Teaching Mathematics and Statistics in Sciences, Modeling and Computer-Aided Approach

IPA HU-SRB/0901/221/088

Experiments in Calculus with GeoGebra

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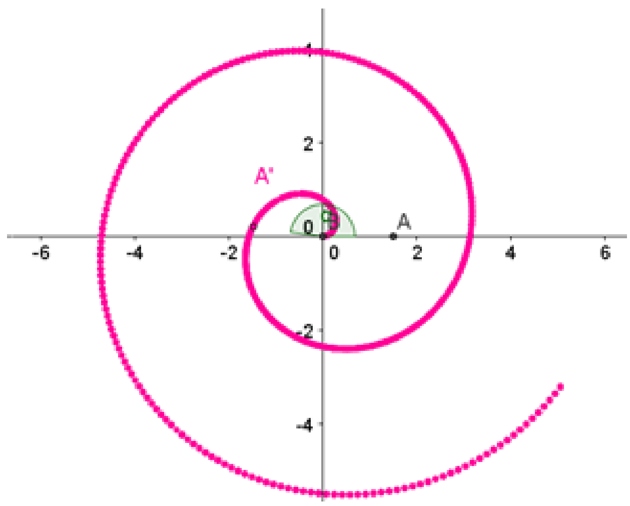

University of Novi Sad
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Preface

In this teaching material the authors' opinion of visualization of calculus by using of package *GeoGebra* is presented. The material can be used for both by complete beginners, as well as by students that already has gone through calculus course. This teaching material represent the modernization of the following course at University of Novi Sad: Calculus for physics, chemistry, mathematics, informatics, pharmacy student.

It is an addendum to the book [4]. All definitions and exercises can be found in this book in Serbian..

1. Functions

1.1. Basic Notions

Definition of Functions

Let A and B be two nonempty sets. By definition, a relation f from A into B is a subset of the direct product $A \times B$.

A relation f is a function which maps the set A into the set B if the following two conditions hold:

- *for every $x \in A$ there exists an element $y \in B$ such that the pair (x, y) is in f;*
- *if the pairs (x, y_1) and (x, y_2) are in f, then necessarily $y_1 = y_2$.*

EXAMPLE 1

Determine the domain D_f for the following functions:

$$a)\ a(x) = \sqrt{2x-5};\quad b)\ b(x) = \sqrt{9-x^2};\quad c)\ c(x) = \sqrt[3]{x+4};$$

$$d)\ d(x) = \sqrt{2x-5};\quad e)\ e(x) = \frac{4}{x^2+4}.$$

SOLUTION:

On Figure 1.1 (linked by Domain) we visualized exercises. Each graphs is presented in different colors and the intervals on *x*-axes, representing the corresponding domain.

The GeoGebra Applets can be obtained by using the hyperlink Domain**.**

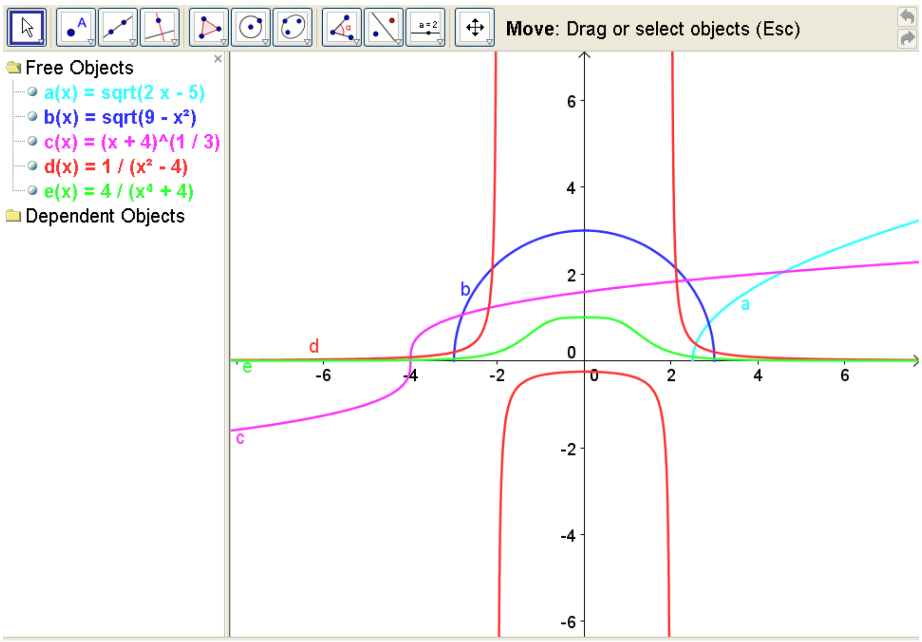


Figure 1.1

Composite of Functions

Let two functions $f : A \to B$ and $g : B \to C$ be given. Then the function $g \circ f : A \to C$ given by

$$g \circ f(x) = g(f(x)), \quad x \in A$$

is called the composite function of the functions g and f.

EXAMPLE 2

The functions f and g are given as:

a) $f(x) = x+3,\ g(x) = 2x - \sqrt{x};$ b) $f(x) = x^2 + 1,\ g(x) = 2x - 3.$

Determine $(f \circ g)(x),\ (g \circ f)(x),\ (f \circ f)(x)$ and $(f \circ f \circ f)(x)$.

SOLUTION

On Figure 1.2 (linked by Composite-a) we denoted by:

$h(x) = (f \circ g)(x),\ \ p(x) = (g \circ f)(x),\ \ p(x) = (f \circ f)(x)\ \ r(x) = (f \circ f \circ f)(x).$

The graphs of these functions are presented in different colors.

On Figure 1.3 (linked by Composite-b) we denoted by:

$h(x) = (f \circ g)(x),\ \ p(x) = (g \circ f)(x),\ \ p(x) = (f \circ f)(x)\ \ r(x) = (f \circ f \circ f)(x).$

The graphs of these functions are presented in different colors.

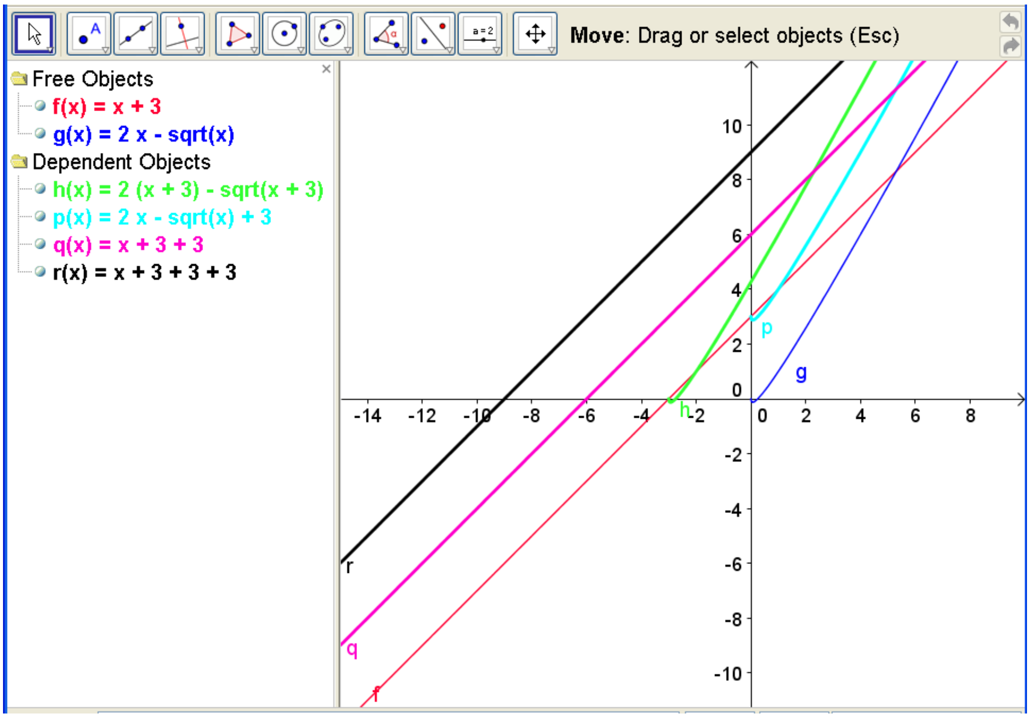


Figure 1.2.

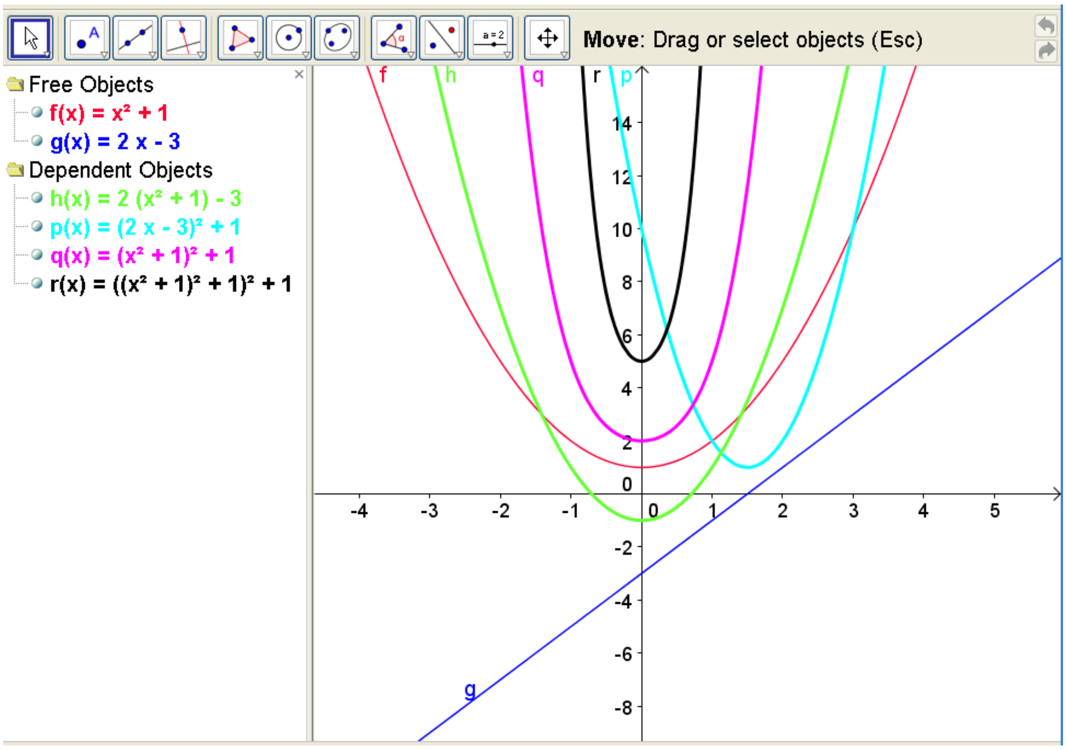


Figure 1.3

Inverse Function

Suppose the function $f: A \to B,$ is a bijection. Then, for every $y \in B$ there exists a unique element $x \in A$ such that $y = f(x)$. Now, the relation from B to A given by

$$f^{-1} := \{(y,x) \in B \times A \mid y = f(x)\}$$

is a function on B which will be called the inverse function for f. The inverse function is also a bijection and it holds

$$f^{-1}(y) = x \iff f(x) = y \quad (x,y) \in A \times B.$$

EXAMPLE 3

Determine the inverse function g, for the given function f,

a) $a(x) = 3x + 1, \quad x \in (-\infty, +\infty);$ b) $f(x) = x^2, \quad x \in (-\infty, 0);$

c) $f(x) = x^2, \quad x \in (0, +\infty);$ d) $d(x) = \dfrac{1-x}{1+x}, \quad x \in (-\infty, -1) \cup (1, +\infty).$

SOLUTION

On Figure 1.4 (linked on Inverse) the graphs of the given functions are drawn in different colors, the function g is inverse to the function a, the points A and A' are symmetric in respect to line given by $y = x$. The slider h enables moving the points A and B and their symmetric points.

