10. Limits involving infinity

It is known from the limit rules for fundamental arithmetic operations (+,-,×,÷) that if two functions have finite limits at a (finite or infinite) point x_0 , that is, they are *convergent*, the limit of their sum, product, etc. will be equal to the sum, product, etc. of their limits (except in case of a quotient with zero denominator). In other words, the operation of finding the limit is interchangeable with the fundamental arithmetic operations.

In the table below we consider those cases, when one of the two functions (or both) tends to infinity (is *proper divergent*), or the denominator function tends to 0. In such situations two cases are possible: either the limit of the expression of fundamental arithmetic can be given in advance (*definite* limit expression), or it cannot be given in advance in lack of additional information, and the limit can be anything or even may not exist (*indefinite* limit expression), in the latter type of cases the unknown limits are symbolized by question marks.

Theorem (limit rules involving infinity)

ASSUMPTION	STATEMENT [1]	SYMBOLIC NOTATION [2]
If $\lim\limits_{x\to x_0} f(x)=\infty$ and $\lim\limits_{x\to x_0} g(x)=\infty$, then	$\lim\limits_{x\to x_0}(f(x)+g(x))=\infty$	$\infty+\infty=\infty$
	$\lim\limits_{x\to x_0}(f(x)-g(x))=?$	$\infty-\infty=?$ [3]
	$\lim\limits_{x\to x_0} f(x)\cdot g(x)=\infty$	$\infty\cdot\infty=\infty$
	$\lim\limits_{x\to x_0}\dfrac{f(x)}{g(x)}=?$	$\dfrac{\infty}{\infty}=?$ [3]

If $\lim\limits_{x\to x_0} f(x)=c$ and $\lim\limits_{x\to x_0} g(x)=\infty$, then ($c$ real constant)	$\lim\limits_{x\to x_0}(f(x)+g(x))=\infty$ $\lim\limits_{x\to x_0}(f(x)-g(x))=-\infty$ $\lim\limits_{x\to x_0} f(x)\cdot g(x)=\begin{cases}-\infty, \text{ if } c<0\\ \infty, \text{ if } c>0,\\ ?, \text{ if } c=0\end{cases}$ $\lim\limits_{x\to x_0}\dfrac{f(x)}{g(x)}=0$	$c+\infty=\infty$ $c-\infty=-\infty$ $c\cdot\infty=-\infty$, if $c<0$ $c\cdot\infty=\infty$, if $c>0$ $0\cdot\infty=?$ [3] $\dfrac{c}{\infty}=0$
If $\lim\limits_{x\to x_0} f(x)=\infty$ or $\lim\limits_{x\to x_0} f(x)=c>0$, and $\lim\limits_{x\to x_0} g(x)=0+$, then (0+: $g(x)\to 0$ and $g(x)>0$ around x_0, that is, it tends to 0 from the positive side)	$\lim\limits_{x\to x_0}\dfrac{f(x)}{g(x)}=\infty$	$\dfrac{\infty}{0+}=\infty$, $\dfrac{c}{0+}=\infty$, if $c>0$
If $\lim\limits_{x\to x_0} f(x)=0$ and $\lim\limits_{x\to x_0} g(x)=0$, then	$\lim\limits_{x\to x_0}\dfrac{f(x)}{g(x)}=?$	$\dfrac{0}{0}=?$ [3]

[1] All statements of the theorem remain valid, if $x\to x_0$ if replaced by the one sided limits $x\to x_0-$, $x\to x_0+$, $x\to\infty$ or $x\to -\infty$ (so in the assumption as in the corresponding statement).

[2] The equations in the symbolic notation must not be considered as ordinary equations (e.g, you must not subtract from both sides ∞, etc.). The left side of the equation symbolizes the *type* of the limit expression, the right side gives the limit in definite cases or a question mark symbolizes uncertainty in the indefinite cases.

