Problems in Mathematics & Experiments with Mathematica

4. The derivative and its applications

4.2 Differentiation rules

Summary of the rules

- **Rules for operation on functions**

$$
\begin{array}{ccc}
(c\,f(x))' & = & c f'(x) \\
(f(x)+g(x))' & = & f'(x)+g'(x) \\
(f(x)\,g(x))' & = & f'(x)\,g(x)+f(x)\,g'(x) \\
\left(\frac{1}{f(x)}\right)' & = & -\frac{f'(x)}{f^2(x)} \\
\left(\frac{f(x)}{g(x)}\right)' & = & \frac{f'(x)\,g(x)-f(x)\,g'(x)}{g^2(x)} \\
(f(g(x)))' & = & f'(g(x))\,g'(x) \\
(f^{-1}(x))' & = & \frac{1}{f'(f^{-1}(x))}
\end{array}
$$

- **Derivative of elementary functions**

The rule	StandardForm	TraditionalForm
$(x^\alpha)' = \alpha\, x^{\alpha-1}$	D[x^α, x]	$\frac{\partial x^\alpha}{\partial x}$
$(\sin x)' = \cos x$	Sin'[x]	$\frac{\partial \sin(x)}{\partial x}$
$(\cos x)' = -\sin x$	Cos'[x]	$\frac{\partial \cos(x)}{\partial x}$
$(tg\,x)' = \frac{1}{\cos^2 x}$	Tan'[x]	$\frac{\partial \tan(x)}{\partial x}$
$(ctg\,x)' = -\frac{1}{\sin^2 x}$	Cot'[x]	$\frac{\partial \cot(x)}{\partial x}$
$(a^x)' = a^x \ln a$	D[a^x, x]	$\frac{\partial a^x}{\partial x}$
$(\log_a x)' = \frac{1}{x \ln a}$	D[Log[a, x], x]	$\frac{\partial \log_a(x)}{\partial x}$

Mathematica statements

Inputform	Trad.Form	Meaning
D[f, x]	$\frac{\partial f}{\partial x}$	The first derivative of the expression *f* by the variable *x*
D[f, {x, 2}]	$\frac{\partial^2 f}{\partial x^2}$	The second derivative of the expression *f* by the variable *x*
D[f, {x, n}, {y, m}]	$\frac{\partial^{m+n} f}{\partial x^n \partial y^m}$	The (n + m) – th partial derivative by *x* and *y*.
f'[x]	$\frac{\partial f}{\partial x}$	The first derivative of the function f[x]
f''[x]	$\frac{\partial^2 f}{\partial x^2}$	The second derivative of f[x]

Exercises and problems

PROBLEM 4.2.1

Apply the rules for sums, products and fractions.

(1) $f(x\_) := 10\,x^2;\quad g(x\_) := 0.5\,\sqrt{x};\quad h(x\_) := -6\,x^2 + 4\,x - \log(6)$

(2) $f(x\_) := 3\,x^5 + \log(3)\,x^3 - 6\,x^2 - x + \frac{1}{4}\,(x+2)^1 + \sin\left(\frac{\pi}{6}\right)$

(3) $f(x\_) := 3\,x + \sin(x) + \sqrt{2};\quad g(x\_) := \frac{x}{3} - \frac{3}{x};\quad h(x\_) := 3\,x^2 + \sqrt{x};$

(4) $f(x\_) := \frac{1}{x} - \frac{x^2}{4};\quad g(x\_) := x\sin(x);\quad h(x\_) := x\,(2\,x+3);$

(5) $f(x\_) := \frac{1}{x} - \frac{1}{3}\,(2\,x^2 - 7\,x)\,x;\quad g(x\_) := \tan(x)\cos(x);$

(6) $f(x\_) := 3^x - \frac{1}{2}\,x^2\log(3);\quad g(x\_) := 53^x - x^{53};\quad h(x\_) := x^2\sin(x);$

(7) $f(x\_) := \frac{-x^2\log(2) + x^{1/3} - \log(2)}{x^{1/2}};\quad f(x\_) := \frac{\sin(x)}{\cos(x)};$

(8) $f(x\_) := \tan(x)\,10^x;\quad g(x\_) := \frac{e^x}{x+2};\quad g(x\_) := 3\,x^3\cot(x);$

(9) $f(x\_) := -\log(\pi)\log(x) + \sin(\pi)\log(x) - x\sin(\pi) + \log(\pi)\sin(x);$

(10) $f(x\_) := \frac{\cos(x)}{e^x} - \frac{1}{3}\,x\tan(2);\quad g(x\_) := 2^x\,(1 - x^3);$

(11) $f(x\_) := 3^{-x}\,(\sin(x) - x)$; $g(x\_) := 2^{x}\,(1 - x^{3})$;

PROBLEM 4.2.2

Apply the chain rule.