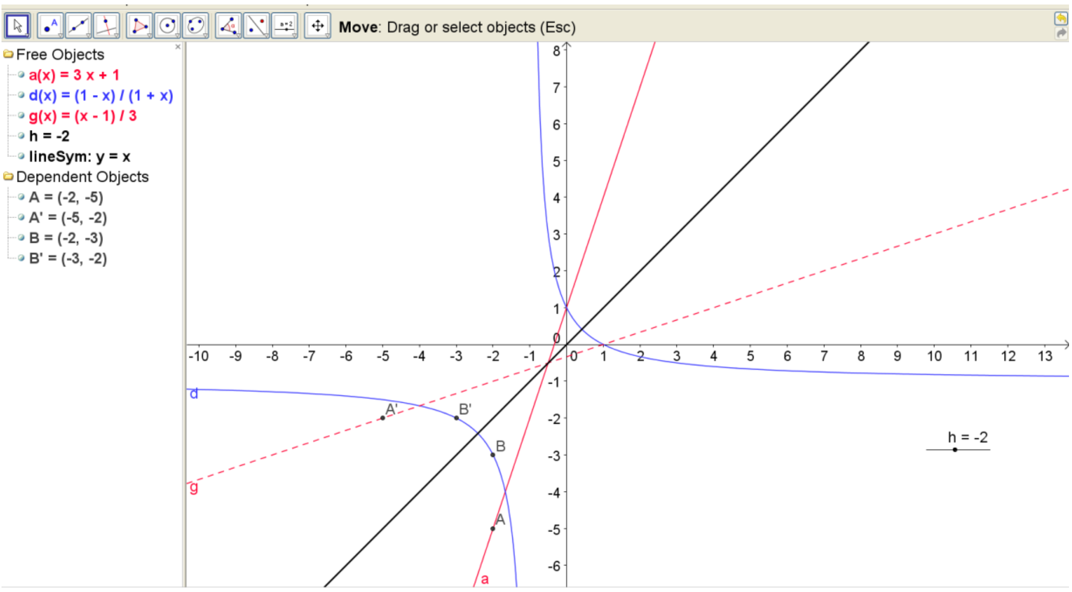


Figure 1.4

Odd and Even Function

Let us suppose that the domain A of a function $f : A \to B$, is *symmetric*. Then f is an

- even function, if for every $x \in A$ it holds $f(-x) = f(x)$;
- odd function, if for every $x \in A$ it holds $f(-x) = -f(x)$.

Geometrically, the graph of an even function is symmetric to the $y-$ axis, while the graph of an odd function is symmetric to the origin.

EXAMPLE 4

Determine which of given functions are odd or even functions:

a) $f(x) = x^2 + 1$; b) $f(x) = \sin x + x$;

c) $f(x) = \sqrt{x^2 + x + 1} - \sqrt{x^2 - x + 1}$; d) $f(x) = \ln \frac{1+x}{1-x}$;

e) $f(x) = e^{x^2} + x^4 + 1$; f) $f(x) = e^{1/x} + 3x$.

SOLUTION

The graph of function in exercise a) is drawn and the point *A'* is symetric with the point *A(a,f(a))*, with respect to *y*-axes. The coordinates of the points *A* can be changed by using slider *a* (Figure 1.5 EVEN).

The functions in a), and e) are even and it can be visualized by changing function.

The functions in b) and d) are odd (Figure 1.6 Odd1.

The function in f) is neither even nor odd.

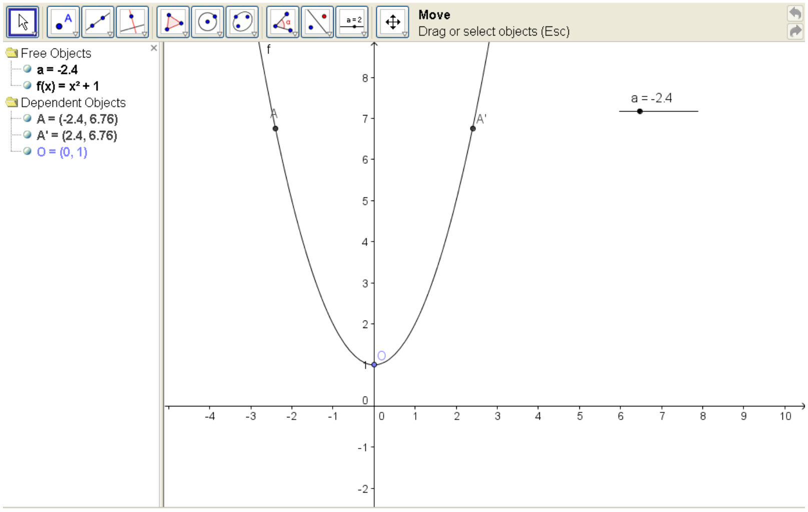


Figure 1.5

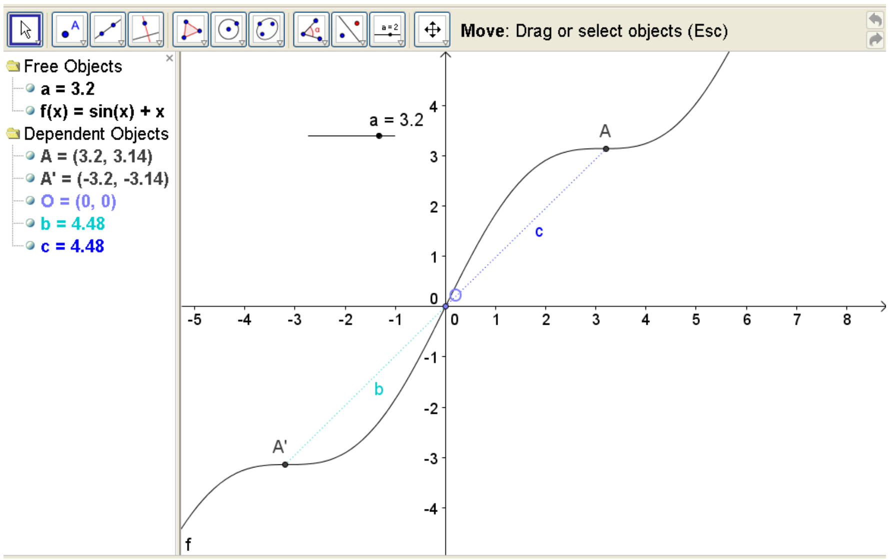


Figure 1.6

Extreme

A function $f : A \to B$ is monotonically increasing (resp. monotonically decreasing) on the set $X \subset A$ if for every pair of elements x_1 and x_2 from the set X it holds

$$x_1 < x_2 \Rightarrow f(x_1) < f(x_2) \quad x_1 < x_2 \Rightarrow f(x_1) > f(x_2)).$$

A function $f : A \to B$ has a local maximum (resp. local minimum) in the point $x_0 \in A$ if there exists a number $\varepsilon > 0$ such that

$$(\forall x \in (x_0 - \varepsilon, x_0 + \varepsilon) \cap A) \quad f(x) \leq f(x_0) \ (f(x) \geq f(x_0)).$$

A function $f : A \to B$ has a global maximum (resp. global minimum) in the point $x_0 \in A$ if

$$(\forall x \in A)\ \ f(x) \le f(x_0)\ \ (\ \ f(x) \ge f(x_0)).$$

On Figure 1.7 (linked on Exstreme) Extreme the function h has local minimums at $A,\ C,\ E,\ G,\ I,$ and local maximums at the points $B,\ D,\ F,\ H,\ J$. Global minimum is at I, and global maximum is at the point H,

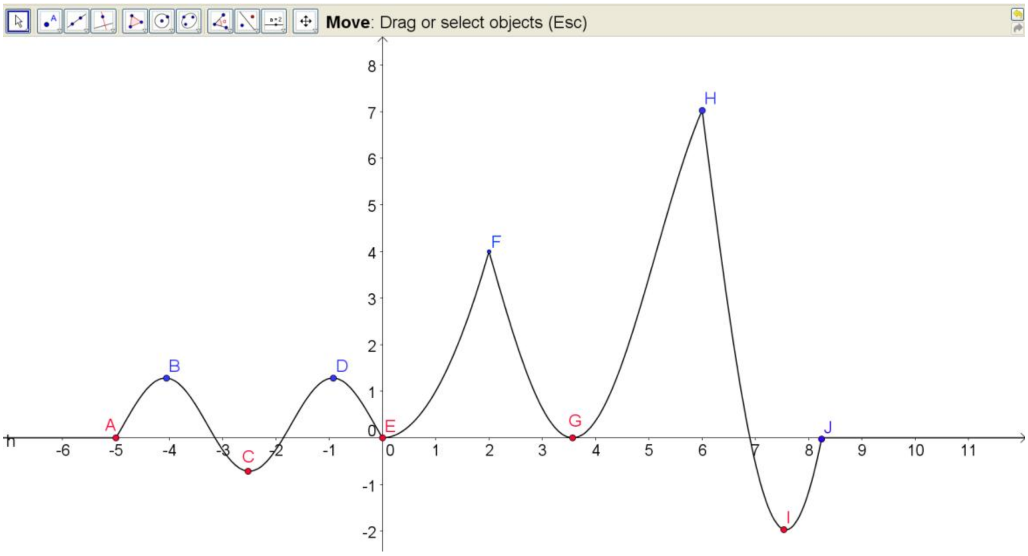


Figure 1.7

Concave functions

A function $f: A \to B,$ is called concave upward on the interval $(a,b) \subset A$ if for every pair $x_1, x_2 \in (a,b)$ and for every $\alpha \in (0,1)$ it holds

$$f(\alpha x_1 + (1-\alpha)x_2) \le \alpha f(x_1) + (1-\alpha) f(x_2).$$

Geometrically, if a function f is concave upward on the interval $(a,b),$ then the segment connecting any two points on its graph is above the graph.

A function $f: A \to B,$ is called concave downward on the interval $(a,b) \subset A$ if for every pair $x_1, x_2 \in (a,b)$ and for every $\alpha \in (0,1)$ it holds

$$f(\alpha x_1 + (1-\alpha)x_2) \ge \alpha f(x_1) + (1-\alpha) f(x_2).$$

Geometrically, if a function f is concave downward on the interval $(a,b),$ then the segment connecting any two points on its graph is under the graph.

On Figure 1.8 (linked on Concave) two functions are consider: $f(x) = x^2,\ g(x) = -x^2$. The first one is concave upward and the second one is concave downward. The points $A, B,$ and $C, D,$ belong to the graphs of f and g, respectively. The segment connecting points $A, B,$ on f is *above* the graph, while the segment connecting points $C, D,$ on g is *above* the graph.

Let us remark that the tangent line on the graph of f, at the point B, is under the graf, while the tangent line on the graph of g, at the point D, is above the graph.

The points A, C, and B, D, can be changed by using sliders a and b, respectively.

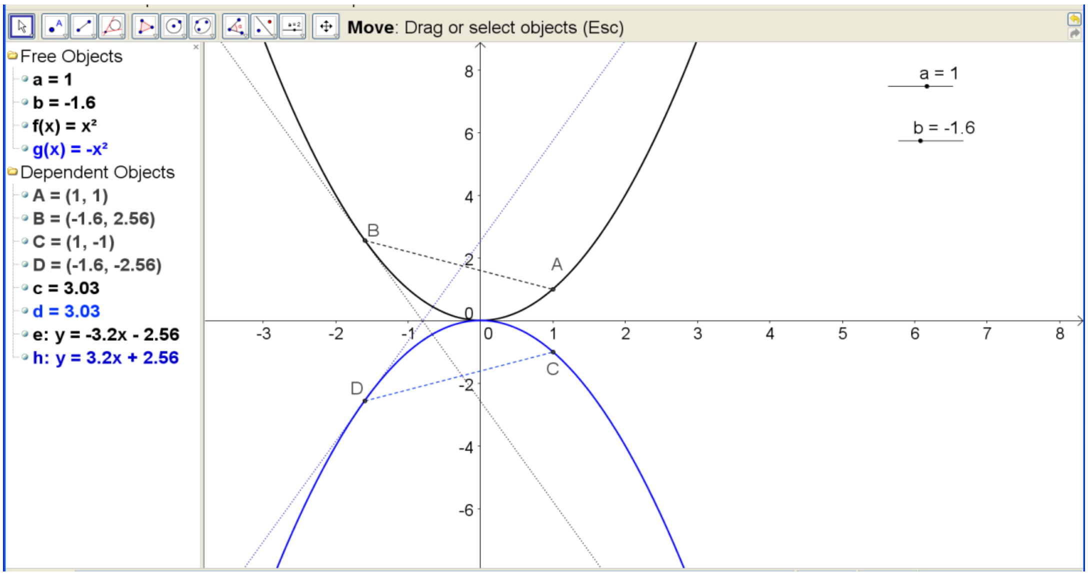


Figure 1.8

Periodic Function

A number $\tau \neq 0$ is called the period of the function $f : A \to B$ if for all $x \in A$ the points $x+\tau$ and $x-\tau$ are also in A and it holds

$$(\forall x \in A) \;\; f(x+\tau) = f(x).$$

The smallest positive period, if it exists, is called the basic period of the function f. Clearly, if we know the basic period T of a function, then it is enough to draw its graph on any set $X \subset A$ of the length T.