[3] The indefinite expressions of type ∞/∞ and 0/0 can be handled by means of l'Hospital's rule. The type $0\cdot\infty$ can be reduced to type 0/0 via the transformation $f(x)\cdot g(x)= f(x)/(1/g(x))$ The difference of type $\infty-\infty$ can be reduced to a quotient type by bringing to a common denominator or using other transformations.

Limits of composite functions

Theorem. *If* $\lim\limits_{x\to x_0} g(x)=u_0$ *and* $\lim\limits_{u\to u_0} f(u)=L$ *, the*

$$\lim_{x\to x_0} f(g(x))=\lim_{g(x)\to u_0} f(g(x))=\lim_{u\to u_0} f(u)=\lim_{x\to u_0} f(x)=L\,.$$

If, in addition, u_0 is finite and f(x) is continuous at u_0, then

$$\lim_{x\to x_0} f(g(x)) = f(\lim_{x\to x_0} g(x)) = f(u_0)$$

Remark. The theorem also holds true for one-sided limits including limits at infinity (x_0,. u_0,.L can be ∞ or $-\infty$). Without „lim"notation we can also write: $x\to x_0 \Rightarrow u=g(x)\to u_0 \Rightarrow f(u)=f(g(x))\to L$.

Example: $\lim\limits_{x\to\infty} \cot(e^{-x}) = \lim\limits_{u\to 0+} \cot(u) = \infty$, because: $x\to\infty \Rightarrow u=e^{-x}\to 0+ \Rightarrow$ $\cot(u)=\cot(e^{-x})\to\infty$.

L'Hospital's rule

Theorem. *If* $\lim\limits_{x\to x_0} f(x) = \lim\limits_{x\to x_0} g(x) = 0$ (*type* 0/0) *or* $\lim\limits_{x\to x_0} f(x) = \lim\limits_{x\to x_0} g(x) = \infty$ (*type* ∞/∞), *and both functions are differentiable around* x_0, *then* $\lim\limits_{x\to x_0} \dfrac{f(x)}{g(x)} = \lim\limits_{x\to x_0} \dfrac{f'(x)}{g'(x)}$, *provided the latter limit exists.*

Remark. The theorem also holds true for one-sided limits including limits at infinities ("around x_0" means then a one-sided environment). In addition to the type ∞/∞ it can also be applied to the types $-\infty/\infty$, $\infty/(-\infty)$ and $-\infty/(-\infty)$, as well.

Important: Always check the conditions of applicability before using the rule's formula! If not used for the proper type of expression, false results are to be expected.

Examples:

a) type 0/0: $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = \lim\limits_{x\to 0} \dfrac{\cos x}{1} = \cos 0 = 1$.

b) type $0\cdot(-\infty)$, change into type $-\infty/\infty$, use l'Hospital's, then simplify and substitute:

$$\lim_{x\to 0+} x\ln x = \lim_{x\to 0+} \frac{\ln x}{\frac{1}{x}} = \lim_{x\to 0+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0+} (-x) = 0 \ .$$

c) type $\infty-\infty$, reduce to quotient, first use composite's and then l'Hospital's rule:

$$\lim_{x\to\infty} (\ln(1+e^x) - x) = \lim_{x\to\infty} (\ln(1+e^x) - \ln e^x) = \lim_{x\to\infty} \ln \frac{1+e^x}{e^x} = \ln\left(\lim_{x\to\infty} \frac{1+e^x}{e^x} \right) = \ln\left(\lim_{x\to\infty} \frac{e^x}{e^x} \right) = \ln 1 = 0$$

Power expressions: use the exponential transscription $(f(x))^{g(x)}=e^{\ln(f(x))^{g(x)}}=e^{g(x)\ln f(x)}$ and find the limit of the formula in the exponent. For example, utilizing the result of the previous example *b*) (substituting $u=x\ln x\to 0$) we can find this limit:

$$\lim_{x\to 0+} x^x = \lim_{x\to 0+} e^{\ln x^x} = \lim_{x\to 0+} e^{x\ln x} = \lim_{u\to 0} e^u = e^0 = 1 \ .$$

