9. Taylor-polinomal, approximation

PROBLEM 1

Determine the n-th order Taylor-polynomial of the function $f(x)$ around the given point x_0, and use it for the approximation of the given value of $f(x)$!

a) $f(x) = e^{1-x}$, $n=3$, $x_0=1$; $f(0.7)\approx?$

b) $f(x) = 3 - \ln(1-x)^3$, $n=4$, $x_0=0$; $f(0.2)\approx?$

c) $f(x) = \sqrt{x^3}$, $n=4$, $x_0=1$; $f(0.6)\approx?$

d) $f(x) = \sqrt{2x+2}$, $n=3$, $x_0=1$; $f(1.4)\approx?$

e) $f(x) = \sin x$, $n=4$, $x_0=\pi$; $f(\pi-0.3)\approx?$

f) $f(x) = 6\sin^2 x$, $n=4$, $x_0=\frac{\pi}{2}$; $f(\frac{\pi}{2}+0.01)\approx?$

g) $f(x) = \operatorname{tg} x$, $n=3$, $x_0=\frac{\pi}{4}$; $f(\frac{\pi}{4}+0.03)\approx?$

h) $f(x) = \operatorname{tg} x^2$, $n=2$, $x_0=0$; $f(-0.1)\approx?$

i) $f(x) = 1.5(e^x+e^{-x})$, $n=4$, $x_0=0$; $f(0.01)\approx?$

j) $f(x) = e^{-x^2/2}$, $n=3$, $x_0=0$; $f(0.1)\approx?$

k) $f(x) = e^{1-\cos x}$, $n=3$, $x_0=0$; $f(0.1)\approx?$

l) $f(x) = \ln(8\sin^6 x)$, $n=3$, $x_0=\frac{\pi}{4}$; $f(\frac{\pi}{4}+0.01)\approx?$

SAMPLE SOLUTION:

c) $f(x) = x^{\frac{3}{2}}$, $f(x_0) = f(1) = 1^{\frac{3}{2}} = 1$

$f'(x) = \frac{3}{2}x^{\frac{1}{2}}$, $f'(x_0) = f'(1) = \frac{3}{2}1^{\frac{1}{2}} = \frac{3}{2}$

$f''(x) = \frac{3}{4}x^{-\frac{1}{2}}$, $f''(x_0) = f''(1) = \frac{3}{4}1^{-\frac{1}{2}} = \frac{3}{4}$

$f'''(x) = -\frac{3}{8}x^{-\frac{3}{2}}$, $f'''(x_0) = f'''(1) = -\frac{3}{8}1^{-\frac{3}{2}} = -\frac{3}{8}$

$f^{IV}(x)=\frac{9}{16}x^{-\frac{5}{2}}$, $f^{IV}(x_0)=f^{IV}(1)=\frac{9}{16}1^{-\frac{5}{2}}=\frac{9}{16}$, which, substituting into the Taylor–formula, gives:

$$T_n(x)=T_4(x)=f(x_0)+f'(x_0)(x-x_0)+\frac{f''(x_0)}{2}(x-x_0)^2+\frac{f'''(x_0)}{6}(x-x_0)^3+\frac{f^{IV}(x_0)}{24}(x-x_0)^4=$$

$$=1+\tfrac{3}{2}(x-1)+\frac{\frac{3}{4}}{2}(x-1)^2+\frac{-\frac{3}{8}}{6}(x-1)^3+\frac{\frac{9}{16}}{24}(x-1)^4$$ (Here, we leave the powers of $(x–x_0)$

$$=1+\tfrac{3}{2}(x-1)+\tfrac{3}{8}(x-1)^2-\tfrac{1}{16}(x-1)^3+\tfrac{3}{128}(x-1)^4$$

as is and do not apply the raising to square, cube, etc. identities!)

Approximation at x=0.6: since $(x–x_0)=(x–1)=0.6–1=–0.4$, so:

$$0.6^{\frac{3}{2}}=f(0.6)\approx T_4(0.6)=1+\tfrac{3}{2}(-0.4)+\tfrac{3}{8}(-0.4)^2-\tfrac{1}{16}(-0.4)^3+\tfrac{3}{128}(-0.4)^4=$$

$$T_4(0.6)=1-\tfrac{3}{2}\cdot 0.4+\tfrac{3}{8}\cdot 0.16+\tfrac{1}{16}\cdot 0.064+\tfrac{3}{128}\cdot 0.0256=1-0.6+0.06+0.004+0.0006=0.4646\,.$$