EXAMPLE 5

Determine which of given functions are periodic:

a) $f(x) = 2\sin 5x;$ b) $f(x) = \sin^2 x;$

c) $f(x) = \sqrt{\tan x};$ d) $f(x) = \cos 3x + 3\sin 3x.$

SOLUTIONS A)

The function is periodic with period $2k\pi/5, \;\; k \in Z$. On Figure 1.9 linked on Period the graph of function is drawn and the points

$$A(a, f(a)), \;\; B(a+2k\pi/5, f(a+2k\pi/5)), \;\; C(a+k\pi, f(a+k\pi)),$$

$$D(a+2k\pi/3, f(a+2k\pi/3)).$$

Let us remark that points A and B have the same second coordinate, meaning that $2k\pi/5$ is the period of the function. The points C and D illustrate that $k\pi$ and $k\pi/3$ are not the periods for the given function (the second coordinate differ). By using slider *k* we change the integers *k*, and by using slider *a* we change the value of *x*.

The values $k\pi$ and $k\pi/3$ are periods for the function in b) c) and d), respectively. Using the same *GeoGebra* applets one can illustrate it by changing the function. Of course, some other function can be considered.

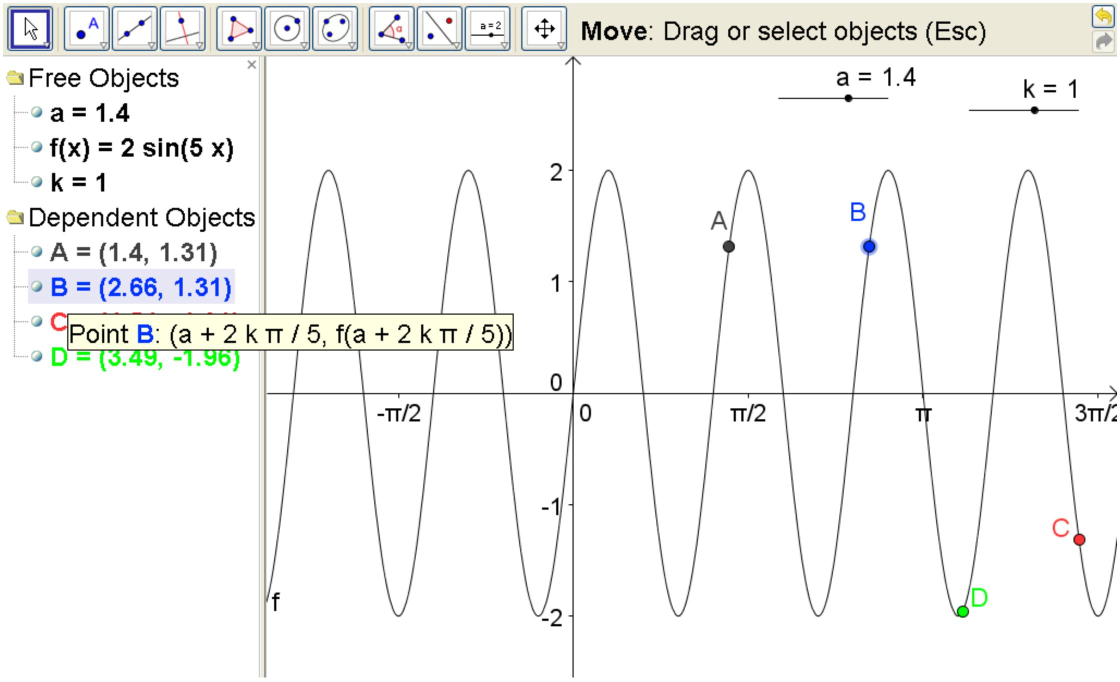


Figure 1.9

1.2. Polynomials

The function

$$P_n(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0, x \in \mathbf{R}, \quad (\ x \in \mathbf{C}),$$

where the coefficients a_j, $j = 0,1,\ldots,n$, are real numbers, is called polynomial of degree $n \in \mathbf{N}$, if $a_n \neq 0$.

By definition, the constant function is a polynomial of degree zero.

On Figure 1.10 we consider the graph of polynomial of six degree (linked on Poly6)

$$P_6(x) = ax^6 + bx^5 + cx^4 + dx^3 + ex^2 + fx + h, \qquad x \in \mathbf{R},$$

For different values of coefficients

$$a, \quad b, \quad c, \quad d, \quad e, \quad f, \quad h,$$

gave by sliders with different color. For

$$a = 1, \quad b = 0, \quad c = -14, \quad d = 0, \quad e = 49, \quad f = 0, \quad h = -36,$$

we have

$$x^6 - 14x^4 + 49x^2 - 36 = p(x) = (x-1)(x+1)(x-2)(x+2)(x-3)(x+3)$$

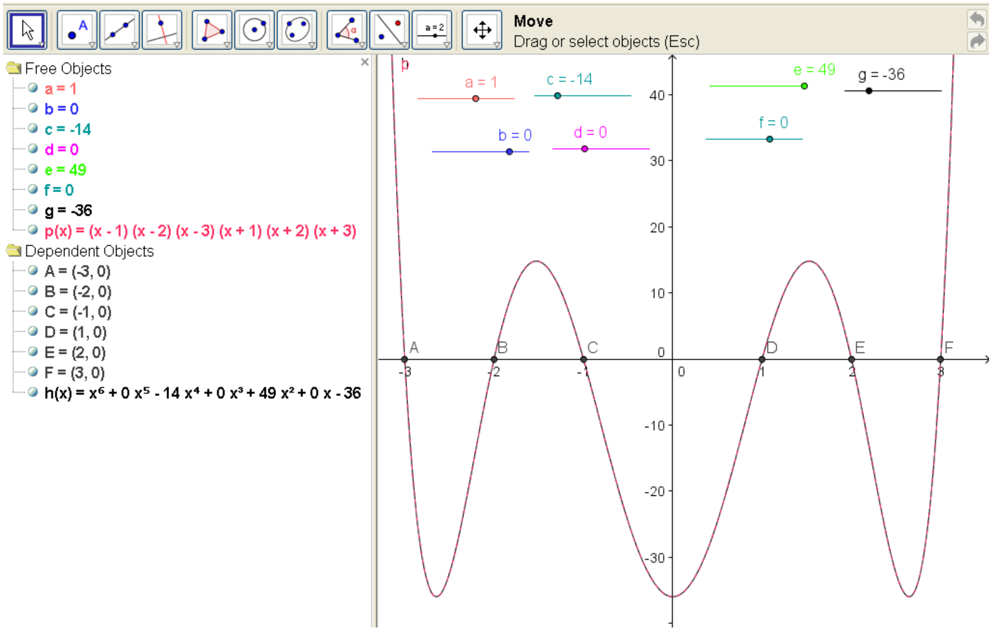


Figure 1.10

EXAMPLE 6

Draw the graphs of the following functions:

a)	$f(x) = x^2$,	$g(x) = x^4$,	$h(x) = x^6$;
b)	$f(x) = -x^2$,	$g(x) = -x^4$	$h(x) = -x^6$;
c)	$f(x) = x$,	$g(x) = x^3$,	$h(x) = x^5$;
e)	$f(x) = -x$,	$g(x) = -x^3$,	$h(x) = -x^5$.

SOLUTION

In general, (Figure 1.11, linked on PolyK) we consider the function $f(x) = a\,x^k$, and by using sliders one can change the coefficients and the powers a, k, of polynomial f respectively.

On Figure 1.12 (linked on Poly246) the polynomials of degrees *2*, *4*, and *6* are drawn, for *k=1*. Taking *k=-1* one can get the solution for b). But the changing of graphs with the change of coefficients can be followed in *GeoGebra* files.

On Figure 1.13 (linked on Poly135) the polynomials of degrees *1*, *3* and *5* are drawn.

Let us remark that all four functions have the 2 common points *A* and *B*.

