7. Formal differentiation rules

Derivatives of elementary functions

Elem. function $f(x)$	**Its derivative** $(f(x))'=f'(x)$	**Composed with inner function** $(f(g(x))'=f'(g(x))\cdot g'(x)$	Composite function example
Power function x^α (α: real constant)	$(x^\alpha)'=\alpha x^{\alpha-1}$	$((g(x))^\alpha)'=\alpha\,(g(x))^{\alpha-1}\cdot g'(x)$	$\left((1-x^2)^{\frac{1}{2}}\right)'=$ $\frac{1}{2}(1-x^2)^{-\frac{1}{2}}(-2x)$
Root function $\sqrt[\beta]{x}$ ($\beta\neq 0$ is real)	Use $\sqrt[\beta]{x}=x^{\frac{1}{\beta}}$ to reduce roots to power functions		
Exponential function a^x ($0<a\neq 1$ real constant)	$(a^x)'=a^x\cdot\ln a$ $(e^x)'=e^x$	$(a^{g(x)})'=(a^{g(x)}\ln a)\cdot g'(x)$ $(e^{g(x)})'=e^{g(x)}\cdot g'(x)$	$\left(2^{\mathrm{tg}\,x}\right)'=\left(2^{\mathrm{tg}\,x}\ln 2\right)\cdot\frac{1}{\cos^2 x}$ $\left(e^{-x}\right)'=e^{-x}\cdot(-1)=-e^{-x}$
Logarithm function $\log_a x$ ($0<a\neq 1$ real constant)	$(\log_a x)'=\dfrac{1}{x\ln a}$, if $x>0$	$(\log_a g(x))'=\dfrac{g'(x)}{g(x)\ln a}$, $g(x)>0$ *logarithmic derivative of $g(x)$:*	$(\lg(\cos x))'=\dfrac{-\sin x}{\cos x\cdot\ln 10}$
$\ln x=\log_e x$	$(\ln x)'=\dfrac{1}{x}$, if $x>0$	$(\ln g(x))'=\dfrac{g'(x)}{g(x)}$, ha $g(x)>0$	$(\ln(1+e^x))'=\dfrac{e^x}{1+e^x}$
$\sin x$	$(\sin x)'=\cos x$	$(\sin g(x))'=\cos g(x)\cdot g'(x)$	$(\sin(\ln x))'=\cos(\ln x)\cdot\dfrac{1}{x}$
$\cos x$	$(\cos x)'=-\sin x$	$(\cos g(x))'=-\sin g(x)\cdot g'(x)$	$(\cos(x^2-1))'=$ $-\sin(x^2-1))\cdot 2x$
$\tan x$	$(\mathrm{tg}\,x)'=\dfrac{1}{\cos^2 x}$ $=\mathrm{tg}^2 x+1$	$(\mathrm{tg}\,g(x))'=\dfrac{g'(x)}{\cos^2(g(x))}$	$(\mathrm{tg}(\mathrm{tg}\,x))'=\dfrac{\mathrm{tg}^2 x+1}{\cos^2(\mathrm{tg}\,x)}$

$\operatorname{ctg} x$	$(\operatorname{ctg} x)' = -\dfrac{1}{\sin^2 x}$ $\equiv -\operatorname{ctg}^2 x - 1$	$(\operatorname{ctg} g(x))' = -\dfrac{g'(x)}{\sin^2(g(x))}$	$\left(\operatorname{ctg}\left(x+\sqrt{\pi}\right)\right)' = -\dfrac{1}{\sin^2\left(x+\sqrt{\pi}\right)}$

General rules

1. Constant factor rule: $(cf(x))' = cf'(x)$
2. Sum rule: $(f(x)+g(x))' = f'(x)+g'(x)$
3. Difference rule: $(f(x)-g(x))' = f'(x)-g'(x)$
4. Product rule: $(f(x)\cdot g(x))' = f'(x)g(x)+f(x)g'(x)$
5. Quotient rule: $\left(\dfrac{f(x)}{g(x)}\right)' = \dfrac{f'(x)g(x)-f(x)g'(x)}{(g(x))^2}$
6. Chain rule (composite function): $(f(g(x))' = f'(g(x))\cdot g'(x)$
7. Logarithmic differentiation: $f'(x) = f(x)\cdot(\ln f(x))'$

   Using logarithm we can reduce product to sum, quotient to difference, power to product, e. g.::

   a) $f(x) = \dfrac{(x^3-1)\sin^2 x}{e^x+e^2} \quad\Rightarrow\quad f'(x) = f(x)\cdot\left(\ln(x^3-1)+\ln(\sin^2 x)-\ln(e^x+e^2)\right)' =$

   $$= \frac{(x^3-1)\sin^2 x}{e^x+e^2}\cdot\left(\frac{3x^2}{x^3-1}+\frac{2\sin x\cos x}{\sin^2 x}-\frac{e^x}{e^x+e^2}\right)$$

   b) $f(x) = \sqrt[x]{x} = x^{\frac{1}{x}} \quad\Rightarrow\quad f'(x) = f(x)\cdot\left(\ln x^{\frac{1}{x}}\right)' = f(x)\cdot(x^{-1}\ln x)' = f(x)\cdot(-x^{-2}\ln x + x^{-1}\tfrac{1}{x})$

   $$= \sqrt[x]{x}\cdot\frac{1-\ln x}{x^2}$$

PROBLEM 1

Differentiate these functions using rules for fundamental operations:

