Problems in Mathematics
&
Experiments with Mathematica

4. The derivative and its applications

4.1 Definition, geometrical meaning, tangent line

Theory

DEFINITION 4.1.1

If the function *f(x)* is defined at and around the place x_0, and there exists the limit

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0},$$

then this limit is the differential quotient (derivative) of *f(x)* at x_0. If it exists at every point of the interval *(a, b)*, then *f(x)* is called differentiable on the interval *(a, b)*.

- **Notations**

$$f'(x), \quad \frac{df}{dx}$$

- **Geometrical meaning**

The quotient $\frac{f(x)-f(x_0)}{x-x_0}$ is called difference quotient, and it is the slope of the straight line passing through the points $(x_0, f(x_0))$ and *(x, f(x))*.

If there exists a limit position of the secant lines as $x \to x_0$, then it is the tangent line to the graph of $f(x)$ passing through the point $(x_0, f(x_0))$.

The equation of the tangent is

$$y = f(x_0) + (x - x_0) f'(x_0).$$

DEFINITION 4.1.2 Higher order derivatives

If the derivative function *f'(x)* is also differentiable, then *(f'(x))'=f''(x)* is the second derivative of $f(x)$. Similarly, if the n-th derivative is differentiable, then $(f^{(n)}(x))' = f^{(n+1)}(x)$ is the (n+1)-th derivative of *f(x)*.

Links

Differentiation rules

Linear approximation

Taylor polynomials

Exercises and problems

Mathematica initialization

```
SetDirectory["FileName" /.
    NotebookInformation[EvaluationNotebook[]] /.
    Which[
            $System == "Microsoft Windows",
                FrontEnd`FileName[{_, d___}, nam_, ___] :>
     ToFileName[{d}] ,
             $System != "Microsoft Windows", FrontEnd`FileName[
                d_List, nam_, ___]     :> ToFileName[d] ]
];
```

```
<< "package//plotslop.m"
```

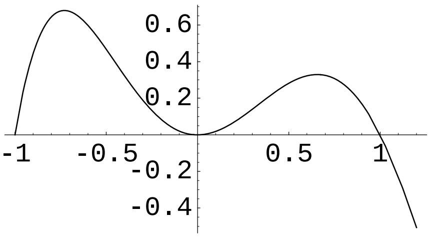
```
<< package//npow.m;
```

```
Abs'[x_] := Which[x < 0, -1, x > 0 , 1, True, 0]
Sign'[x_] := 0
```

```
tangent[x_, x0_] := f[x0] + f'[x0] (x - x0)
```

SOLVED PROBLEM 4.1.1 Graphical finding of the derivative

Sketch the graph of the derivative of the following graphically given function.



○ SOLUTION

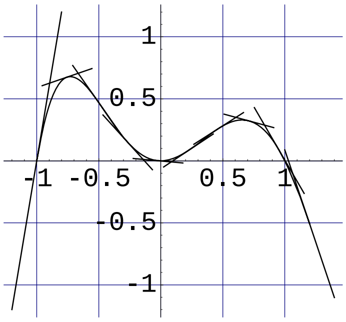
The *Mathematica* statements used are

```
xmin = -1;
xmax = 1.2;
f[x_] = (x + 1) x^2 (x - 1) (x - 2);
```

```
pl1 = Plot[f[x], {x, xmin, xmax}, AspectRatio -> Automatic];
```

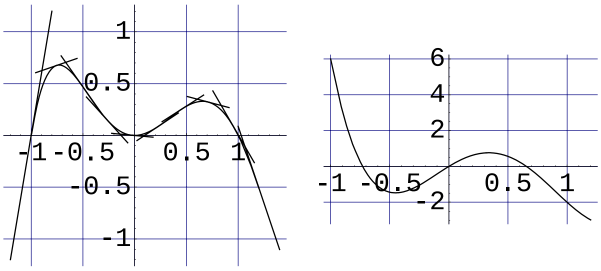
Draw tangent to "significant" points of the graph.

```
plsl = Show[pl1,
PlotSlope[f[x], {x, xmin, xmax, (xmax - xmin)/9}, DxSlope → 0.2,
              AspectRatio -> Automatic],
        DisplayFunction → $DisplayFunction, GridLines → Automatic];
```



Estimate the slope of the tangent lines, and plot the values in the Cartesian system. Join the points.

