Content

Problems in Mathematics
&
Experiments with Mathematica

4. The derivative and its applications

4.3 Linear approximation

Theory

THEOREM 4.3.1

If the function f is differentiable at and around the place x_0, then the tangent line at x_0 (the equation is $y = f(x_0) + f'(x_0)(x - x_0)$) is the best approximating straight line among the lines of form $y = f(x_0) + m(x - x_0)$ passing through the point $(x_0, f(x_0))$ as $x \to x_0$.

- *The error of the approximation*

If the slope of the equation $y = f(x_0) + m(x - x_0)$ is not equal to $f'(x_0)$, then the error is proportional with the expression $|(m - f'(x_0))(x - x_0)|$, i.e., it is linear. If the slope is equal to $f'(x_0)$, then the error is of higher order. In addition, if *f(x)* is twice differentiable on some interval $[x_0 - h, x_0 + h]$, then the error can be estimated as follows:

$$|f(x) - f(x_0) - f'(x_0)(x - x_0)| \le \frac{1}{2}\left\{\operatorname*{Max}_{u \in [x_0 - h, x_0 + h]} \left|f''(u)\right|\right\} |x - x_0|$$

Exercises and problems

Mathematica initialization

```
SetDirectory["FileName" /.
    NotebookInformation[EvaluationNotebook[]] /.
    Which[
            $System == "Microsoft Windows",
                FrontEnd`FileName[{_, d___}, nam_, ___] :>
     ToFileName[{d}] ,
             $System != "Microsoft Windows", FrontEnd`FileName[
                d_List, nam_, ___]     :> ToFileName[d]] ];

<< "package//plotslop.m"

<< package//npow.m;

Abs'[x_] := Which[x < 0, -1, x > 0 , 1, True, 0]
Sign'[x_] := 0
```

```
tangent[x_, x0_] := f[x0] + f'[x0] (x - x0)
```

SOLVED PROBLEM 4.3.1

Find the equation of the tangent line to the graph of the function *f(x)* at the point $(x_0,\ f(x_0))$. Draw the graphs with the help of *Mathematica*. Approximate the function with the tangent line at x_1. Calculate the error.

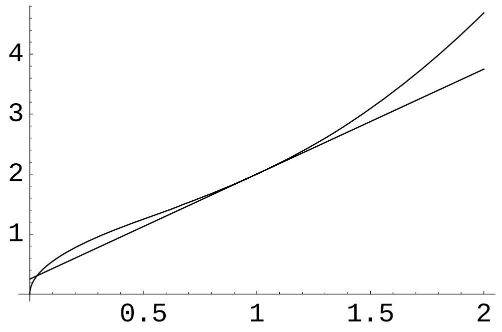
$$f(\text{x\_}) := \sqrt{x^4 + 3x};\ x_0 = 1;\ \ x_1 = 1.01;$$

○ SOLUTION

```
y = tangent[x, x0]
```

$2 + \frac{7}{4}\ (-1 + x)$

```
Plot[{f[x], y}, {x, x0 - 1, x0 + 1}];
```




```
error = f[x1] - tangent[x1, x0]
```

0.000073793

○

PROBLEM 4.3.2

Follow the instructions in the above solved problem for the following functions.

(1) $f(\text{x\_}) := \sqrt{x^2 - 3x};\ x_0 = -1;\quad x_1 = -0.97;$

(2) $f(\text{x\_}) := \sqrt{5x - x^2};\ x_0 = 1;\quad x_1 = 0.95;$

(3) $f(\text{x\_}) := \sqrt{x^3 + \frac{x}{2}};\ x_0 = 2;\quad x_1 = 2.01;$

(4) $f(\text{x\_}) := \sin(x);\ x_0 = 0;\ x_1 = 0.1;$

(5) $f(\text{x\_}) := \sin(x);\ x_0 = \frac{\pi}{3};\ x_1 = \frac{\pi}{3} + 0.05;$

(6) $f(\text{x\_}) := \log(x);\ x_0 = 1;\ x_1 = 1.1;$

(7) $f(\text{x\_}) := 2^x;\ x_0 = 0;\ x_1 = 0.5;$

(8) $f(\text{x\_}) := \sqrt{x^3 + 1};\ x_0 = 2;\ x_1 = 2.1;$

(9) $f(\mathrm{x\_}) := \sqrt{5\,x^2 - x^4};\ x_0 = 1;\ x_1 = 0.9;$