Comparing the rates of convergence or proper divergence

Definition 1. *Assume that* $\lim\limits_{x\to x_0} f(x)=\lim\limits_{x\to x_0} g(x)=0$. *Then we use the following terminology:*

if $\lim\limits_{x\to x_0}\left|\dfrac{f(x)}{g(x)}\right| =$

- 0 : *f*(*x*) *tends to* 0 *faster (in order of magnitude) than* *g*(*x*), *as* $x\to x_0$
- $c\neq 0$: *f*(*x*) *and* *g*(*x*) *tend to* 0 *in the same order of magnitude* (*c*=1: *equally fast*), *if* $x\to x_0$
- ∞ : *f*(*x*) *tends to* 0 *slower (in order of magnitude) than* *g*(*x*), *as* $x\to x_0$

Definition 2. Assume that $\lim_{x\to x_0} f(x) = \lim_{x\to x_0} g(x) = \infty$. Then we use the following terminology:

$$\text{if } \lim_{x\to x_0}\frac{f(x)}{g(x)} = \begin{cases} \infty: f(x) \text{ tends to } \infty \text{ faster (in order of magnitude) than } g(x), \text{ as } x\to x_0 \\ c\neq 0: f(x) \text{ and } g(x) \text{ tend to } \infty \text{ in the same order of magnitude} \\ (c=1\text{: equally fast}), \text{ if } x\to x_0 \\ 0: f(x) \text{ tends to } \infty \text{ slower (in order of magnitude) than } g(x), \text{ as } x\to x_0 \end{cases}$$

It is exactly l'Hospital's rule to use for such rate comparisons, because the assumptions are the same. For example, by the above example *a*), $f(x) = \sin x$ and $g(x) = x$ tend equally fast to 0 as $x\to 0$, because their quotient tends to $c=1$.

PROBLEM 1 LIMITS OF RATIONAL FRACTIONAL FUNCTIONS AT A FINITE POINT

Determine the following limits:

a) $\lim_{x\to 2}\frac{x^2-4x+4}{x^2-4}$ *b)* $\lim_{x\to 1}\frac{x^2+2x+1}{x^2+1}$

c) $\lim_{x\to 0}\frac{x^2+2x-2}{3x^2}$ *d)* $\lim_{x\to 0}\frac{3x^3+2x^2+x}{x^2}$

PROBLEM 2 LIMITS OF RATIONAL FRACTIONAL FUNCTIONS AT INFINITY

Determine the following limits:

a) $\lim_{x\to -\infty}\left(1+x+\frac{x^2}{2}+\frac{x^3}{6}\right)$ *b)* $\lim_{x\to\infty}(4x^2-2x^4)$ *c)* $\lim_{x\to\infty}\frac{x^2+2x-2}{3x^2}$

d) $\lim_{x\to -\infty}\frac{3x^3-2x^2+x}{4x+3}$ *e)* $\lim_{x\to -\infty}\frac{6x^5-4x^3+2x}{x^6-x^4+x^2-1}$ *f)* $\lim_{x\to\infty}\left(\frac{2}{x}-\frac{x}{2}\right)^9$

g) $\lim_{x\to\infty}\frac{(2x^2-x)^2}{1-3x^2-5x^4}$ *h)* $\lim_{x\to -\infty}\frac{2007x^{2006}-2006x}{x^{-2008}}$

PROBLEM 3 LIMITS OF ELEMENTARY FUNCTIONS

Give the limits of the elementary functions (sketch the graph and observe the limit):

a) $\lim_{x\to -3} 1$ *b)* $\lim_{x\to\infty} 2$ *c)* $\lim_{x\to\infty} x^2$ *d)* $\lim_{x\to -\infty} x^3$ *e)* $\lim_{x\to 0+}\sqrt{x}$

f) $\lim_{x\to -64}\sqrt{x}$ *g)* $\lim_{x\to -64}\sqrt[3]{x}$ *h)* $\lim_{x\to -\infty}\sqrt[3]{x}$ *i)* $\lim_{x\to -\infty} 2^x$ *j)* $\lim_{x\to\infty} e^x$