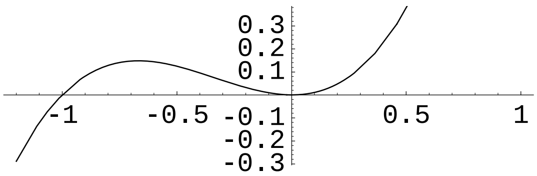
```
pl2 = Plot[f′[x], {x, xmin, xmax},
        DisplayFunction → Identity, GridLines → Automatic];
Show[GraphicsArray[Partition[{plsl, pl2}, 2]],
 DisplayFunction → $DisplayFunction];
```



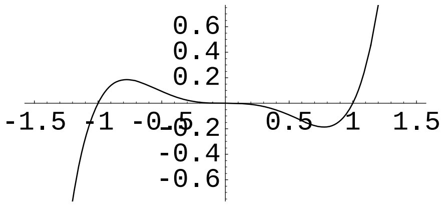
PROBLEM 4.1.2

Based on the previous exercise, sketch the graph of the derivative of the functions given below. The differentiability is not guaranted at every point. Take care of the possible breaks and tears.

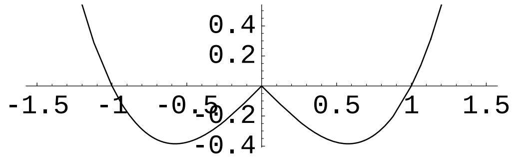
(1)



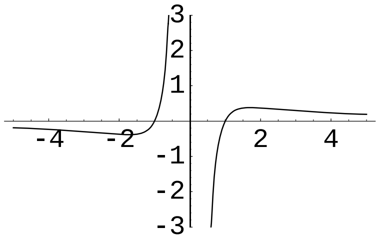
(2)



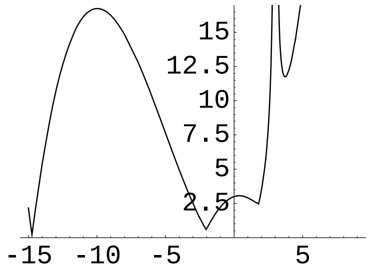
(3)



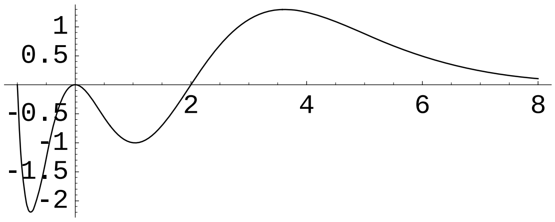
(4)



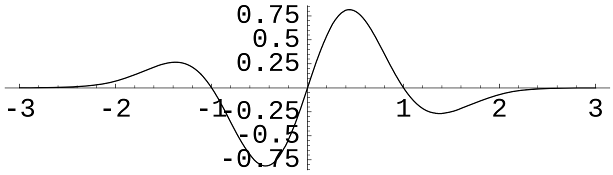
(5)



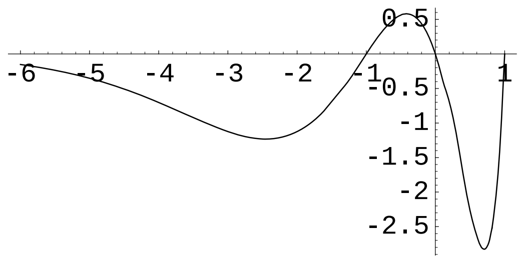
(6)



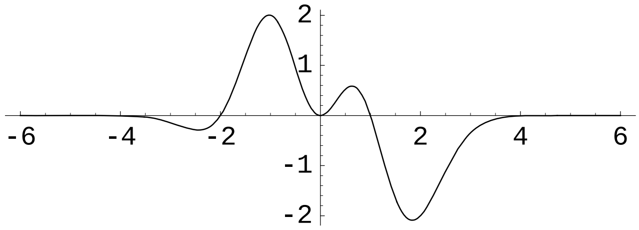
(7)



(8)

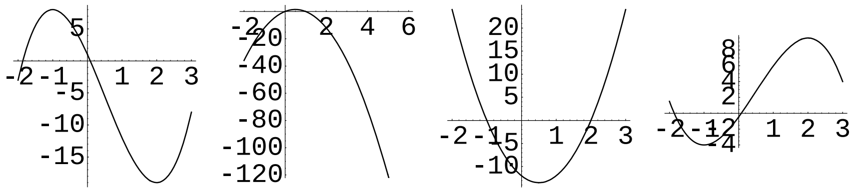


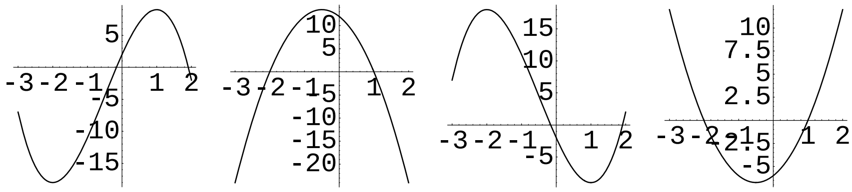
(9)

