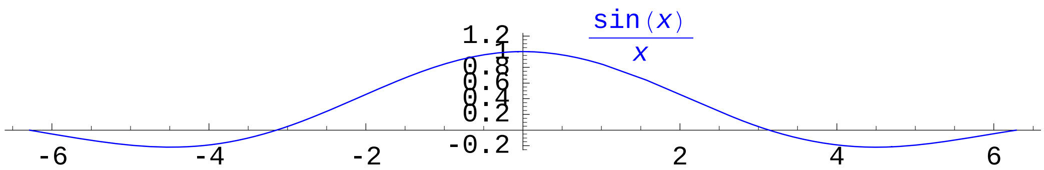
$$\boxed{\lim_{x\to 0} \frac{\sin(x)}{x} = 1}$$

The proof is based on the inequality

$$\sin(x) \le x \le \tan(x)$$

and the ``Pinching Theorem".





Problems and exercises

Solved Problem 3.3.1

Apply $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$ to find the following limits. Sketch the graph of the functions.

(1) $\displaystyle \lim_{x\to 0} \frac{\sin(x)}{2\,x}$

○ Solution

$$\lim_{x\to 0} \frac{\sin(x)}{2\,x} = \frac{1}{2} \lim_{x\to 0} \frac{\sin(x)}{x} = \frac{1}{2}\,.$$

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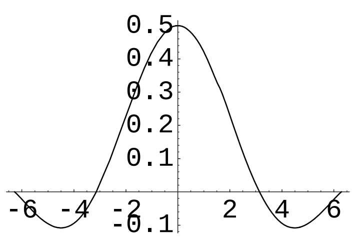
Plot[Sin[x]/(2 x), {x, -2 π, 2 π}];
```



(2) $\displaystyle \lim_{x\to 0} \frac{\sin(2\,x)}{x}$

◦ SOLUTION

$$\lim_{x\to 0}\frac{\sin(2\,x)}{x} = 2\lim_{x\to 0}\frac{\sin(2\,x)}{2\,x} = 2.$$

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Plot[Sin[2 x]/x, {x, -2 π, 2 π}, PlotRange → All];
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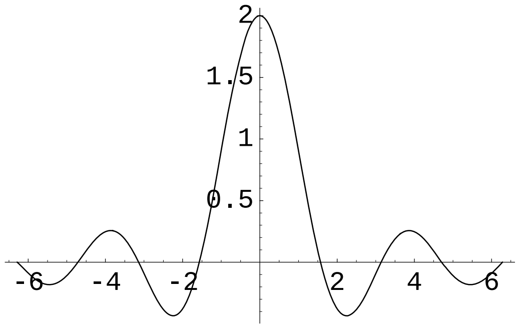


◦

(3) $\lim_{x\to 0}\frac{\sin(x)}{x^2}$

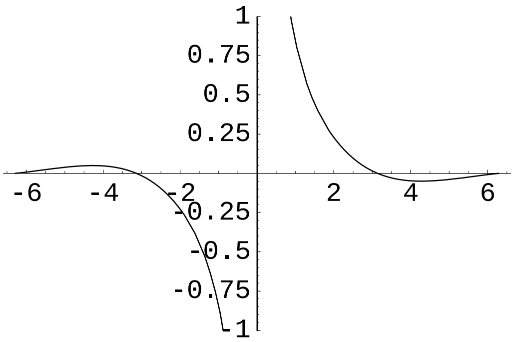
◦ SOLUTION

$$\lim_{x\to 0}\frac{\sin(x)}{x^2} = \lim_{x\to 0}\frac{1}{x}\;\lim_{x\to 0}\frac{\sin(x)}{x} = \lim_{x\to 0}\frac{1}{x}.$$

This limit does not exist, only from left and righ, and so

$$\lim_{x\to 0^+}\frac{\sin(x)}{x^2} = \infty, \quad \text{and} \quad \lim_{x\to 0^-}\frac{\sin(x)}{x^2} = -\infty.$$

Plot$\left(\frac{\sin(x)}{x^2}, \{x, -2\,\pi, 2\,\pi\}, \text{PlotRange} \to \{-1, 1\}\right)$;



◦

PROBLEM 3.3.2

(1) $\lim_{x\to 0}\frac{\sin(x^3)}{x^2}$

(2) $\lim_{x\to \frac{\pi}{2}}\frac{\cos(x)}{x}$

(3) $\lim_{x\to 0}\frac{\mathrm{Tan}(x)}{x^2}$

(4) $\lim_{x \to 0} \frac{\sin(x)^3}{x^2}$

PROBLEM 3.3.3

Apply the ``Pinching Theorem" to find the following limits. Sketch the graph of the functions.

(1) $\lim_{x \to \infty} \frac{\sin(x)}{x}$

(2) $\lim_{x \to 0} x \sin\left(\frac{1}{x}\right)$