k) $\lim\limits_{x\to-\infty} e^x$ *l)* $\lim\limits_{x\to 0+} 10^x$ *m)* $\lim\limits_{x\to 0+} \lg x$ *n)* $\lim\limits_{x\to-\infty} \ln x$ *o)* $\lim\limits_{x\to\infty} \ln x$

p) $\lim\limits_{x\to 0+} \log_{\frac{1}{2}} x$ *q)* $\lim\limits_{x\to\infty} \sin x$ *r)* $\lim\limits_{x\to 0} \cos x$ *s)* $\lim\limits_{x\to\frac{\pi}{2}} \tan x$ *t)* $\lim\limits_{x\to\frac{-\pi}{2}} \cot x$

u) $\lim\limits_{x\to 10} \lg x$ *v)* $\lim\limits_{x\to 0} \tan x$ *w)* $\lim\limits_{x\to\frac{\pi}{4}} \tan x$ *x)* $\lim\limits_{x\to 0-} \cot x$

PROBLEM 4 DEFINITE LIMIT EXPRESSIONS; COMPOSITE FUNCTIONS

- Determine the following limits:

a) $\lim\limits_{x\to\infty}(x^2+e^x)$ *b)* $\lim\limits_{x\to-\infty}(x^2+e^x)$ *c)* $\lim\limits_{x\to 0+}(\ln x-\cot x)$

d) $\lim\limits_{x\to\frac{\pi}{2}} \dfrac{\tan x}{2x-\pi}$ *e)* $\lim\limits_{x\to 1} \dfrac{x}{\ln x}$ *f)* $\lim\limits_{x\to-\infty} x^3(\tfrac{1}{2})^x$

g) $\lim\limits_{x\to 0+} \dfrac{\log_{0,5} x}{\cos^2 x}$ *h)* $\lim\limits_{x\to\infty} x^3 \ln x$ *i)* $\lim\limits_{x\to\infty} \dfrac{1}{\ln x}$

j) $\lim\limits_{x\to 0-}\left(\dfrac{1}{\sin x}+\dfrac{1}{x}\right)$

- Determine the limits of the following composite functions:

a) $\lim\limits_{x\to 1-} \ln(1-x^2)$ *b)* $\lim\limits_{x\to\infty} \sin\left(\dfrac{1}{\ln x}\right)$ *c)* $\lim\limits_{x\to 0-} \sqrt[x]{e}$

d) $\lim\limits_{x\to 0+} \sqrt[x]{e}$ *e)* $\lim\limits_{x\to-\infty} \cot e^x$ *f)* $\lim\limits_{x\to 0} \tan\left(\dfrac{\pi}{1+\cos x}\right)$

g) $\lim\limits_{x\to\pi} \dfrac{\pi}{1+\cos x}$ *h)* $\lim\limits_{x\to\infty} \ln\left(\dfrac{x}{x+1}\right)$ *i)*

$\lim\limits_{x\to 0} \lg(\cos x)$

j) $\lim\limits_{x\to 0+} \sqrt[x]{x}$

PROBLEM 5 INDEFINITE LIMIT EXPRESSIONS: L'HOSPITAL'S RULE

- Determine the following limits:

a) $\lim\limits_{x\to 0} \dfrac{1-e^{-2x}}{x}$ *b)* $\lim\limits_{x\to\infty} \dfrac{\lg x}{x^{10}}$ *c)* $\lim\limits_{x\to 1} \dfrac{\sin \pi x}{\ln x}$

d) $\lim\limits_{x\to-\infty} xe^x$ *e)* $\lim\limits_{x\to\pi}(x-\pi)\cot x$ *f)* $\lim\limits_{x\to\pi} \dfrac{\sin 2x}{1+\sin x+\cos x}$