a) $1+x+\dfrac{x^2}{2}+\dfrac{x^3}{6}$

b) $3\sin x - \operatorname{tg} x + 2\pi^2$

c) $2^x(\sqrt{2x}-3\sqrt[3]{x})$

d) $\left(3\operatorname{tg} x - \dfrac{2}{x^3}\right)\cdot \lg x$

e) $\sin^2 x + \cos^2 x$

f) $\dfrac{e^x - x^{2005}}{\sqrt[2006]{x}}$

g) $\dfrac{\sin x - \cos x}{\sin x + \cos x}$

h) $\dfrac{x^4+\pi x^2 - \frac{e}{2}}{2x^5 - x^3 + 9x}$

i) $\dfrac{x\operatorname{ctg} x}{x+\ln x} - \sqrt{2}$

j) $\dfrac{\sqrt{x}(e^x+1)}{\log_{0,5} x}$

PROBLEM 2

Differentiate these functions using the chain rule for composite functions:

a) $\log_2(-x)$ b) e^{-2x} c) $\cos\frac{x}{2}$ d) $\sqrt{-4x}$

e) $(e^x - x^e)^{2,71828}$ f) $e^{-\cos x+\sin x}$ g) $\sin 2x + \cos 2x$ h) $\sqrt[2005]{x^{2006} - x}$

i) $\cos(\cos x)$ j) $\exp(-\frac{1}{2}x^2)$ k) $\exp(e^x)$ l) $\exp(-\frac{1}{2}x^2)$

m) $\sqrt{1+\sqrt{1+\sqrt{x}}}$ n) $\ln\sqrt{\frac{1-x}{1+x}}$ o) $\sqrt[-1]{\log_2\frac{x}{2}}$ p) $\left(1+\frac{x}{100}\right)^{100}$

q) $\sin(\cos(\pi^x))$ r) $\operatorname{tg}(\operatorname{ctg}(x^\pi))$

PROBLEM 3

Differentiate these functions using the appropriate rules:

a) $\frac{1}{2}e^{4x} - \frac{1}{3}4^{3x-1}$ b) $x^3(\operatorname{tg}3x - \operatorname{ctg}2x)$ c) $\ln\sqrt{\frac{e^{2x}-e^{-2x}}{e^{2x}+e^{-2x}}}$

d) $3^{-x}\log_3(1+x^3)$ e) $\frac{x^3-x^2+x-1}{2x^2+2}$ f) $(1+x^2)\sqrt{1-x^2}$

g) $\frac{\sin 2x}{\cos^2 x}$ h) $e^{-x}\sin 2x$ i) $x^e\ln(1+e^x) - e\cdot x^{2\pi}$

j) $(2\operatorname{ctg}x - \lg x)^{-2}$ k) $\frac{\sqrt[4]{x^4 - x^{-4}}}{\operatorname{tg}4x}$ l) $\frac{\cos^2 x + \cos x^2}{x^2}$

m) $e^{\sin x}\ln(\sin x)$ n) $\operatorname{ctg}(\operatorname{tg}x) - \operatorname{tg}(\operatorname{ctg}x)$ o*) $\sqrt[3]{\frac{3^x\operatorname{tg}x}{\ln 2x}}$

p*) $\frac{(1-2x^2)\sin x}{e^{2x}\cdot\sqrt{x^2+1}}$ q*) $(\cos x)^{\operatorname{tg}x}$ r*) $x^x\cdot\sqrt[x]{2}$

s*) $(\sin x + \cos x)^{\sin x - \cos x}$ t*) $\sqrt[x]{x}$

Domain of definition, differentiability, derivative

PROBLEM 4

What is the domain of definition and where is it differentiable? Determine the derivative!

(also plot the graph of the original function and give its range.)

a+) $\sqrt[3]{x-1}$ b) $\log_2(1-\cos^2 x)$ c+) $e^{2\cdot\ln(1-x)}$ d) $\operatorname{ctg}\sqrt{x}$

e) $\sqrt{\sin x}$ f+) $\sqrt{\cos x - 1} + 1$ g) $\sqrt{\dfrac{1-x}{2+x}}$ h) $\dfrac{1}{x^2 - 3x + 2}$

i) $2^{-x} \cdot x^{-2}$ j) $\ln(\operatorname{tg} x)$ k+) $\dfrac{\sqrt{-x}+1}{-1+\sqrt{x}}$ l+) $\sqrt[\ln x]{x}$

Equation of tangent line

PROBLEM 5

Detemine the tangent line of $f(x)$ at x_0! (+-problem: also plot the graph of $f(x)$ and its tangent line.) Approximate the value of $f(x)$ using the tangent line equation at $x=x_0+0.1$ or at $x=x_0-0.1$.

a+) $f(x) = \sqrt{x-1},\ \ x_0 = 5$

b+) $f(x) = -\dfrac{x^2}{2} + 3x - 4,\ \ x_0 = 1$

c+) $f(x) = \dfrac{2x+5}{x+2},\ \ x_0 = 0$

d+) $f(x) = 1 - \cos 2x,\ \ x_0 = \frac{\pi}{4}$

e) $f(x) = \ln(x^2 + 2x - 2),\ \ x_0 = 1$

f) $f(x) = 2xe^{x+1} + 1,\ \ x_0 = -1$