13. Definite Integral

PROBLEM 1 CALCULATION OF DEFINITE INTEGRALS

a) $\int\limits_{-1}^{-e} \frac{1}{x}\,\mathrm{d}x$ *b)* $\int\limits_{1}^{256} \frac{\sqrt{\sqrt{\sqrt{x}}}}{x}\,\mathrm{d}x$ *c)* $\int\limits_{0}^{4} (\sqrt{x}-2x+3x^2-6x^{\frac{3}{2}})\,\mathrm{d}x$

d) $\int\limits_{-1}^{1} \frac{e^x-2e^{2x}+3e^{3x}}{e^{2x}}\,\mathrm{d}x$ *e)* $\int\limits_{-\pi/2}^{-\pi/4} \mathrm{ctg}^2 x\,\mathrm{d}x$ *f)* $\int\limits_{1/4}^{1/2} \frac{\pi}{\sin^2 \pi x}\,\mathrm{d}x$

g) $\int\limits_{0}^{\pi} (\tfrac{1}{2}\sin x + \cos\tfrac{x}{2})\,\mathrm{d}x$ *h)* $\int\limits_{0}^{\pi/4} \frac{3-4\cos^3 x}{\cos^2 x}\,\mathrm{d}x$ *i)* $\int\limits_{-1/2}^{1/2} \cos^2 \pi x\,\mathrm{d}x$

j) $\int\limits_{1}^{e^2} \frac{\ln\sqrt{x}}{\sqrt{x}}\,\mathrm{d}x$ *k)* $\int\limits_{0}^{\pi/2} (\sin x + 1)^3 \cos x\,\mathrm{d}x$ *l)* $\int\limits_{-\pi/4}^{\pi/4} \frac{1}{e^{\mathrm{tg}\,x}\cos^2 x}\,\mathrm{d}x$

m) $\int\limits_{1}^{e} x\ln x\,\mathrm{d}x$ *n)* $\int\limits_{0}^{1} xe^x\,\mathrm{d}x$ *o)* $\int\limits_{0}^{\pi} x\sin x\,\mathrm{d}x$

p) $\int\limits_{\pi}^{0} x\cos x\,\mathrm{d}x$

PROBLEM 2 AREA CALCULATION

Calculate the area enclosed by the two curves! *a)--g):* also draw the graph!

a) $\sin\pi x$ és $2x$ *b)* $x+1$ és 2^x *c)* $\sqrt{x}$ és x^2 *d)* $4-x^2$ és $(x-2)^2$

e) $3-x$ és $2/x$ *f)* $\sqrt[3]{x}$ és x^3 *g)* x^3-1 és $3x^2-2x-1$

h) x^3+2x^2-2x és x^3-2x^2+2x

PROBLEM 3 VOLUME OF REVOLUTION

- Calculate the volume of revolution obtained by revolving around the x-axis the area between the curve of $f(x)$ and the x-axis. Plot the rotated area!

a) $[a,b]=[0,1]$ és $f(x)=\sqrt{x}$

b) $[a,b]=[0,2\pi]$ és $f(x)=1+\frac{1}{2}\sin x$

c) $[a,b]=[\frac{1}{e},e]$ és $f(x)=\ln x$

d) $[a,b]=[-\frac{\pi}{2},\frac{\pi}{2}]$ és $f(x)=(\cos\frac{1}{2}x)^{-1}$

e) $[a,b]=[0,h]$ és $f(x)=\frac{r}{h}x$ (cone)

f) $[a,b]=[0,h]$ és $f(x)=r\sqrt{\frac{x}{h}}$ (paraboloid)

- Calculate the volume of revolution obtained by revolving around the x-axis the area between the curves $f(x)$ and $g(x)$ over the interval $[a,b]$. Make a graph of the rotated area!

  a) $[a,b]=[0,2],\ f(x)=\sqrt{x}+1,\ g(x)=\sqrt{x}$

  b) $[a,b]=[0,\pi],\ f(x)=2\sin x,\ g(x)=\sin x$

  c)* $[a,b]=[-r,r],\ f(x)=R+\sqrt{r^2-x^2},\ g(x)=R-\sqrt{r^2-x^2},\ R\geq r$ (torus–ring)

PROBLEM 4 INTEGRAL MEAN VALUE, INITIAL CONDITIONS

- Let $M(t)$ be a time-dependent quantity and $v(t)=M'(t)$ its rate of change. Determine the formula of $M(t)$ under the given initial conditions, and calculate its average rate of change over the interval $[t_0, t_1]$! Also, calculate the mean value (average) of $M(t)$ over the interval $[t_0, t_1]$!

  a) $v(t)=\sin\frac{\pi}{12}t,\ M(6)=24,\ [t_0,t_1]=[0,24]$

  b) $v(t)=\mathrm{tg}^2 t,\ M(0)=2,\ [t_0,t_1]=[-\frac{\pi}{4},\frac{\pi}{4}]$

  c) $v(t)=\frac{1}{t},\ M(1)=1,\ [t_0,t_1]=[1,e]$

  d) $v(t)=te^t,\ M(0)=1,\ [t_0,t_1]=[1,\ln 9]$

- At any instant t, denote the position (distance) of a motion by $s(t)$, its velocity by $v(t)$ and its acceleration by $a(t)$. Determine the formula of $v(t)$ and $s(t)$ under the given initial conditions, and calculate the average acceleration and velocity and the distance traveled on $[t_0, t_1]$!