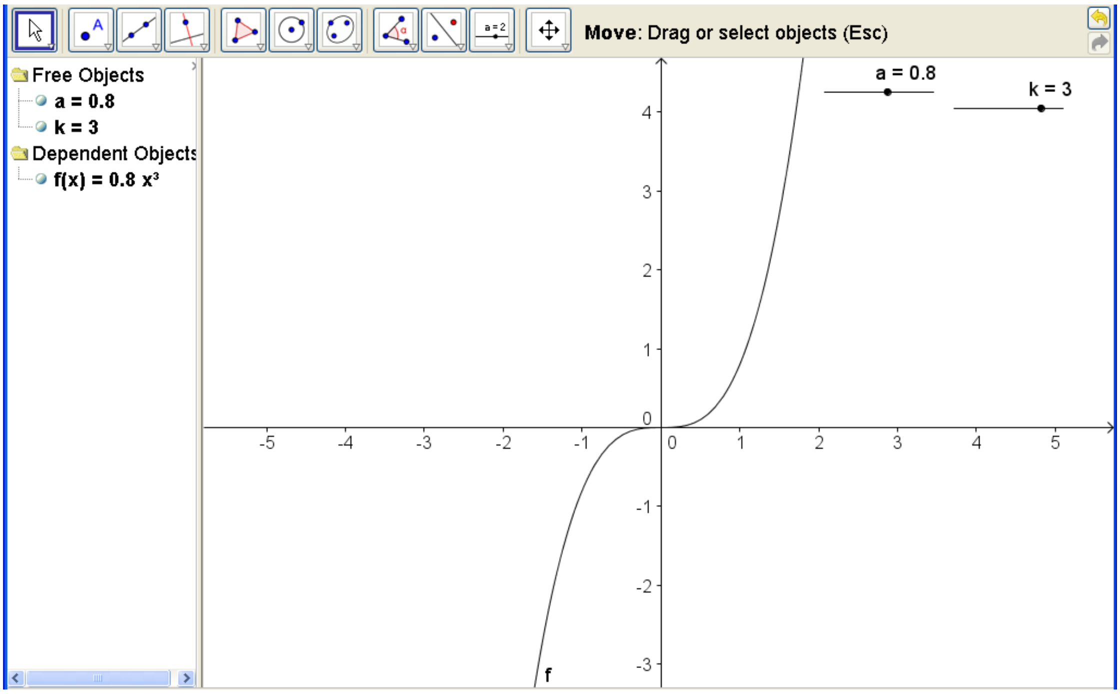


Figure 1.11

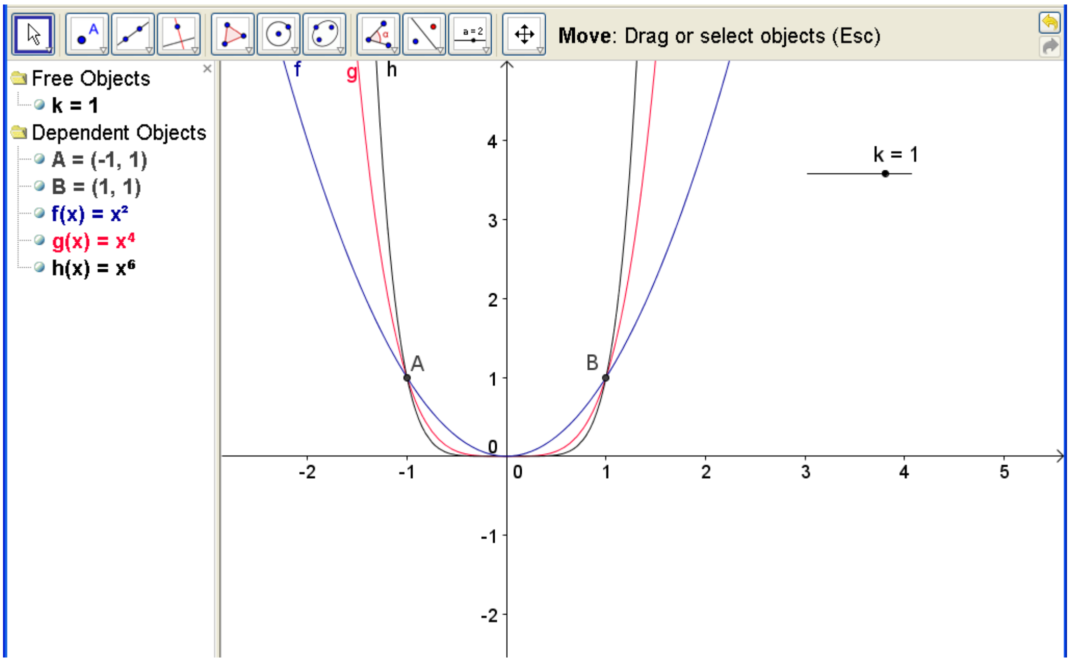


Figure 1.12

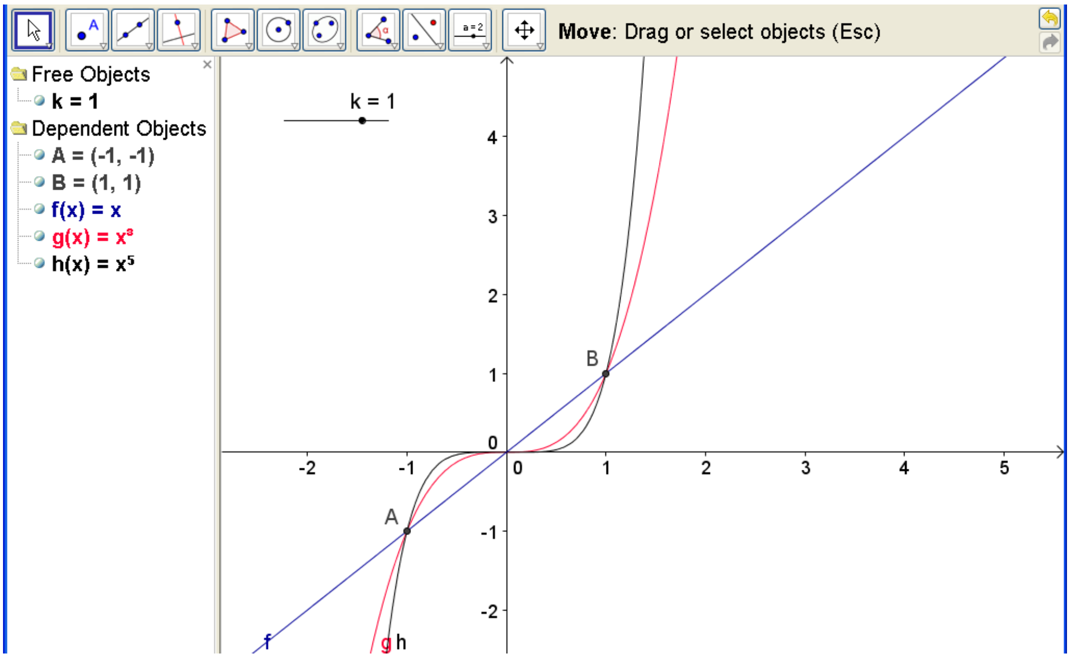


Figure 1.13

1.3. Rational Functions

The **rational function** is the quotient of functions

$$R(x) = \frac{P_n(x)}{Q_m(x)}, \qquad Q_m(x) \neq 0,$$

where $P_n(x)$ and $Q_m(x)$ are polynomials of degree n and m.

EXAMPLE 7

Draw the graphs of the following functions on the same pictures:

$$\text{a)} \quad f(x) = \tfrac{1}{x}, \quad g(x) = \tfrac{1}{x^3}, \quad h(x) = \tfrac{1}{x^5};$$
$$\text{b)} \quad k(x) = \tfrac{1}{x^2}, \quad l(x) = \tfrac{1}{x^4}, \quad m(x) = \tfrac{1}{x^6}.$$

SOLUTION

On Figure 1.13 (linked on RacFunc) both examples are drawn and with two sliders *a* and *k*, one can analyze rational functions $R(x) = a \ x^k$, for different values *a* and *k* (negative in this case).

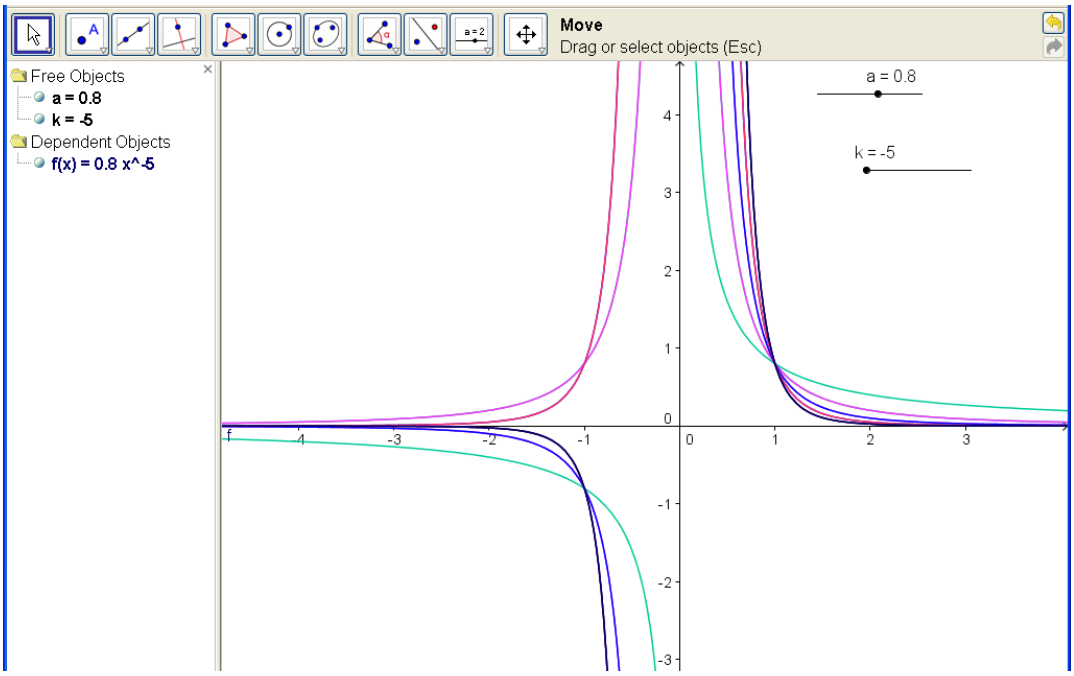


Figure 1.14

1.4. Exponential and logarithmic function

Exponential functions are the functions of the form $f(x) = a^x, \quad x \in R, \quad a > 0, \quad a \neq 1.$

Logarithmic function $f(x) = \log_a x, \quad x \in (0,\infty), \quad a > 0, \quad a \neq 1.$

EXAMPLE 8

On Figure 1.15 (linked on explog) the graphs of functions

$$f(x) = ab^x, \quad g(x) = c\log_d(hx), \ c = \ln a, \quad d = e, \quad h = 1/a$$

are drawn. Let us remark that the functions f and g are inverse, and it is visualized such that the point E' is symmetric with the point E with respect to the line $d: y = x.$

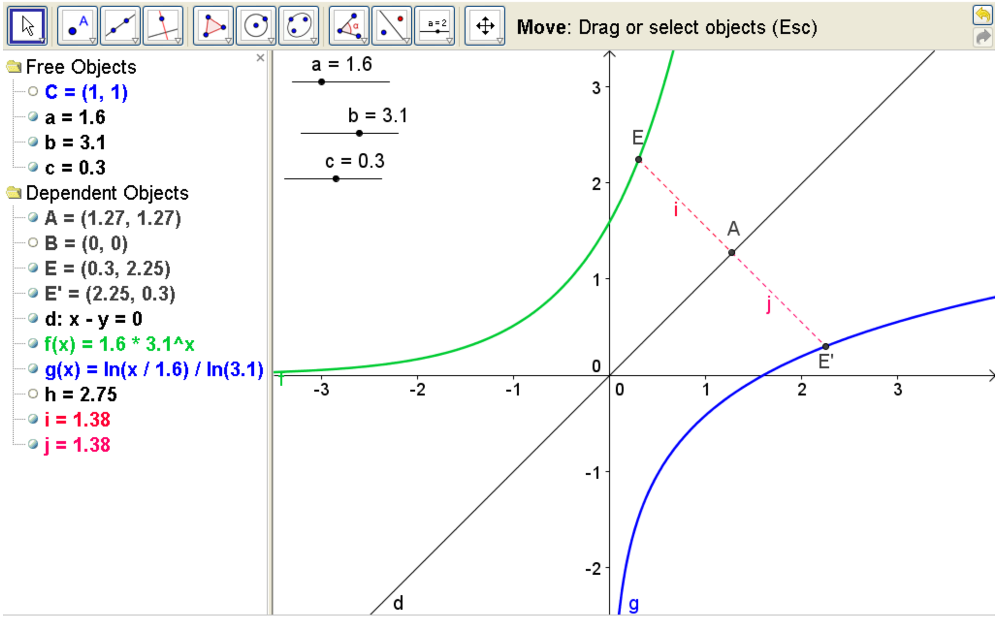


Figure 1.15

The graph of the function $f(x) = be^{-(x-a)^2}$ is drawn on Figure 1.16 (linked on Normras), with its maximum at $A(a, f(a))$.

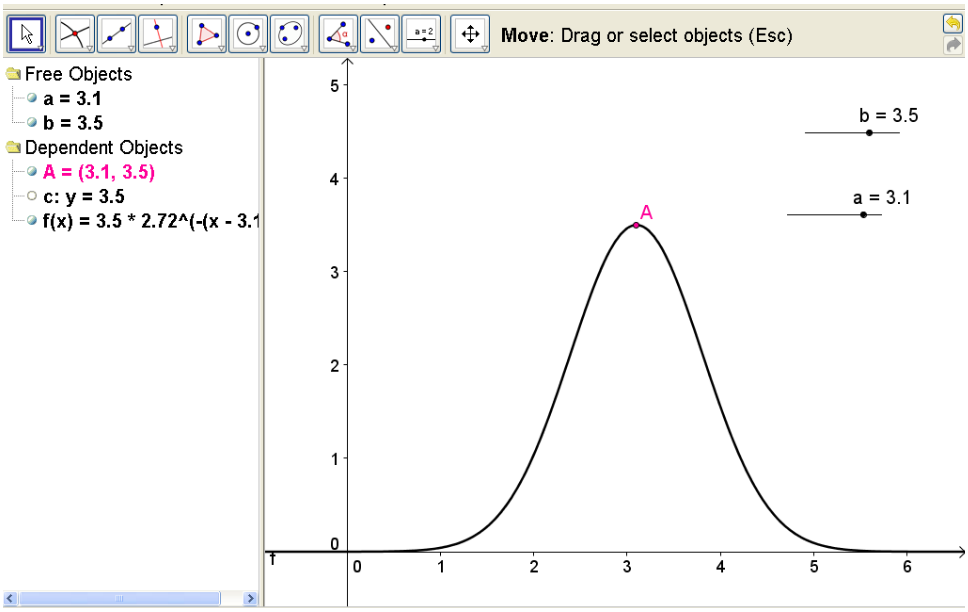


Figure 1.16

1.5. Trigonometric Functions

Trigonometric functions are of the form

$$y = \sin x, \quad x \in R$$
$$y = \cos x, \quad x \in R$$
$$y = \tan x, \quad x \neq k\pi/2, \ k \in Z,$$
$$y = \cot x, \quad x \neq k\pi, \ k \in Z,$$

On Figure 1.17 (linked on Sin) the graph of the function $f(x) = \sin x$ is drawn by using trigonometric circle. The point $M(\alpha, d)$ is on this given circle, corresponding to the angle alpha. The point A has the absisa $x = \alpha$, and the length of ordinate is the same as $MC = \sin\alpha$. By using slider α we change the angle and the point A, with trace included is drawing the of the function $f(x) = \sin x$.

