Problems in Mathematics & Experiments with Mathematica

5. Indefinite Integral

5.2 Integration techniques

Summary of the rules

- **Integral of elementary functions**

Remember that the *Mathematica* notation sometimes different form the traditional notation. See the chapter A short introduction to *Mathematica*.

$$\int x^{\alpha}\, dx = \frac{x^{\alpha+1}}{\alpha+1} + C \qquad (\alpha \neq -1)$$

$$\int \frac{1}{x}\, dx = \log|x| + C$$

$$\int \sin x\, dx = C - \cos x$$

$$\int \cos x\, dx = C + \sin x$$

$$\int \frac{1}{\cos^2 x}\, dx = C + \tan x$$

$$\int \frac{1}{\sin^2 x}\, dx = C - \operatorname{cotan} x$$

$$\int a^{x}\, dx = \frac{a^{x}}{\ln a} + C$$

• Rules for operation on functions

$\int c f = c \int f$	multiplication by *a* constant
$\int (f + g) = \int f + \int g$	sum
$\int f' g' = f g - \int f g'$	integration by parts (conversion of the differentiation of *a* product)
$\int f(g(x))\, g'(x)\, dx = \int f(y)\, dy$ $(y = g(x),$ $dy = g'(x)\, dx)$	integration by substitution (conversion of the chain rule)
$\int \frac{f'(x)}{f(x)}\, dx =$ $\log \lvert f(x) \rvert + C$	substitution : $y = f(x)$
$\int f^n(x)\, f'(x)\, dx =$ $\frac{1}{n+1} f^{n+1}(x) + C$	substitution : y = f(x)

• Integration of rational fractions. Theorem of expansion to fractions with minimal denominators

Let $P_n(x)$ and $Q_m(x)$ are polynomials, where $n<m$. Let

$$Q_m(x) = \left(\prod_{k=0}^{m}{}_1 (x - x_k)^{\alpha_k}\right)\left(\prod_{l=0}^{m}{}_2 (x^2 + a_l)^{\beta_l}\right),\ (m_1 + 2\, m_2 = m)$$

Then

$$\frac{P_n[x]}{Q_m[x]} = \sum_{k=0}^{m}{}_1 \left(\sum_{i=1}^{\alpha}{}_k \frac{A_{k,i}}{(x - x_k)^i}\right) + \sum_{l=0}^{m}{}_2 \left(\sum_{j=1}^{\beta}{}_l \frac{A_{l,j}}{(x - x_l)^j}\right).$$

Mathematica interpretation: `Apart` function.

- *Examples.*

```
Apart[1/(x (x + 1))]
```

$$\frac{1}{x} - \frac{1}{1 + x}$$

```
Apart[1/((x - 1) (x + 2))]
```

$$\frac{1}{3\ (-1 + x)} - \frac{1}{3\ (2 + x)}$$

The Mathematica statements

InputForm	Trad. Form	Meaning
Integrate[f[x], x]	$\int f(x)\,dx$	The indefinite integral of $f(x)$

Exercises and problems

PROBLEM 5.2.1 Use elementary techniques.

(1) $\int 10\,x^2\,dx$

(2) $\int 0.5\,\sqrt{x}\,dx$

(3) $\int (-6\,x^2 + 4\,x - \log(6))\,dx$

(4) $\int \left(\frac{x}{3} - \frac{3}{x}\right) dx$

(5) $\int \left(3\,x^5 + x^3\log(3) - 6\,x^2 - x + \frac{1}{4}(x+2)^3 + \sin\left(\frac{\pi}{6}\right)\right) dx$

(6) $\int \left(3\,x + \sin(x) + \sqrt{2}\right) dx$

(7) $\int \frac{3\,x^2 + \sqrt{x}}{\sqrt[3]{x}}\,dx$

(8) $\int \left(\frac{1}{x} - \frac{x^2}{4}\right) dx$

(9) $\int x\,(2\,x+3)\,dx$

(10) $\int \left(\frac{1}{x} - \frac{1}{3}(2\,x^2 - 7\,x)\,x\right) dx$

(11) $\int \tan(x)\cos(x)\,dx$

(12) $\int (53^x - x^{53})\,dx$

(13) $\int \left(3^x - \frac{1}{2}\,x^2\log(3)\right) dx$

(14) $$\int \frac{-x^2 \log(2) + \frac{1}{3} x - \log(2)}{x^{12}}\, dx$$

(15) $$\int (-\log(\pi)\log(x) + \sin(\pi)\log(x) - x\sin(\pi) + \log(\pi)\sin(x))\, dx$$

(16) $$\int \left(x^2 + \frac{1}{x^2}\right)^2 dx$$

(17) $$\int \frac{x^2 + x + 2}{\sqrt{x^3}}\, dx$$

(18) $$\int \frac{x^3 - 2x + 1}{\sqrt[3]{x^2}}\, dx$$

(19) $$\int x^2 \left(x - \frac{1}{\sqrt[5]{x}}\right) dx$$

(20) $$\int x^2 \left(\sqrt{x} - \frac{1}{x}\right) dx$$

(21) $$\int \frac{x^2 + 2x - 1}{x^3}\, dx$$

(22) $$\int x^4 \left(1 - \frac{1}{\sqrt[3]{x}}\right) dx$$

(23) $$\int \frac{x^3 + 3x - 1}{x^2}\, dx$$

(24) $$\int x^2 \left(1 - \frac{1}{\sqrt[5]{x}}\right) dx$$

$$\int x^2\, dx - \int x^2 \frac{1}{\sqrt[5]{x}}\, dx$$

$$-\frac{5\, x^{14/5}}{14} + \frac{x^3}{3}$$

PROBLEM 5.2.2 Integrate by substitution.