  a) $a(t)=2-e^{-t},\ v(0)=0,\ s(0)=1,\ [t_0,t_1]=[0,2]$

  b) $a(t)=t^{-\frac{3}{2}},\ v(1)=0,\ s(4)=4,\ [t_0,t_1]=[1,9]$

  c) $a(t)=t\cos t,\ v(\pi)=0,\ s(0)=0,\ [t_0,t_1]=[0,\pi]$

  d) $a(t)=e^t(e^t+1)^{-2},\ v(0)=\frac{1}{2},\ s(0)=0,\ [t_0,t_1]=[0,1]$

SOLUTIONS

Calculation of definite integrals

a) 1 *b)* 8 *c)* $-\frac{352}{15}=-23\frac{7}{15}=-23{,}46\dot{6}\ldots$ *d)* $4(e-\frac{1}{e}-1)$ *e)* $1-\frac{\pi}{4}$ *f)* 1 *g)* 3

h) $3-2\sqrt{2}$ *i)* $\frac{1}{2}$ *j)* 2 *k)* 3,75 *l)* $e-\frac{1}{e}$ *m)* $\frac{e^2+1}{4}$ *n)* 1 *o)* π *p)* 2

Area calculation

a) $T_{[-\frac{1}{2},0]}+T_{[0,\frac{1}{2}]}=\frac{2}{\pi}-\frac{1}{2}$ *b)* $T_{[0,1]}=\frac{3}{2}-\frac{1}{\ln 2}=\frac{\ln(8/e^2)}{\ln 4}$ *c)* $T_{[0,1]}=\frac{1}{3}$ *d)* $T_{[0,2]}=\frac{8}{3}$

e) $T_{[1,2]}=\frac{3}{2}-2\ln 2$ *f)* $T_{[-1,0]}+T_{[0,1]}=1$ *g)* $T_{[0,1]}+T_{[1,2]}=\frac{1}{4}+\frac{1}{4}=\frac{1}{2}$ *h)* $T_{[0,1]}=\frac{2}{3}$

Volume of revolution

a) $\frac{\pi}{2}$ *b)* $(\frac{3}{2}\pi)^2=2{,}25\pi^2$ *c)* $\frac{e^2-5}{e}\pi$ *d)* 4π *e)* $\frac{h}{3}r^2\pi$ *f)* $\frac{h}{2}r^2\pi$

a) $\frac{44}{3}\pi$ *b)* $\frac{3}{2}\pi^2$ *c)* $(2R\pi)(r^2\pi)=2Rr^2\pi^2$

Integral mean value, initial conditions

a) $M(t)=24-\frac{12}{\pi}\cos\frac{\pi}{12}t,\ \bar{v}_{[0,24]}=0,\ \overline{M}_{[0,24]}=24$ *b)* $M(t)=\mathrm{tg}\,t-t+2,\ \bar{v}_{[-\frac{\pi}{4},\frac{\pi}{4}]}=\frac{4}{\pi}-1,\ \overline{M}_{[-\frac{\pi}{4},\frac{\pi}{4}]}=2$

c) $M(t)=\ln|t|+1,\ \bar{v}_{[1,e]}=\frac{1}{e-1},\ \overline{M}_{[1,e]}=\frac{e}{e-1}$ *d)* $M(t)=e^t(t-1)+2,\ \bar{v}_{[1,\ln 9]}=9,\ \overline{M}_{[1,\ln 9]}=11-\frac{9-e}{\ln 9-\ln e}$

a) $v(t)=2t-1+e^{-t},\ \bar{a}_{[0,2]}=\frac{3+e^{-2}}{2},\ s(t)=t^2-t+2-e^{-t},\ \bar{v}_{[0,2]}=\frac{3-e^{-2}}{2},\ s(2)-s(0)=3-e^{-2}$

b) $v(t)=2-\frac{2}{\sqrt{t}},\ \bar{a}_{[1,9]}=\frac{1}{6},\ s(t)=2t-4\sqrt{t}+4,\ \bar{v}_{[1,9]}=1,\ s(9)-s(1)=8$

c) $v(t)=t\sin t+1+\cos t,\ \bar{a}_{[0,\pi]}=-\frac{2}{\pi},\ s(t)=t(1-\cos t)+2\sin t,\ \bar{v}_{[0,\pi]}=2,\ s(\pi)-s(0)=2\pi$

d) $v(t)=\frac{e^t}{e^t+1},\ \bar{a}_{[0,1]}=\frac{e-1}{2(e+1)},\ s(t)=\ln\frac{e^t+1}{2},\ \bar{v}_{[0,1]}=s(1)-s(0)=\ln\frac{e+1}{2}$