On Figure 1.18 (linked on CosKx) the graph of the function $f(x) = a\cos(kx)$ is drawn by using sliders a and k.

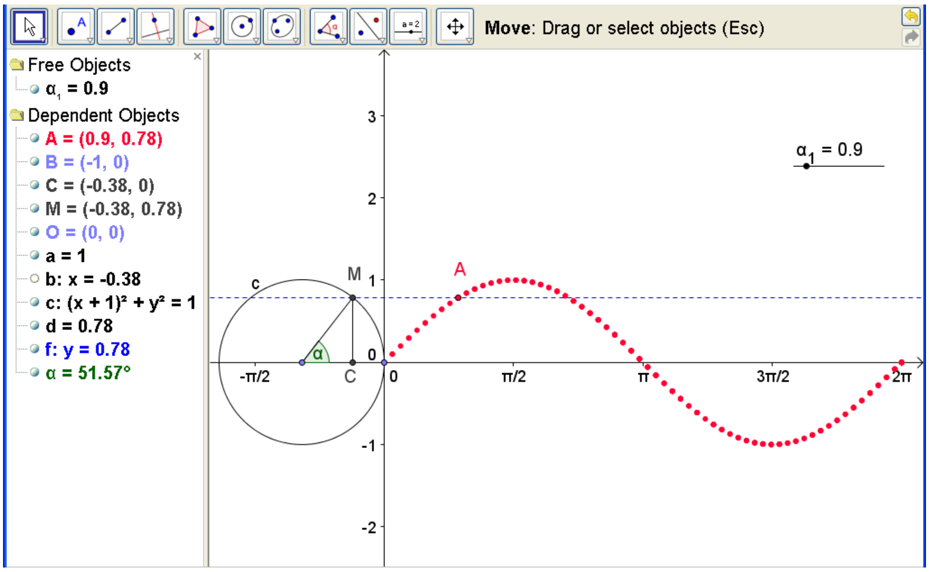


Figure 1.17

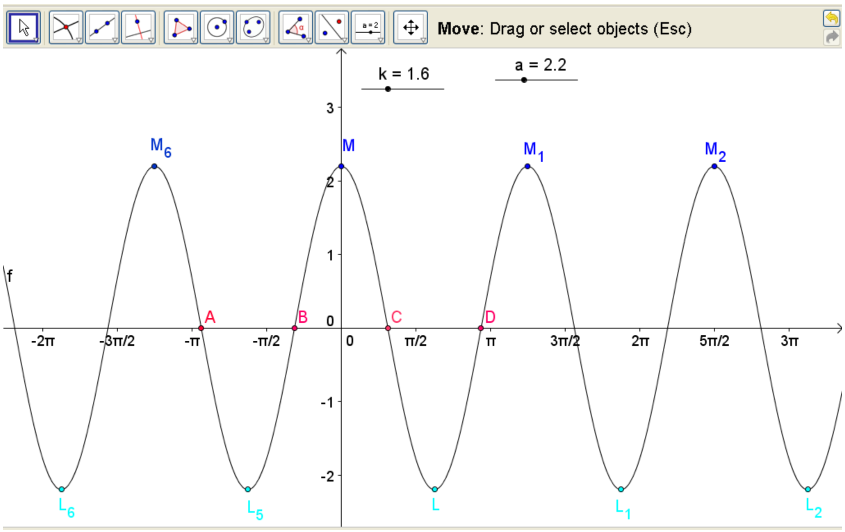


Figure 1.18

1.6. Inverse Trigonometric Functions

Inverse trigonometric functions are of the form

$$y = \arcsin x, \quad D = [0,1], \quad C_D = [-\pi/2, \pi/2],$$
$$y = \arccos x, \quad D = [0,1], \quad C_D = [0, \pi],$$
$$y = \arctan x, \quad D = R, \quad C_D = [-\pi/2, \pi/2],$$
$$y = arc\cot x, \quad D = R, \quad C_D = [0, \pi].$$

On Figure 1.20 (linked on ArcSinCos) the graphs of the functions $f(x) = \arcsin x,\quad f(x) = \arccos x,$ are drawn.

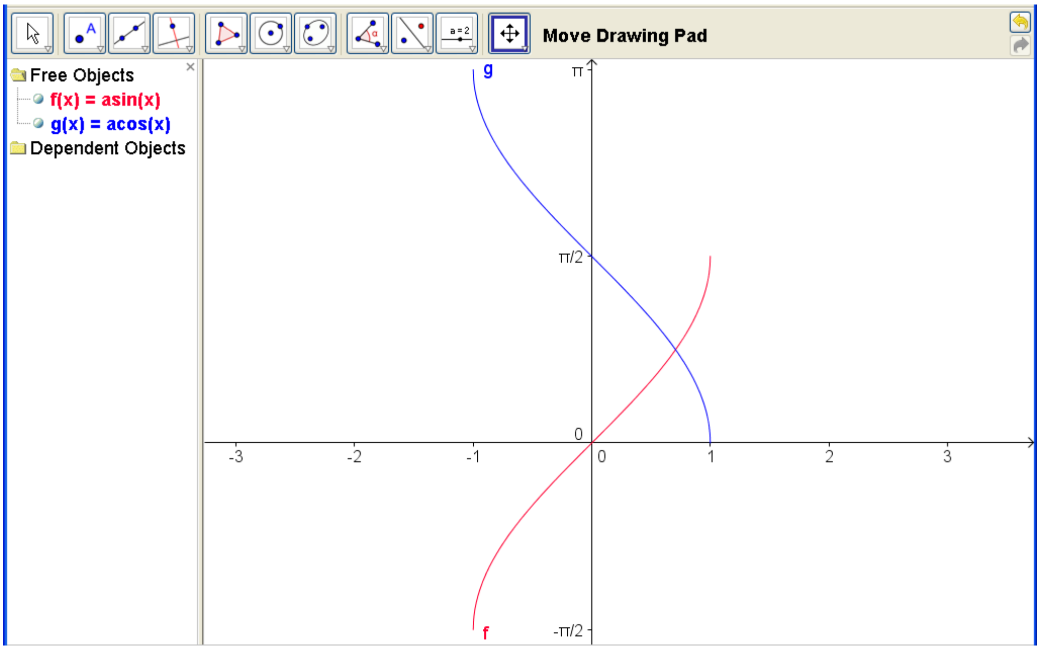
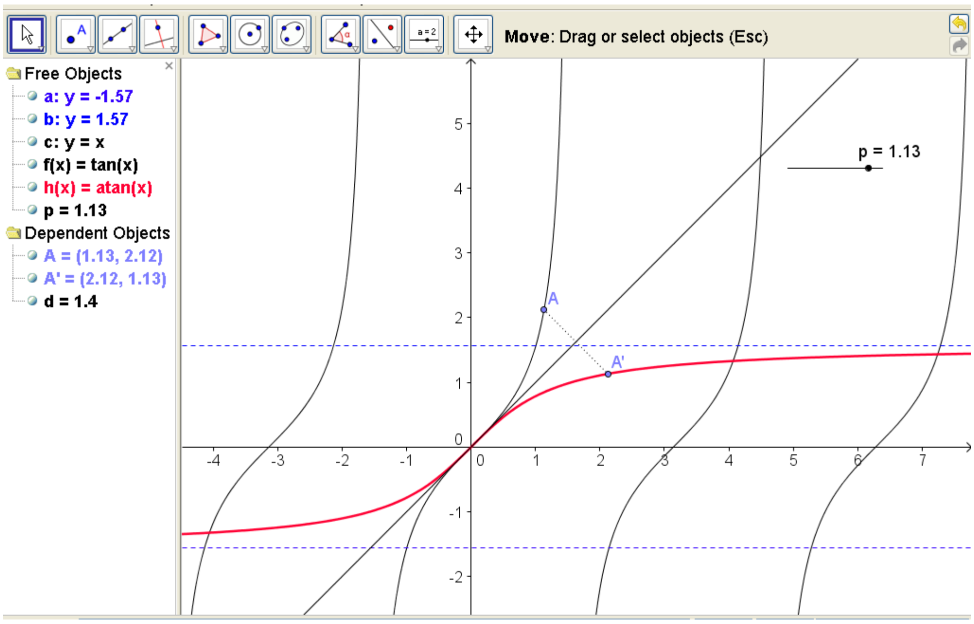


Figure 1.20

On Figure 1.21 Arctan the function $f(x) = \tan x$ and its inverse one $f(x) = \arctan x,$ are drawn, and the points A and B, which are symmetric in respect to the line given by $y = x.$



Fugure 1.21

1.7. Curves given in parametric forms

CYCLOID:

On Figure 1.22 the Cycloid, linked by Cycloid, $x = a(t - \sin t)$, $y = a(1 - \cos t)$, is drawn by using the point $A(a(t - \sin t), a(1 - \cos t))$ and two sliders a, t enabling their changes.

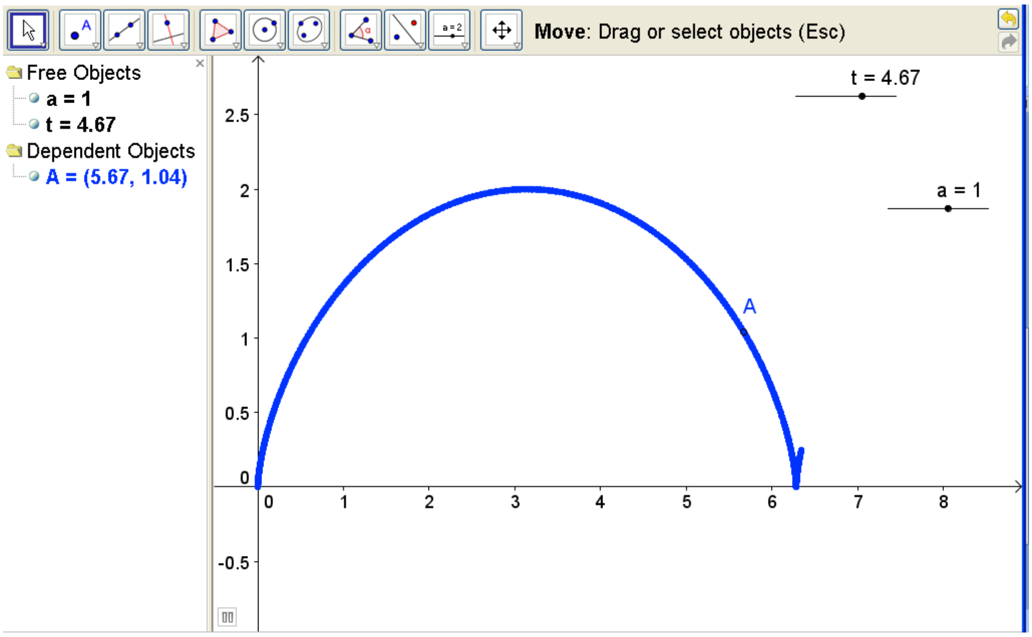


Figure 1.22

ASTROID:

On Figure 1.23 the Astroid, linked by Astroid, $x = a\cos^3 t$, $y = a\sin^3 t$, is drawn

