Problems in Mathematics
&
Experiments with Mathematica

3. Limit and continuity

3.4 Simple and compound growths. The limit $\lim_{x\to 0}\left(1+\frac{1}{x}\right)^x = e$

Theory

- **Simple, arithmetic growth**

Let the initial amount of a material or a population be A_0. Let the amount A_1 after a fixed time (e.g. T) be defined by $A_1 = A_0 + d$. Then, at the n'th moment the amount will be $A_n = A_{n-1} + d = A_0 + n\,d$. This growth is called arithmetic since the sequence A_n is an arithmetic sequence.

- **Properties of the arithmetic growth**

If $d > 0$, then $\lim\limits_{n\to\infty} A_0 + n\,d = \infty$.
If $d < 0$, then $\lim\limits_{n\to\infty} A_0 + n\,d = -\infty$ (restricted meaning in life science models).

- **Geometric growth**

Let the initial amount of a material or a population be A_0. Let the amount A_1 after a fixed time (e.g. T) be defined by $A_1 = q\,A_0$. Then, at the n'th moment the amount will be $A_n = q\,A_{n-1} = q^n\,A_0$. This growth is called geometric since the sequence A_n is a geometric sequence.

- **Reformulation: compound growth**

Let the initial amount of a material or a population be A_0. Let the growth rate be p during a fixed length of time, that is, $A_1 = (1+p)\,A_0$. Then, at the n'th moment the amount will be $A_n = (1+p)\,A_{n-1} = (1+p)^n\,A_0$.

- **Properties of arithmetic and geometric growth**

If $q > 1$and $A_0 > 0$, then $A_{n+1} > A_n$ and $\lim_{n\to\infty} q^n\,A_0 = \infty$.
If $q = 1$, then $A_{n+1} = A_n$ and $\lim_{n\to\infty} q^n\,A_0 = 1$.
If $0<q<1$and $A_0 > 0$, then $0 < A_{n+1} < A_n$ and $\lim_{n\to\infty} q^n\,A_0 = 0$.
If $-1<q<0$, then $0 < |A_{n+1}| < |A_n|$ and $\lim_{n\to\infty} q^n\,A_0 = 0$. The sequence oscillates.
If $q = -1$, then $A_{n+1} = -A_n$.
If $q < 1$and $A_0 > 0$, then $|A_{n+1}| > |A_n|$ and $\lim_{n\to\infty} |q^n\,A_0| = \infty$. The sequence oscillates.

• Compound interest: a problem leading to e

Let the initial amount of our money at the bank be M_0, and let the annual interest be p, that is, our money at the end of the year is $M_1 = (1 + p) M_0$. Calculate the growth twice a year at interest rate $\frac{p}{2}$. Then, our money at the end of the year is $M_2 = (1 + \frac{p}{2})^2 M_0$. It is easy to see that $M_2 > M_1$. Count the growth theee times, ..., n-times a year. We obtain $M_n = (1 + \frac{p}{n})^n M_0$, and it can be proved that

$$M_0 < M_1 < M_2 < \ldots < M_{n-1} < M_n.$$

Can our money grow unbounded? NO! The sequence $\{M_n\}$ is bounded. Since it is monoton increasing and bounded (see the section Definitions,...), the limit

$$\lim_{n\to\infty} (1 + \frac{p}{n})^n$$

exists. In particular, we introduce the following definition:

DEFINITION 3.4.1

The sequence $(1 + \frac{1}{n})^n$ is bounded and monotone increasing. Its limit is called e.

$$e := \lim_{n\to\infty} (1 + \frac{1}{n})^n.$$

In more general,

$$e := \lim_{x\to\infty} (1 + \frac{1}{x})^x = \lim_{x\to 0} (1 + x)^{\frac{1}{x}}.$$

The sequence $(1 + \frac{1}{n})^{n+1}$ is bounded and monotone decreasing. Its limit is also e.

$$e = \lim_{n\to\infty} (1 + \frac{1}{n})^n.$$

An approximate value of e

$N(e, 100)$

```
2.718281828459045235360287471352662497757247093699959574966967
 627724076630353547594571382178525166427
```