(1) $$\int \frac{\sin(x)}{\cos(x)}\, dx$$

(2) $$\int \sin(2x)\, dx$$

(3) $$\int x\cos(x^2)\, dx$$

(4) $\int \sin(x)\ \cos^2(x)\, dx$

(5) $\int x\, e^{-x^2}\, dx$

(6) $\int \cos^{-1}\left(\sqrt{x}\right) dx$

(7) $\int \tan^{-1}\left(\frac{x}{10}\right) dx$

(8) $\int e^{\sin(x^2)}\, x \cos(x^2)\, dx$

(9) $\int \frac{1}{x \log(x)}\, dx$

(10) $\int \cos(\log(\sin(x)))\, \cot(x)\, dx$

PROBLEM 5.2.3 Integrate by parts.

(1) $\int x \sin(x)\, dx$

(2) $\int x^2 \sin(x)\, dx$

(3) $\int e^x (x^2 - x)\, dx$

(4) $\int \cos(x)\, 10^x\, dx$

(5) $\int 3\, x^3 \cos(x)\, dx$

(6) $\int \left(\frac{\cos(x)}{e^x} - \frac{x}{3} \tan(2)\right) dx$

SOLVED PROBLEM 5.2.4

Expand the following fraction to fractions with minimal denominators.

$$f(x\_) := \frac{1}{x\,(x-3)\,(x+2)}$$

- SOLUTION

  - *The way of manual expansion*

    Expand

$$f[x\_] := \frac{1}{x\,(x-3)\,(x+2)}$$

Consider the equation

```
eqn = 1/(x (x - 3) (x + 2)) == A/x + B/(x - 3) + C/(x + 2)
```

$$\frac{1}{(-3+x)\,x\,(2+x)} == \frac{B}{-3+x} + \frac{A}{x} + \frac{C}{2+x}$$

```
Together[eqn[[2]]]
```

$$\frac{-6A - Ax + 2Bx - 3Cx + Ax^2 + Bx^2 + Cx^2}{(-3+x)\,x\,(2+x)}$$

The coefficients on the left- and right-hand sides must be identical. We obtain a system of linear equations for the variables A, B and C.

```
Collect[Numerator[Together[eqn[[2]]]], {x, x^2}]
```

$$-6A + (-A + 2B - 3C)\,x + (A + B + C)\,x^2$$

```
Solve[{-6 A == 1, -A + 2 B - 3 C == 0, (A + B + C) == 0}, {A, B, C}]
```

$$\left\{\left\{A \to -\frac{1}{6},\ B \to \frac{1}{15},\ C \to \frac{1}{10}\right\}\right\}$$

Compare the obtained coefficients with the result obtained by *Mathematica*:

```
Apart[f[x]]
```

$$\frac{1}{15\,(-3+x)} - \frac{1}{6x} + \frac{1}{10\,(2+x)}$$

PROBLEM 5.2.5

Expand the following fraction to fractions with minimal denominators.

(1) $\dfrac{1}{x\,(x-3)\,(x+2)}$

(2) $\dfrac{2x+1}{x\,(1-x)^2\,(x+3)}$

(3) $\dfrac{x^2+x-1}{x\,(x-3)^2\,(x^2+3)}$

(4) $\dfrac{1}{(x+2)\,(x^2+2)}$

SOLVED PROBLEM 5.2.6 Use the method of minimal denominators

$$\int \frac{1}{x\,(x-1)^2}\,dx$$

○ SOLUTION

Expand the fraction to partial fractions

```
Apart[1 / (x (x - 1)^2 (x^2 + 2) (x^2 + 3)^2)]
```

$$\frac{1}{48\,(-1+x)^2}-\frac{1}{18\,(-1+x)}+\frac{1}{18\,x}+\frac{4+x}{18\,(2+x^2)}+\frac{-3-x}{24\,(3+x^2)^2}+\frac{-27-8\,x}{144\,(3+x^2)}$$

Now, each term can be integrated independently.

$$\int \frac{1}{x\,(x-1)^2}\,dx$$

$$-\frac{1}{-1+x}-\text{Log}[-1+x]+\text{Log}[x]$$

○

PROBLEM 5.2.7

(1) $\int \frac{1}{(-x+2)\,(x-1)}\,dx$

(2) $\int \frac{1}{x\,(x+2)^2}\,dx$

(3) $\int \frac{x+2}{(x+3)\,(x-1)}\,dx$

(4) $\int \frac{x^2+x}{x\,(x-1)^2}\,dx$

PROBLEM 5.2.8 Mixed problems

(1) $\int \cos(x)\left(x-\frac{1}{(\sin(x)-1)^3}\right)dx$

(2) $\int e^x\left(x-\frac{1}{(e^x+1)^2}\right)dx$

(3) $\int e^x\left(x-\frac{1}{e^x-1}\right)dx$

(4) $\int x\left(\log(x)-\sqrt{x^2+1}\right)dx$

(5) $\int x\left(\sin(x)+\frac{1}{\sqrt{x}+1}\right)dx$

(6) $$\int \sin(x)\left(\cos(x) - \frac{1}{(\cos(x)-1)^3}\right) dx$$