Content

Problems in Mathematics
&
Experiments with Mathematica

4. The derivative and its applications

4.5 L'Hospital rule to find limit of functions

Theory

The L'Hospital rule helps to find the indetermined limits of form $\frac{0}{0}$ or $\frac{\infty}{\infty}$. See the sections Basic rules and techniques for limits and Comparison of growth rates.

THEOREM 4.5.1

Let the functions f and g be differentiable at and in a neighborhood (x_1, x_2) of the point x_0 (i.e. $x_0 \in (x_1, x_2)$), and $g'(x) \neq 0$ in (x_1, x_2). Assume that

$$\lim_{x \to x_0} f(x) = 0 \text{ and } \lim_{x \to x_0} g(x) = 0$$

or

$$\lim_{x \to x_0} f(x) = \infty \quad \text{and} \quad \lim_{x \to x_0} g(x) = \infty.$$

If $\lim_{x \to x_0} \frac{f'(x)}{g'(x)}$ exists, then $\lim_{x \to x_0} \frac{f(x)}{g(x)}$ also exists and

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}.$$

The theorem also holds for half-side limits. In these cases, the conditions have to be satisfied in an appropriate half-side neighborhood of x_0.

Exercises and problems

SOLVED PROBLEM 4.5.1

Use the L'Hospital rule to find the following limits. Verify the calculations by findig the limits immediately with *Mathematica.*

(1) $\lim_{x \to \infty} \frac{e^x}{x}$

◆ **SOLUTION**

Since $\lim_{x\to\infty} e^x = \infty$ and $\lim_{x\to\infty} x = \infty$, using the L'Hospital rule, we obtain

$$\lim_{x\to\infty} \frac{e^x}{x} = \lim_{x\to\infty} \frac{(e^x)'}{(x)'} = \infty$$

The limit with *Mathematica*

$$\lim_{x\to\infty} \frac{e^x}{x}$$

∞

(2) $\lim_{x\to 0^+} x \log(x)$

○ **SOLUTION**

The limit is a product of indetermined form 0*∞. The function can be transformed to a fraction that satisfies the conditions of the L'Hospital rule.
Since $\lim_{x\to\infty} e^x = \infty$ and $\lim_{x\to\infty} x = \infty$, using the L'Hospital rule yields

$$\lim_{x\to 0^+} x \log(x) = \lim_{x\to 0^+} \frac{\log(x)}{\frac{1}{x}} = \lim_{x\to 0^+} \frac{(\log(x))'}{\left(\frac{1}{x}\right)'} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0^+} -x = 0.$$

The limit with *Mathematica* is

$$\lim_{x\to 0^+} x \log(x)$$

0

PROBLEM 4.5.2

Use the L'Hospital rule or other methods to find the following limits. Verify the calculations with *Mathematica*.

(1) $\lim_{x\to\infty} x\, e^{-x^2}$; $\lim_{x\to-\infty} x\, e^{-x^2}$;

(2) $\lim_{x\to\infty} \frac{e^{-x^2}}{x^2}$; $\lim_{x\to-\infty} \frac{e^{-x^2}}{x^2}$; $\lim_{x\to 0} \frac{e^{-x^2}}{x^2}$;

(3) $\lim_{x\to 0} \frac{\log(x)}{x}$; $\lim_{x\to\infty} \frac{\log(x)}{x}$;

(4) $\lim_{x\to 0} x \log(|x|)$; $\lim_{x\to-\infty} x \log(|x|)$;

(5) $\lim_{x\to 1} (x-1)^2 \log(x-1)$;

(6) $\lim_{x\to 0} \frac{\sin(x)}{x}$; $\lim_{x\to 0} \frac{\sin(x)}{x^3}$; $\lim_{x\to 0} \frac{\sin(x^2)}{x}$; $\lim_{x\to 0} \frac{\tan(x)}{x}$;

(7) $\lim_{x\to 0} \sin(x)\log(x)$; $\lim_{x\to \infty} \sin(x)\log(x)$;