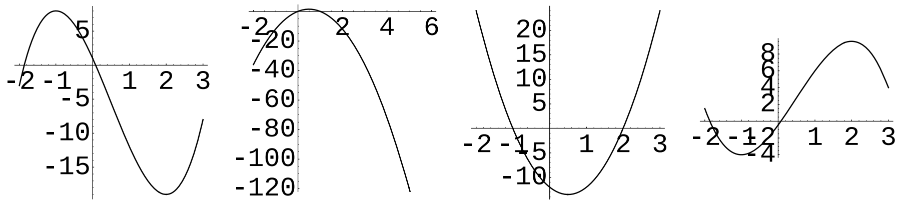


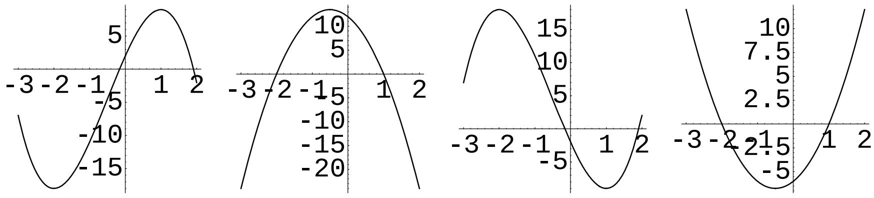
PROBLEM 4.1.3

The first figure contains the graph of *f(x)*. Choose its derivative from the other graphs.

(1) 

(2) 

(3) 

(4) 

SOLVED PROBLEM 4.1.4

Find the equation of the tangent line to the point $(x_0,\ f(x_0))$ of the graph of the function $f(x)$. Draw the graphs with the help of *Mathematica*.

$$f(x\_) := x^2 - x - 2;\ x_0 = 1;$$

○ SOLUTION

The value of $f(x)$ at x_0:

```
f[x0]
```

```
-2
```

The value of $f'(x)$ at x_0:

```
f'[x0]
```

```
1
```

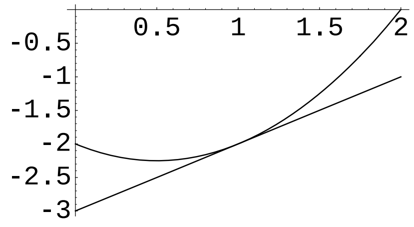
The derivative is calculated using the differentiation rules. See the chapter [Differentiation rules.]

The equation of the tangent is

```
y = tangent[x, x0]
```

```
-3 + x
```

```
Plot[{f[x], y}, {x, x0 - 1, x0 + 1}];
```



Problem 4.1.5

Find the equation of the tangent line to the point $(x_0, f(x_0))$ for the function $f(x)$. Draw the graphs with the help of *Mathematica.*

(1) $f(x\_) := x^3 - x^2 + 1;\ x_0 = -2;$

(2) $f(x\_) := -x^4 + 2x^2 + x - 1;\ x_0 = -1;$

(3) $f(x\_) := \dfrac{-x^4 + x^2 - 2x}{x - 1};\ \ x_0 = 3;$

Problem 4.1.6

Certify that the parabolas $x^2 - 4y = 4$ and $x^2 + 16y = 64$ intersect each other at angle $\pi/2$.

Problem 4.1.7

Find the point of the parabola $y = 2x - x^2$ where the tangent line is parallel to the secant line that crosses the graph $x_1 = 1$ and x_2=3.

Problem 4.1.8

Certify that the parabolas $x^2 - 4y = 4$ and $x^2 + 16y = 64$ intersect each other at degree $\pi/2$.

Problem 4.1.9

Find the angle between the graphs of the following functions.

(1) $f(x\_) := 0.5x^2;\ g(x\_) := 4 - 0.5x^2;$

(2) $f(x\_) := x^2 + 4;\ g(x\_) := (x + 2)^2;$

(3) $f(x\_) := x^2 + 4;\ g(x\_) := x^2 + 4x;$

(4) $f(x\_) := \sqrt{|x|};\ g(x\_) := \dfrac{x^3}{8};$

Problem 4.1.10

What are the points of the curve $y = x^3 - x^2$, where the angle between the tangent line and the x-axis is equal to $\pi/4$ radian?

PROBLEM 4.1.11

Find the tangent line of the curve $y = x^3$ that crosses the curve at the point $P = (1,1)$.

PROBLEM 4.1.12

Give the numbers a and b such that the parabola $y = ax^2 + bx + c$ intersects the point $P=(2,4)$, and the straight line $2x - y=1$ is its tangent at $x_0 = 1$.

PROBLEM 4.1.13

See further problems in the chapter Linear approximations.