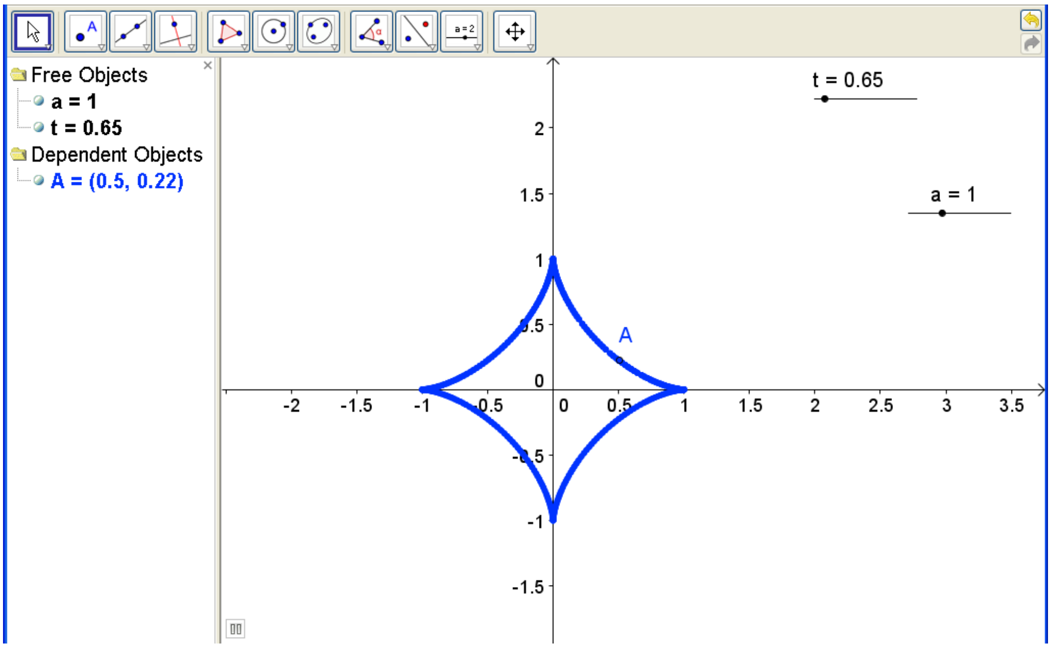


Figure 1.23

DESCATES CURVE:

On Figure 1.24 the Decartes leaves, linked on DecLeav, $x = \dfrac{at}{1+t^3}, \quad y = \dfrac{at^2}{1+t^3}$. is drawn

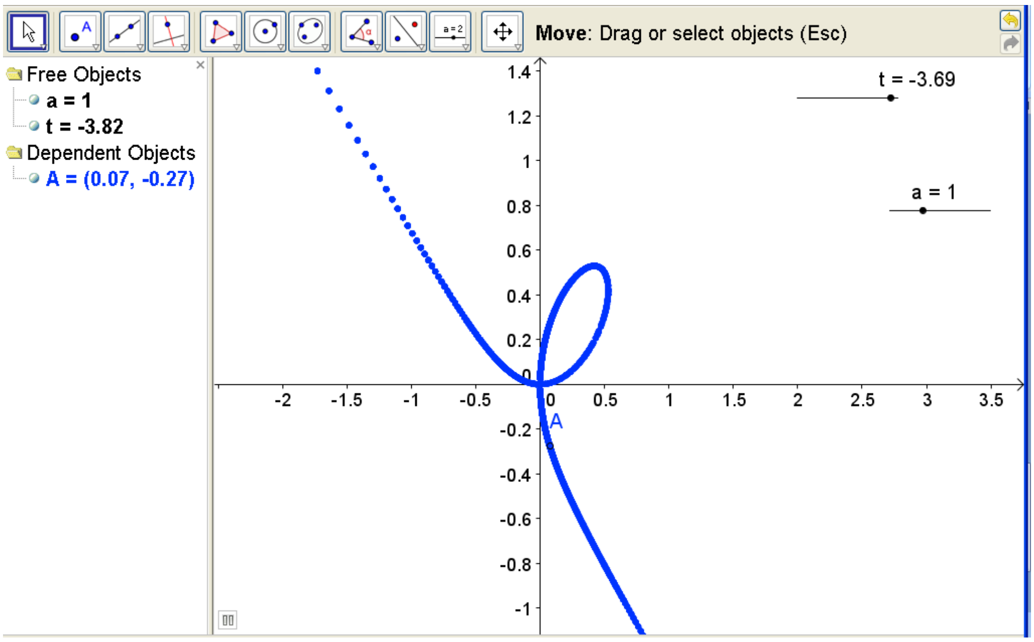


Figure 1.24

1.8. Curves given in polar Coordinates

On Figure 1.25 Lemniscata Bernoulli, linked on BernLemnis, $r = a^2\cos(2t)$ is drawn.

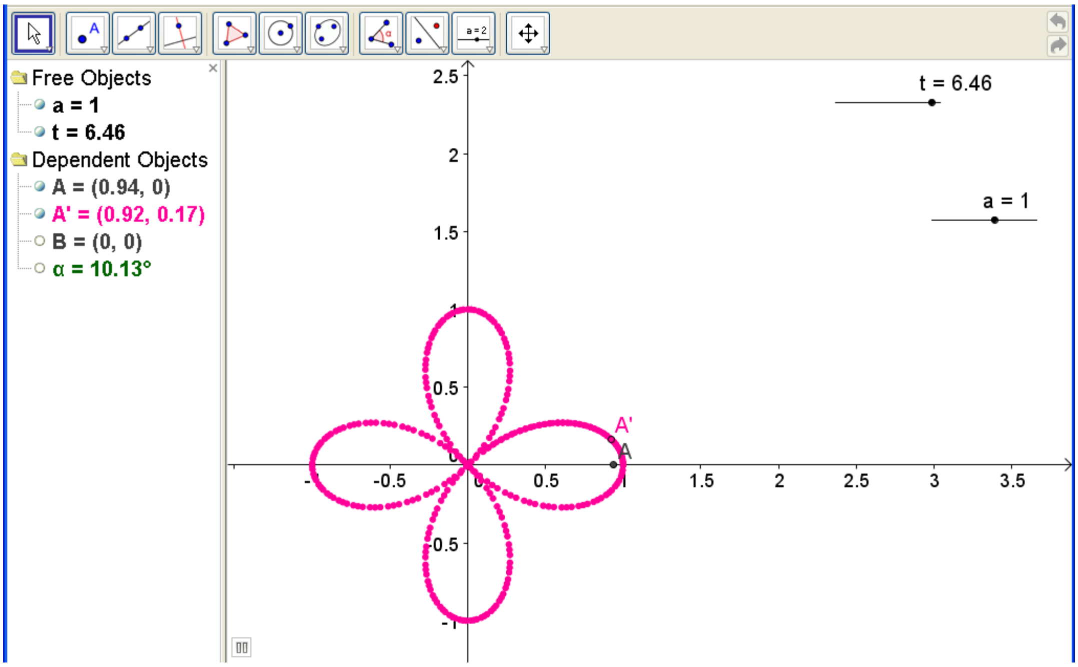


Figure 1.25

On Figure 1.26 Cardioid : $r = a(1+\cos t)$ linked on Cardioid, is drawn.

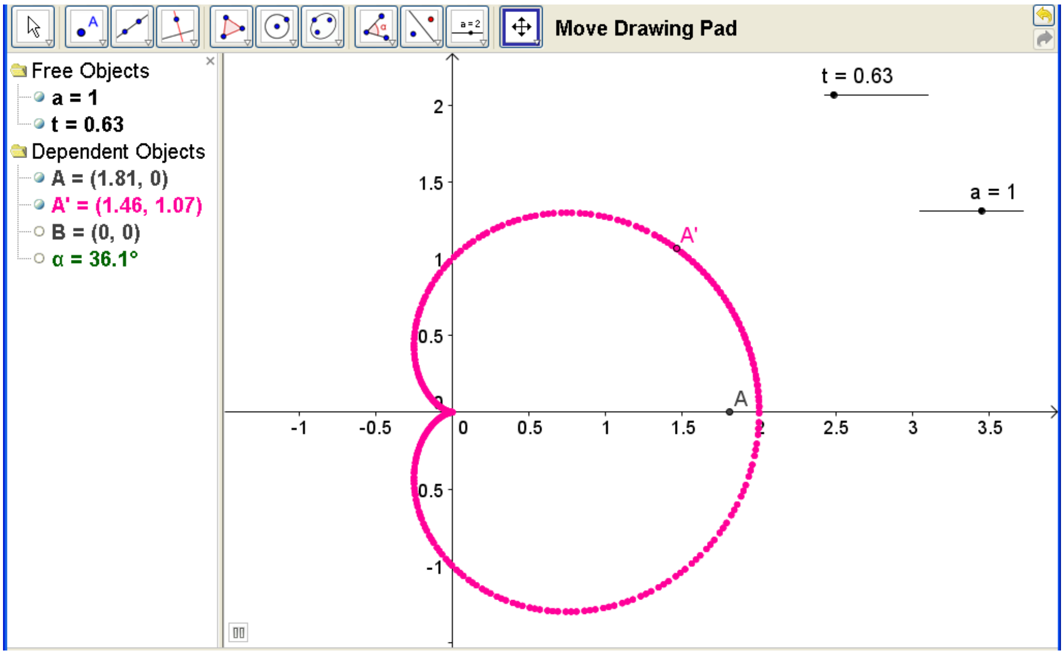


Figure 1.26

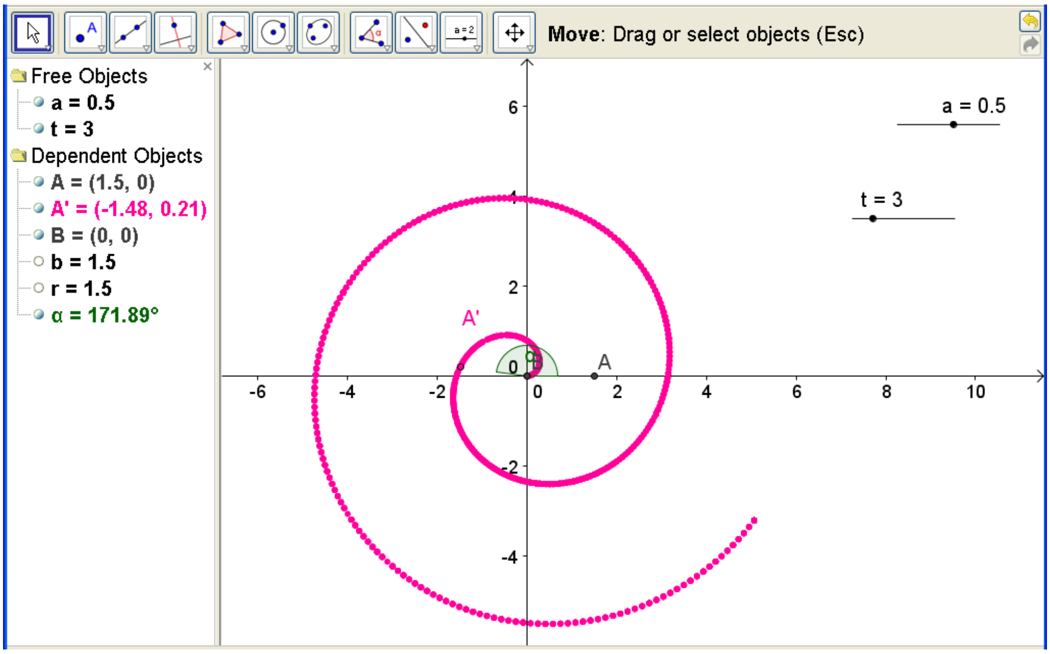


Figure 1.27

On Figure 1.27 Archimedean Spiral linked on, Spiral: $r = at,$, is drawn.

2. Limits and Continuity

2.1. Sequences

A **sequence** is a function $a \ : \ \mathbf{N} \to \mathbf{R}$. It is usual to write

$$a_n \ := a(n), \ n \in \mathbf{N} \quad a = (a_n)_{n \in \mathbf{N}}.$$

In package *Geogebra* the sequences can be visualized by using sliders, and animations. On Figure 2.1 (linked onSeq) we drew the graph of the sequence $a_n = \frac{1}{n}$, by using slider n, and the point $A(n, \frac{1}{n})$, with the trace on. In fact the point A has the $A(n, f(n))$, coordinates meaning that one can change the function f and the sequences is changed also, as on Figure 2.2.