g) $\lim_{x\to\infty} \frac{e^x + 4x^2}{e^x + 2x}$

h) $\lim_{x\to 0} x \ln x^2$

i) $\lim_{x\to 0} \frac{1-\cos x}{x \sin x}$

j) $\lim_{x\to\pi} \frac{e^x - e^\pi}{\operatorname{tg} x}$

k) $\lim_{x\to 0} \frac{\sin(\pi \cos x)}{\sin^2 x}$

l) $\lim_{x\to 1-} \ln x \cdot \ln(1-x)$

m) $\lim_{x\to 1+} \frac{\log_2 x}{\sqrt{x-1}}$

n) $\lim_{x\to \frac{\pi}{2}} \frac{\ln \sin x}{\cos^2 x}$

o) $\lim_{x\to\infty} \sqrt[x]{x}$

p) $\lim_{x\to 0} \sqrt[x]{1+x}$

q) $\lim_{x\to 0+} (\cos x)^{x^{-2}}$

r) $\lim_{x\to 1} \left(\frac{1}{\ln x} - \frac{1}{x-1} \right)$

PROBLEM 6

Find out the limits of the two functions (sketch their graphs) and compare the rate of their convergence (or proper divergence)!

a) x and $(1-\cos x)$, as $x\to 0$

b) $\ln x$ and x, as $x\to\infty$

c) e^x and x, as $x\to\infty$

d) x and $\frac{1}{\ln x}$, as $x\to 0+$

e) $\log_{\frac{1}{2}} x$ and $\cot x$, as $x\to 0+$

f) 2^x and x^2, as $x\to\infty$

g) $\ln x$ and $(x-1)^2$, as $x\to 1$

h) $\cot x$ and $\frac{1}{x}$, as $x\to 0$

i) $\tan x$ and $\frac{1}{2x-\pi}$, as $x\to \frac{\pi}{2}$

j) e^{-x} and x^{-1}, as $x\to\infty$

k) $\cot \pi x$ and $\ln\frac{1}{2x}$, as $x\to\frac{1}{2}$

l) $\ln x$ and $\sqrt[3]{x-1}$, as $x\to 1$

PROBLEM 7 FINDING LIMITS FOR THE EXAMINATION OF FUNCTIONS

Where is the function continuous (domain)? Determine the (possibly one-sided) limits of the function at the end-points of its domain! (Attention: l'Hospital's rule is not always applicable!)

