Content

Problems in Mathematics & Experiments with Mathematica

9. Functions of several variables

9.3 Finding a local extremum

Theoretical basis

THEOREM 9.3.1 Necessary condition for local extremum

Let the function $f(x,y)$ be differentiable at (x_0, y_0). If $f(x,y)$ has a local extremum at (x_0, y_0), then

$$\left(\left[\frac{\partial f(x,y)}{\partial x}\right]\right)_{(x,y)=(x_0,y_0)} = \left(\left[\frac{\partial f(x,y)}{\partial y}\right]\right)_{(x,y)=(x_0,y_0)} = 0$$

The points that satisfy this equality are called critical points.

- *Remark.*

The conversion is not true. Remember the function $f(x, y) = x^2 - y^2$ at $(0,0)$.

THEOREM 9.3.2 A sufficient condition for local extremum

Let the function $f(x,y)$ be defined in a neighborhood (x_0, y_0), the first and second order partial derivatives exist and are continuous at (x_0, y_0). Assume that (x_0, y_0) is a critical point, that is it satisfies the relation

$$\left(\left[\frac{\partial f(x,y)}{\partial x}\right]\right)_{(x,y)=(x_0,y_0)} = \left(\left[\frac{\partial f(x,y)}{\partial y}\right]\right)_{(x,y)=(x_0,y_0)} = 0.$$

1. Assume that

$$\left[\frac{\partial^2 f(x, y)}{\partial x^2}\,\frac{\partial^2 f(x, y)}{\partial y^2} - \left(\frac{\partial^2 f(x, y)}{\partial y\,\partial x}\right)^2\right]_{(x,y)=(x_0,y_0)} > 0.$$

If $\left[\frac{\partial^2 f(x,y)}{\partial x^2}\right]_{(x,y)=(x_0,y_0)} > 0$, then $f(x,y)$ has a local minimum at (x_0, y_0).

If $\left[\frac{\partial^2 f(x,y)}{\partial x^2}\right]_{(x,y)=(x_0,y_0)} < 0$, then $f(x,y)$ has a local maximum at (x_0, y_0).

2. If

$$\left[\frac{\partial^2 f(x, y)}{\partial x^2}\,\frac{\partial^2 f(x, y)}{\partial y^2} - \left(\frac{\partial^2 f(x, y)}{\partial y\,\partial x}\right)^2\right]_{(x,y)=(x_0,y_0)} < 0,$$

then $f(x,y)$ has neither local minimum nor maximum at (x_0, y_0).

THEOREM 9.3.3 A special case for polinomials

Let $f(x, y) = a_1(x + b_1)^{2n} + a_2(y + b_2)^{2m}$, where n and m are positive integers. If $a_1>0$ and $a_2 > 0$, then $f(x,y)$ has a local minimum. If $a_1 < 0$ and $a_2 < 0$, then $f(x,y)$ has a local maximum. If $a_1 a_2 < 0$, then $f(x,y)$ is a saddle.

Exercises and problems

SOLVED PROBLEM 9.3.1

Find the critical points of the function

```
f[x_, y_] = (x - 2)^4 + (y - 5)^2;
```

○ SOLUTION

Calculate the first order partial derivatives, and solve the system

$$\frac{\partial f(x,y)}{\partial x} = 0; \quad \frac{\partial f(x,y)}{\partial y} = 0.$$

```
dx = D[f[x, y], x]
dy = D[f[x, y], y]
```

$4\,(-2 + x)^3$

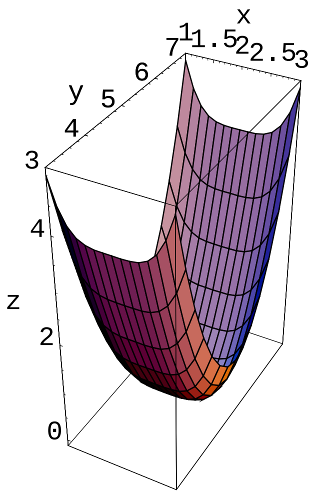
$2\,(-5 + y)$

```
Solve[{dx == 0, dy == 0}, {x, y}]
```

{{x → 2, y → 5}, {x → 2, y → 5}, {x → 2, y → 5}}

Finally, plot the graph around the obtained point:

```
Plot3D[f[x, y], {x, 1, 3}, {y, 3, 7},
  BoxRatios → Automatic, AxesLabel → {x, y, z}];
```



- *Find the critical points using the* `FindRoot` *command*

We read initial values for the critical point from the graph:

```
{x0, y0} = {2.5, 4};
FindRoot[{dx == 0, dy == 0}, {x, x0}, {y, y0}]
{x → 2.00578, y → 5.}
```

PROBLEM 9.3.2

Find the critical points of the function. Use the theorems and/or experiments to decide if they are minimum or maximum points.

(1) $f(x\_, y\_) := x^2 + y^2 - 3\,x\,y$

(2) $f(x\_, y\_) := x^4 - y^3 - x^2\,y$

(3) $f(x\_, y\_) := x^2 - y^2 - 3\,x^2\,y^2$

(4) $f(x\_, y\_) := \sin(x\,y)$

(5) $f(x\_, y\_) := \cos(y^2 + x)$

(6) $f(x\_, y\_) := x\,e^{x^2+y^2}$

(7) $f(x\_, y\_) := y\,e^{x^2-y^2}$

(8) $f(x\_, y\_) := -x^2 + 2\,y\,x + x + 2\,y^2 + y + 6$

(9) $f(x\_, y\_) := x^2 + 2\,x + 2\,y^2 - y - 2\,x\,y$

(10) $f(x\_, y\_) := -x^2 + 2\,y\,x + 2\,x + 2\,y^2 + 7\,y + 6$

(11) $f(x\_, y\_) := x^2 + 2\,x + 2\,y^2 - y - x\,y;$