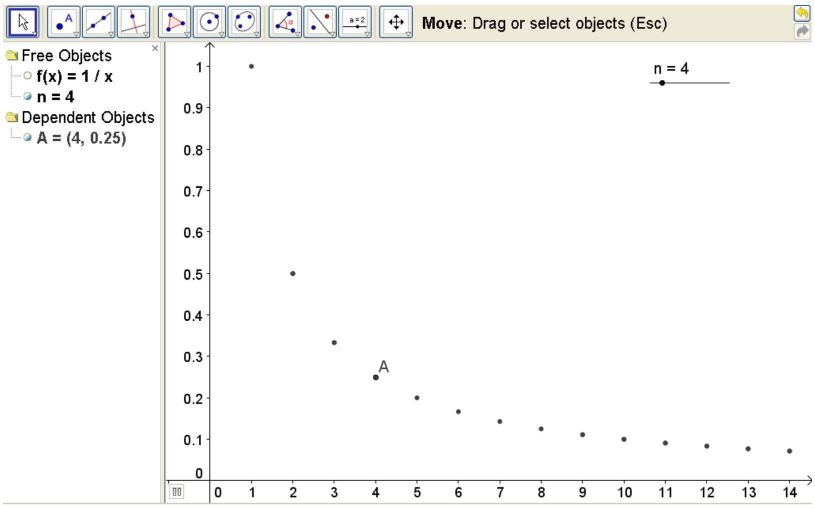


Figure 2.1

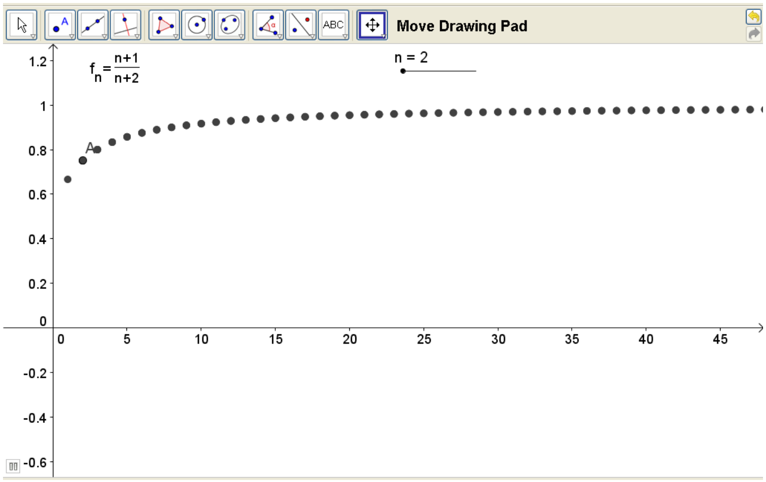


Figure 2.2

EXAMPLE 9

Show that the following holds:

$$\lim_{n\to\infty}\frac{2n^5-3n^2+1}{n^5+3n+2}=2,\quad \lim_{n\to\infty}\frac{3n^4+2n^2+1}{n^3+1}=\infty,\quad \lim_{n\to\infty}\frac{8n^2+3n+1}{n^3+2}=0.$$

SOLUTION

Three points A, B, C, with trace included and with the coordinates

$$A(n,\frac{2n^5-3n^2+1}{n^5+3n+2}),\ B(n,\frac{3n^4+2n^2+1}{n^3+1})\ \ C(n,\frac{8n^2+3n+1}{n^3+2})$$

are considered. The slider n, has included animation and the graph of the three sequences

$$f_n=\frac{2n^5-3n^2+1}{n^5+3n+2},\quad g_n=\frac{3n^4+2n^2+1}{n^3+1},\quad h_n=\frac{8n^2+3n+1}{n^3+2}.$$

It is visualized (linked on Exampl9) that

$$\lim_{n\to\infty} f_n = \lim_{n\to\infty} \frac{2n^5-3n^2+1}{n^5+3n+2} = 2, \quad \lim_{n\to\infty} g_n = \lim_{n\to\infty} \frac{3n^4+2n^2+1}{n^3+1} = \infty,$$

$$\lim_{n\to\infty} h_n = \lim_{n\to\infty} \frac{8n^2+3n+1}{n^3+2} = 0$$

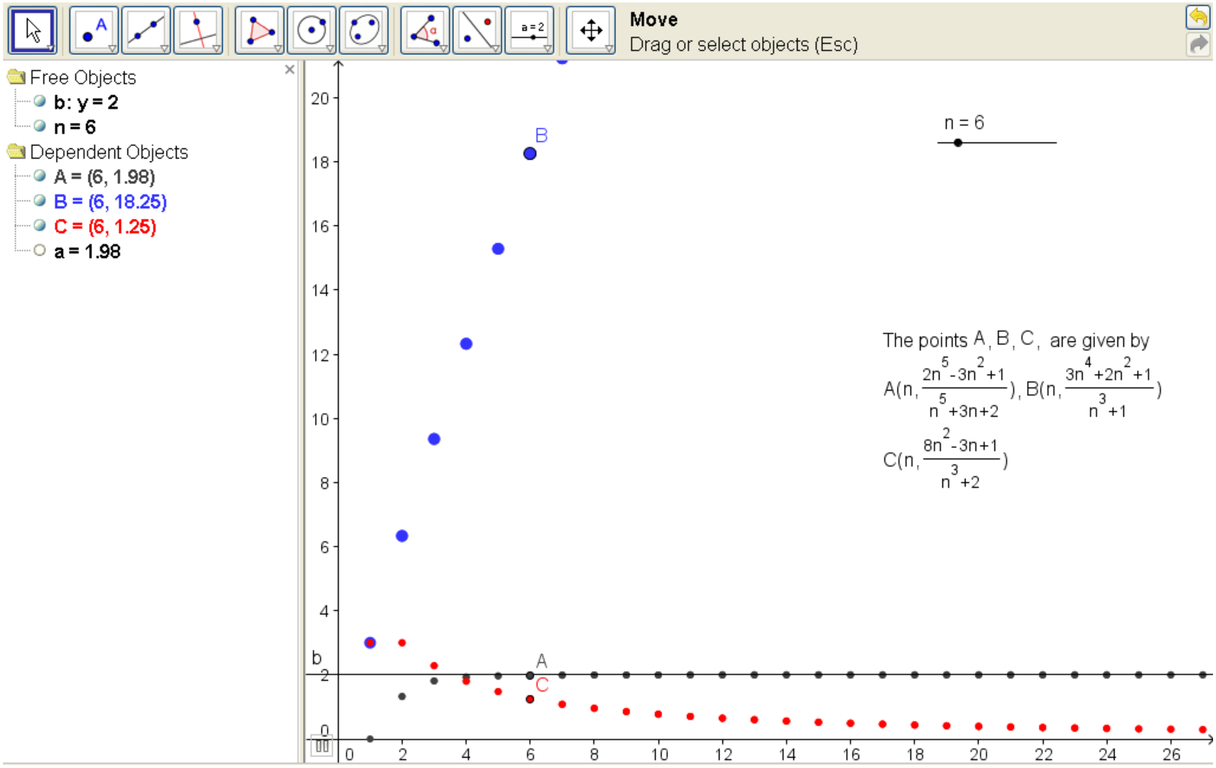


Figure 2.3

EXAMPLE 10

Show

$$\lim_{n\to\infty}(1+\tfrac{1}{n})^{-n} = e^{-1}, \quad \lim_{n\to\infty}(\tfrac{2n+3}{2n})^{-n+2} = e^{-3/2}, \quad \lim_{n\to\infty}(\tfrac{n-1}{n+1})^{n} = e^{-2}$$

SOLUTION

On Figure 2.4 (linked on Example10) three points A, B, C, with trace included and with the coordinates

$$A(n,(1+\tfrac{1}{n})^{-n}),\ B(n,(\tfrac{2n+3}{2n})^{-n+2})\ C(n,(\tfrac{n-1}{n+1})^{n})$$

are considered. The slider n, has included animation and the graph of the three corresponding sequences.

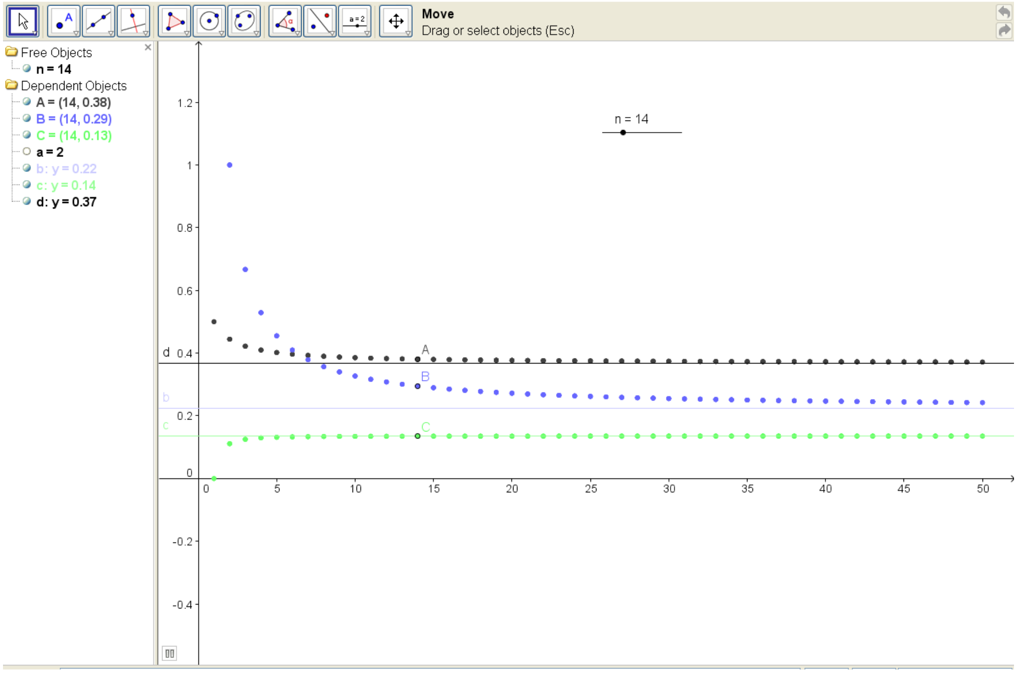


Figure 2.4

EXAMPLE 11

Show

$$\lim_{n\to\infty}\left(\frac{n-1}{n+1}\right)^{n^2}=0, \qquad \lim_{n\to\infty}\left(\frac{n^2-1}{n^2+1}\right)^{n^2}=e^{-2}$$

$$\lim_{n\to\infty}\left(\frac{\ln\sqrt{n+1}-\ln\sqrt{n}}{n}\right)=0, \qquad \lim_{n\to\infty}\left(n(\ln\sqrt{n+1}-\ln\sqrt{n})\right)=\frac{1}{2}$$

SOLUTION

On Figure 2.5 (linked on Example11) the points

$$G\left(n,\left(\tfrac{n^2-1}{n^2+1}\right)^{n^2}\right),\ E\left(n,\left(\tfrac{n-1}{n+1}\right)^{n^2}\right),\ F\left(n,\tfrac{\ln\sqrt{n+1}-\ln\sqrt{n}}{n}\right),\ D(n,n(\ln\sqrt{n+1}-\ln\sqrt{n}))$$

With the trace included, the graph of corresponding sequences can be drawn.

