8. Total examination of functions

Steps for the total examination of the function *f*(*x*)

1. Direct examination of f (x):

a. Determining the domain of definition (D_f) and continuity, finding points of discontinuity

b. Recognizing special properties (odd, even, periodic)

c. Intercepts: y–axis: $x=0 \Rightarrow y=f(0)$; x-axis: $y=0 \Rightarrow x$: $f(x)=0$ (solving equation if possible)

d. Determining (one-sided) limits at the endpoints of the domain (ex.: jump points, or $-\infty$ or $+\infty$)

2. Examination via the 1st derivative: $f'(x) \Rightarrow$ monotonity and local extrema of $f(x)$

a. Determining the derivative $f'(x)$ (differentiation) and finding its domain

b. Finding critical points–I: break points (where $f(x)$ is defined, but $f'(x)$ is not)

c. Finding critical points–II: stationary points: solutions to equation $f'(x)=0$

d. Making a table: separating points are the critical/stationary points and the points of discontinuity. In the intermediate intervals the sign of $f'(x)$ does not change and determines the increasing (+) or decreasing (–) property of $f(x)$ and consequently, which critical point is a local extremum point. Substituting the latter into $f(x)$ we obtain the local extremal values of the function (to be calculated).

3. Examination via the 2nd derivative: $f''(x) \Rightarrow$ convexity/concavity and inflexions of $f(x)$

a. Possible points of inflexion: the solutions to equation $f''(x)=0$

b. Making a table: separating points are: possible points of inflexion and points of discontinuity. In the intermediate intervals the sign of $f''(x)$ does not change and determines the convex (+) or concave (–) property of $f(x)$ and consequently, which separating point is a point of inflexion. Substituting the latter into *f(x)* we obtain the *y*-coordinates of the inflexion points (to be calculated, if not too complicated).

4. Drawing the graph of the function *f(x)*:

a. Points of graph to be marked: axis-intercepts, local extrema (minimum and maximum points) and inflexion points; possibly one or two simply calculable points of $f(x)$ (e.g., $x = \pm 1$, etc.).

b. Connecting the previously marked points by a curve in such a way that the increasing / decreasing as well as the convex / concave properties in the intermediate intervals as determined in steps 2) and 3) are respected.

c. Establishing the range of values (R_f) for the function $f(x)$ on the basis of the graph, respecting the local or global extrema and the limits at the ends of the domain.

PROBLEM 1

Do a total examination of the given function:

a) $2x^2 + 3x - 2$ ***b)*** $x^3 + 3x^2$ ***c)*** $x^3 - 3x$ ***d)*** $2x^3 - 9x^2 + 12x$

e) $x^4 - 4x^3 + 4x^2$ ***f)*** $x + \frac{1}{x}$ ***g)*** $x - \frac{1}{x}$ ***h)*** $2x^2 - \sqrt{x}$

i) $\frac{1}{1-x^2}$ ***j)*** $\ln(4x - x^2)$ ***k)*** $x - \ln x^2$ ***l)*** $\frac{x}{e^x}$ ***m)*** $\frac{e^x}{x}$

m) $\frac{e^x}{x}$ ***n)*** xe^x ***o)*** xe^{-x^2} ***p)*** $x \ln x$

PROBLEM 2

Investigate the properties of the processes described by the following functions:

a) Logistic changes: $\frac{e^{\alpha x}}{A + Be^{\alpha x}}$; $\frac{1}{A + Be^{\alpha x}}$ $A, B, \alpha > 0$

b) Emptying and filling: $Ae^{-\alpha x}$; $A(1 - e^{-\alpha x})$ $A, \alpha > 0$

c) Normal distribution: e^{-x^2}; $Ae^{-\frac{(x-m)^2}{D^2}}$

d) transfer, flow: $e^{-2x} - e^{-3x}$; $e^{-\alpha x} - e^{-\beta x}$; $Ae^{-\alpha x} - Be^{-\beta x}$ $A \geq B \geq 0, \alpha > \beta > 0$