• Compound interest, continued

By the definition of e, we obtain that

$$\lim_{n\to\infty} M_n = \lim_{n\to\infty} (1 + \frac{p}{n})^n M_0 = M_0\, e^p.$$

• Important remark

The definition is not used to approximate e since the convergence is very slow. Look at some members of the sequence

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a[n_] := (1 + 1/n)^n
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```
TableForm[Table[{2^n, N[a[2^n]], N[ⅇ - a[2^n]]}, {n, 1, 10}],
 TableHeadings -> {None, {"n", "a(n)", "ⅇ-a(n)"}}]
```

n	a(n)	ⅇ-a(n)
2	2.25	0.468282
4	2.44141	0.276876
8	2.56578	0.152497
16	2.63793	0.0803533
32	2.67699	0.0412917
64	2.69734	0.0209369
128	2.70774	0.0105428
256	2.71299	0.0052902
512	2.71563	0.00264983
1024	2.71696	0.0013261

A series approximation of form

$$\mathbf{aa[n\_] := \sum_{i=0}^{n} \frac{1}{i!}}$$

is uncomparably faster (see the section Taylor polynomials).

```
TableForm[Table[{n, N[aa[n]], N[ⅇ - aa[n]]}, {n, 1, 10}],
 TableHeadings -> {None, {"n", "aa(n)", "ⅇ-aa(n)"}}]
```

n	aa(n)	ⅇ-aa(n)
1	2.	0.718282
2	2.5	0.218282
3	2.66667	0.0516152
4	2.70833	0.0099485
5	2.71667	0.00161516
6	2.71806	0.000226273
7	2.71825	0.0000278602
8	2.71828	3.05862×10^{-6}
9	2.71828	3.02886×10^{-7}
10	2.71828	2.73127×10^{-8}

Problems and exercises

PROBLEM 3.4.1 Simple growth

Let the initial amount of bacteria be equal to A_0, and it growes by d during every unit time period. Calculate the population at the n'th time unit. Calculate also the doublelife, halflife, and the time when the population dies out (if $d < 0$).

(1) $A_0 = 100;\ d = 10;\ n = 10;$

(2) $A_0 = 500;\ d = 16;\ n = -20;$

(3) $A_0 = 1000;\ d = 30;\ n = 5;$

(4) $A_0 = 200;\ d = -25;\ n = 7;$

PROBLEM 3.4.2 Geometric growth

Let the initial amount be equal to A_0, and the growth rate in a unit time be q, that is, $A_{n+1} = q A_n$. Calculate the n'th member. Calculate the doublelife and halflife.

(1) $A_0 = 2;\ q = 0.5;\ n = 5;$

(2) $A_0 = 10;\ q = -0.5;\ n = 4;$

(3) $A_0 = 100;\ q = 0.9;\ n = 6;$

(4) $A_0 = 500;\ q = -2;\ n = 5;$

(5) $A_0 = 20;\ p = 2;\ n = 4;$

PROBLEM 3.4.3 Geometric growth

Let the initial amount of bacteria be equal to A_0, and the growth rate in time unit be p %. Calculate the population at the n'th time unit. Calculate the doublelife and halflife. Can any of these population die out?

(1) $A_0 = 2;\ p = 10;\ n = 5;$

(2) $A_0 = 100;\ p = -20;\ n = 12;$

(3) $A_0 = 2000;\ p = 5;\ n = 3;$

(4) $A_0 = 500;\ p = -12;\ n = 5;$

(5) $A_0 = 2000;\ p = 8;\ n = 4;$

PROBLEM 3.4.4

Using the theorems for limits and the definition of e, calculate the following limits.

(1) $\lim_{x\to\infty} \left(1 + \frac{2}{x}\right)^x$

(2) $\lim_{x\to\infty} \left(1 + \frac{1}{3x}\right)^x$

(3) $\lim_{x\to\infty} \left(1 + \frac{2}{x}\right)^x$

(4) $\lim_{x\to\infty} \left(1 + \frac{1}{\sqrt{x}}\right)^x$

(5) $\lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^{\sqrt{x}}$