a) $x^2(x-2)^2$

b) $-2x^3+2x^2-x+1$

c) $\frac{1}{x^2+2x-3}$

d) $\frac{x^2}{1+x^2}$

e) $x + \frac{1}{x}$

f) $\sqrt{x-x^2}$

g) xe^x

h) xe^{-x^2}

i) $\frac{e^x}{x}$

j) $e^{1/x}$

k) e^{-1/x^2}

l) $x \ln x$

m) $\frac{\ln x}{x}$

n) $\frac{x}{\ln x}$

o) $\ln\left(\frac{x+1}{x}\right)$

p) $x^2 - \ln x^2$

q) $\cos(\pi e^{-x^2})$

r) $\sqrt[x]{x}$

RESULTS

1. *a)* 0 *b)* 2 *c)* $-\infty$ *d)* does not exist (left: $-\infty$, right: ∞)
2. *a)* $-\infty$ *b)* $-\infty$ *c)* 1/3 *d)* ∞ *e)* 0 *f)* $-\infty$ *g)* $-4/5$ *h)* ∞
3. *a)* 1 *b)* 2 *c)* ∞ *d)* $-\infty$ *e)* 0 *f)* does not exist (not defined) *g)* -4 *h)* $-\infty$ *i)* 0 *j)* ∞ *k)* 0 *l)* 1 *m)* $-\infty$ *n)* does not exist (not defined) *o)* ∞ *p)* $-\infty$ *q)* does not exist (oscillating) *r)* 1
   s) does not exist (left: ∞, right: $-\infty$) *t)* 0 *u)* 1 *v)* 0 *w)* 1 *x)* $-\infty$
4. *a)* ∞ *b)* ∞ *c)* $-\infty$ *d)* $-\infty$ *e)* does not exist (left: $-\infty$, right: ∞) *f)* $-\infty$ *g)* ∞ *h)* ∞ *i)* 0 *j)* $-\infty$
5. *a)* $-\infty$ *b)* 0 *c)* 0 *d)* ∞ *e)* ∞ *f)* $-\infty$ *g)* ∞ *h)* 0 *i)* 0 *j)* 0
6. *a)* 2 *b)* 0 *c)* $-\pi$ *d)* 0 *e)* 1 *f)* -2 *g)* 1 *h)* 0 *i)* 1/2 *j)* e^{π} *k)* $\pi/2$ *l)* 0 *m)* 0 *n)* $-1/2$
   o) 1 *p)* e *q)* $e^{-1/2}$ *r)* 1/2
7. *a)* $\to 0$; x slower *b)* $\to\infty$; $\ln x$ slower *c)* $\to\infty$; e^{x} faster *d)* $\to 0$; x faster
   e) $\to\infty$; $\log_{1/2} x$ slower *f)* $\to\infty$; 2^{x} faster *g)* $\to 0$; $\ln x$ slower
   h) only one-sided limits exist: from the left $\to -\infty$, from the right $\to\infty$; equally fast
   i) only one-sided limits exist: from the left $\to\infty$, from the right $\to -\infty$; equal order of magnitude (c=2)
   j) $\to 0$; e^{-x} faster *k)* $\to 0$; equal order of magnitude ($c=\pi/2$) *l)* $\to 0$; $\ln x$ faster
8. *a)* D $=(-\infty,\infty)$, Lim: both at $-\infty$ and at ∞: $=\infty$ *b)* D $=(-\infty,\infty)$, Lim: at $-\infty$: $=\infty$, at ∞: $=-\infty$
   c) D $=(-\infty,-3)\cup(-3,1)\cup(1,\infty)$, Lim: both at $-\infty$ and at ∞: $=0$; at -3 left: $=\infty$, right: $=-\infty$; at 1 oppositely
   d) D $=(-\infty,\infty)$, Lim: both at $-\infty$ and at ∞: $=1$
   e) D $=(-\infty,0)\cup(0,\infty)$, Lim: at $-\infty$: $=-\infty$, at ∞: $=\infty$; at 0–: $=\infty$, at 0+: $=\infty$
   f) D $=[0,1]$, Lim: both at 0+ and at 1–: $=0$ *g)* D $=(-\infty,\infty)$, Lim: at $-\infty$: $=0$, at ∞: $=\infty$
   h) D $=(-\infty,\infty)$, Lim: both at $-\infty$ and at ∞: $=0$
   i) D $=(-\infty,0)\cup(0,\infty)$, Lim: at $-\infty$: $=0$, at ∞: $=\infty$; at 0–: $=-\infty$, at 0+: $=\infty$
   j) D $=(-\infty,0)\cup(0,\infty)$, Lim: both at $-\infty$ and at ∞: $=1$; at 0–: $=0$, at 0+: $=\infty$
   k) D $=(-\infty,0)\cup(0,\infty)$, Lim: both at $-\infty$ and at ∞: $=1$; at 0: $=0$ *l)* D $=(0,\infty)$, Lim: at 0+: $=0$; at ∞: $=\infty$
   m) D $=(0,\infty)$, Lim: at 0+: $=-\infty$; at ∞: $=0$
   n) D $=(0,1)\cup(1,\infty)$, Lim: at 0+: $=0$; at ∞: $=\infty$; at 1–: $=-\infty$, at 1+: $=\infty$
   o) D $=(-\infty,-1)\cup(0,\infty)$, Lim: both at $-\infty$ and at ∞: $=0$; at -1–: $=-\infty$; at 0+: $=\infty$
   p) D $=(-\infty,0)\cup(0,\infty)$, Lim: both at $-\infty$, at ∞ and at 0: $=\infty$
   q) D $=(-\infty,\infty)$, Lim: both at $-\infty$ and at ∞: $=1$ *r)* D $=(0,\infty)$, Lim: at 0+: $=0$; at ∞: $=1$.