(1) $f(x\_) := \sin(2\,x)$; $g(x\_) := \cos(x^{2})$; $h(x\_) := \cos^{2}(x)$;

(2) $f(x\_) := e^{-x^{2}}$; $g(x\_) := \sqrt{\log(x^{2}+1)}$;

(3) $f(x\_) := 0.5\sin(x^{2}+\log(x)) - \frac{1}{2}\sqrt{3}\cos(x^{2}+\log(x))$;

(4) $f(x\_) := \sin^{-1}(x^{2})$; $g(x\_) := \cos^{-1}\left(\sqrt{x}\right)$; $h(x\_) := \tan^{-1}\left(\frac{x}{10}\right)$;

(5) $f(x\_) := e^{\sin(x^{2})}$; $g(x\_) := \tan^{-1}(\cos(x))$; $h(x\_) := \sin(3\,x) - 3\cos(x)$;

(6) $f(x\_) := \log(\log(x))$; $g(x\_) := \sin(\log(\sin(x)))$;

(7) $f(x\_) := \frac{1}{x^{6}+x^{2}+\frac{1}{x^{3}}}$; $f(x\_) := \frac{1}{2}\left(x^{2}+\frac{1}{x^{2}}\right)^{3}$;

(8) $f(x\_) := x^{x}$; $g(x\_) := \sin^{\frac{1}{\cos(x)}}(x)$; $h(x\_) := x^{1/x}$;

PROBLEM 4.2.3

Differentiate the following functions.

(1) $f(x\_) := \frac{(x+e^{x})^{100}}{\cos(x)}$; $g(x\_) := \frac{(x^{2}+1)\,e^{\sin(x)}}{x-1}$;

(2) $f(x\_) := \frac{(x-1)\cos(e^{x})}{x^{2}-x}$; $g(x\_) := \frac{(x^{2}+e^{-x})^{99}}{\cot(x)}$

(3) $f(x\_) := \sin(\log(x) + 0.5^{x})$; $g(x\_) := (\log(x)\cos(x))^{9}$;

(4) $f(x\_) := \cos(5^{x}\log_{10}(x))$; $g(x\_) := \log(10^{x\cos(x)})$;

(5) $f(x\_) := 3^{x}\,(1 - \tan(x))$; $g(x\_) := 3^{x}\,(1 - \log(x))$;

(6) $f(x\_) := 2^{x}\,(1 - \cos(x))$; $g(x\_) := 2^{x}\,(1 - \cos(\log(x)))$;

(7) $f(x\_) := 10^{-x}\left(1 - \frac{1}{x}\right)$; $g(x\_) := \sqrt[3]{1-x^{5}}$; $h(x\_) := \sqrt{\sin(x)+1}$;

(8) $f(x\_) := \sqrt{1-x^{2}}$; $g(x\_) := \sqrt{x^{3}+x}$; $h(x\_) := \frac{\sin(x^{2})\,(x-1)}{x^{2}-x-1}$;

(9) $f(x\_) := \cos(x)\,10^{x+\log_{10}(x-2)}$; $g(x\_) := \tan(x)\log_{10}\left(x\,10^{x-\frac{3}{x}}\right)$;

(10) $f(x\_) := \frac{\log^{2}(x)}{x^{3}-x^{2}-x}$; $g(x\_) := \frac{\sin(x^{2})\,(x-1)}{x^{2}-x-1}$;

PROBLEM 4.2.4

Differentiate the following functions, piecewise if necessary, and find the domain of the function and the derivative.

(1) $f(x\_) := \log(|x|); \quad g(x\_) := \sqrt{-x^2}; \quad h(x\_) := \sqrt[3]{x};$

(2) $f(x\_) := |x^2 - x^3|; \quad g(x\_) := \sin\left(\frac{1}{x}\right); \quad h(x\_) := x^2 \sin\left(\frac{1}{x}\right);$