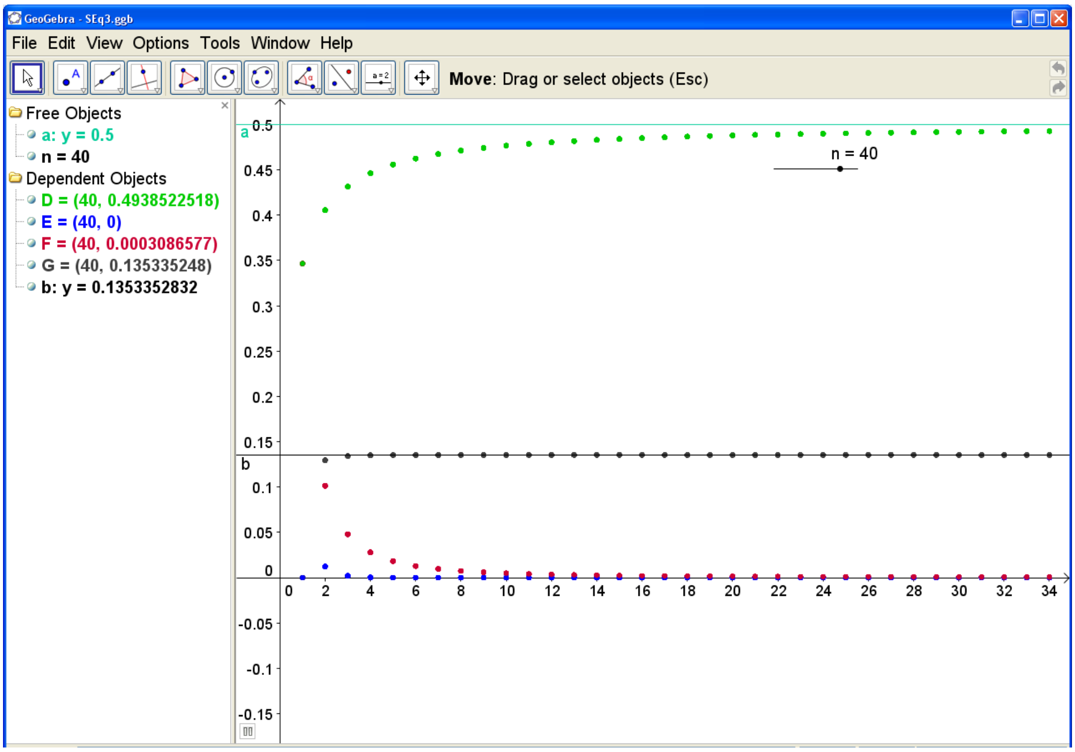


Figure 2.5

2.2. Continuous function

DEFINITION OF CONTINUITY

(the visualization is shown on Figure 2.6, and linked on DefinitionCont). A function $f\ :\ A \subset \mathbf{R} \to \mathbf{R}.$ is continuous at a point $x_0 \in A$ iff for every $\varepsilon > 0,$ there exists a $\delta > 0,\ \delta = \delta(\varepsilon),$ such that for every $x \in A$ it holds:

$$0 < | x - x_0 | < \delta \Rightarrow | f(x) - L | < \varepsilon.$$

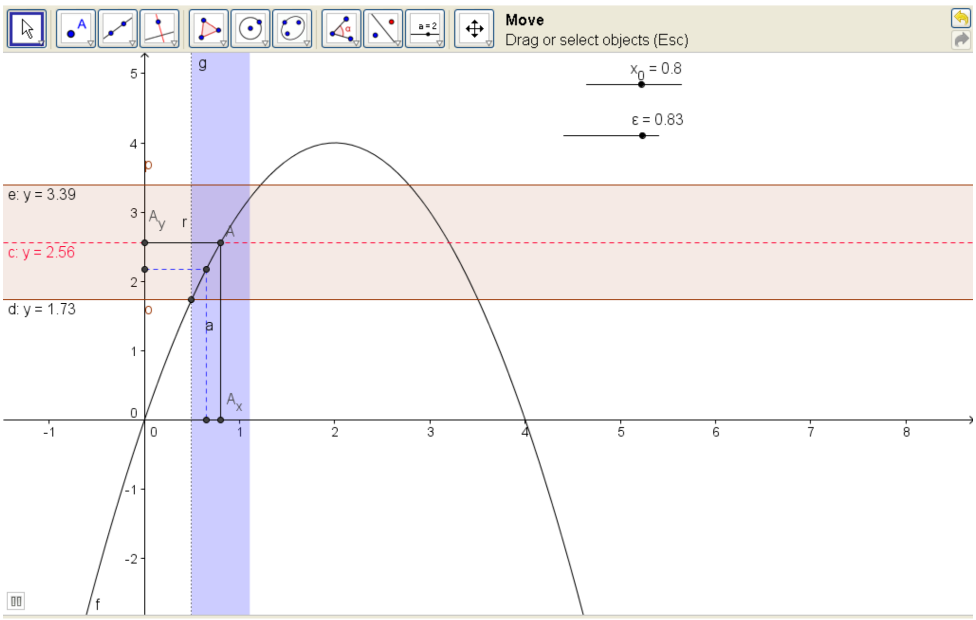


Figure 2.6

If the point $x_0 \in A$ is an accumulation point of the set A, then the following two definitions can also be used:

DEFINITION BY HEINE

It is (linked on Heine), (Figure 2.7). A function $f : A \subset \mathbf{R} \to \mathbf{R}$ is continuous at a point $x_0 \in A$, where x_0 is an accumulation point of the domain A, if for every sequence $(x_n)_{n\in\mathbf{N}}$ of elements from A it holds that

$$\lim_{n\to\infty}(f(x_n) - f(x_0)) = 0,$$

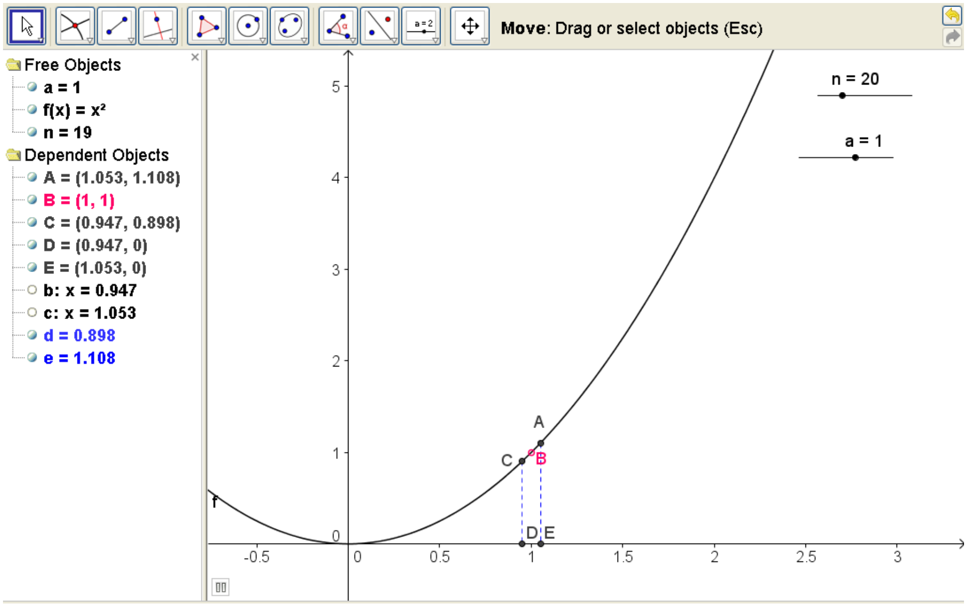


Figure 2.7

3. Derivative of the function

3.1. On the visualization of the first derivative of function

Let f be a real function defined on an open interval (a,b) and let $x_0 \in (a,b)$. Then the following limit

$$f'(x_0) = \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

(provided it exists) is called the first derivative of f at the point x_0. The number h in is called the increment of the independent variable x at the point x_0, while the difference $f(x_0+h) - f(x_0)$ is called the increment of the dependent variable at the point x_0.

On Figure 3.1, linked on Deriv1, we consider the function $f(x) = x^2$, and the points $A(a, f(a))$, and $B\left(a, \dfrac{f(a+h) - g(a)}{h}\right)$, depending on a, and h, which can be changed with the sliders. We can fix one of slider, for example h, and move a, then we can the graph on Figure1. The trace is included for point B. If we fixed a, and change h, , then we are so close to value of the first derivative at the point a.

On Figure 3.2, linked on Deriv2, we consider the function $g(a,h) = \dfrac{f(x+h) - g(x)}{h}$ of two variable representing differential quotation, depending on h, and x. If we move h, with „animation on", we obtain the lines as close to the red line, the graph of first derivative as h.

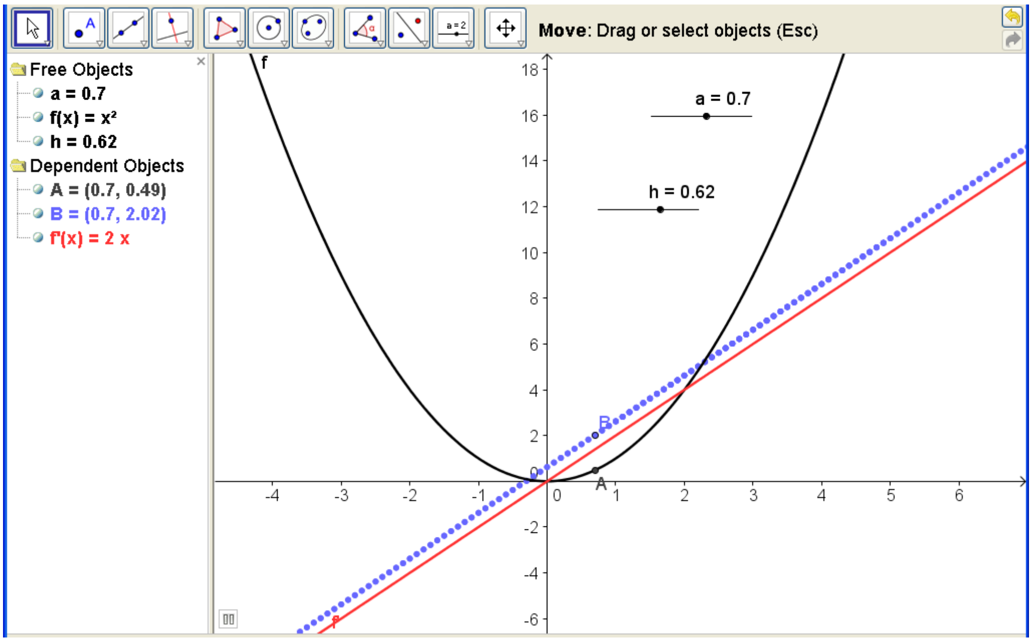


Figure 3.1

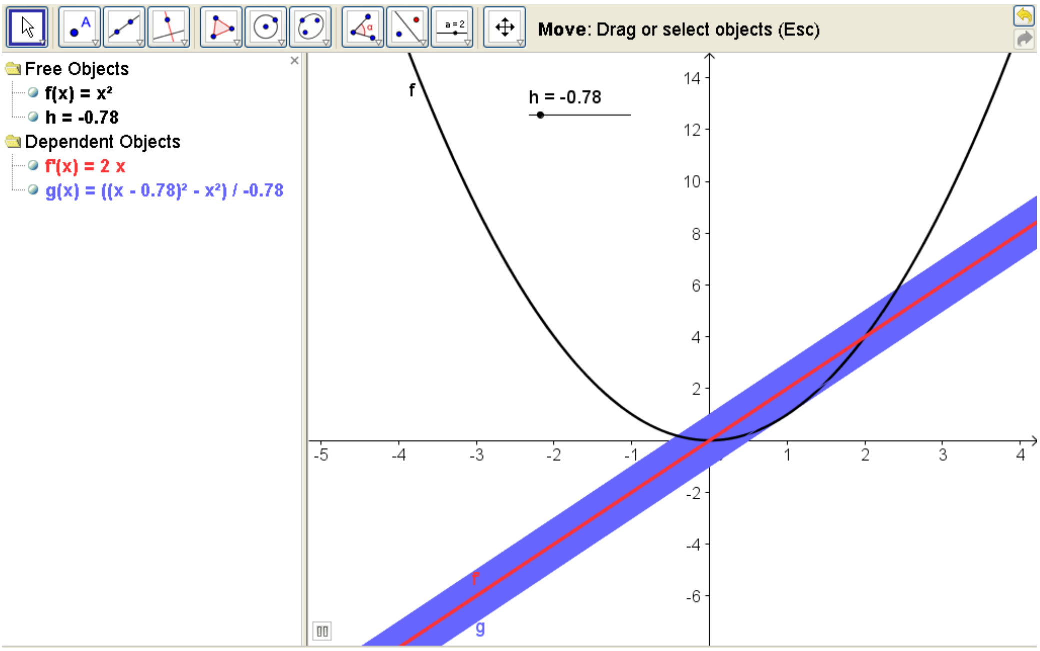


Figure 3.2

Tangent line

If a function $f\ :(a,b)\to\mathbf{R}$ has a first derivative at the point $x_0\in(a,b),$ then the line

$$y-y_0=f'(x_0)(x-x_0),$$

where $y_0=f(x_0),$ is the tangent line of the graph of the function f at the point $T(x_0,f(x_0)).$

If it holds $f'(x_0)\neq 0,$ the line

$$y-y_0=-\frac{1}{f'(x_0)}(x-x_0)$$

is the perpendicular line of the graph of the function f at the point $T(x_0,f(x_0))$.

If a function f has a first derivative at a point x_0, and $0\leq\alpha<\pi$ is the angle between the tangent line at the point x_0 and the positive direction of the $x-$ axis, then it holds

$\tan\alpha=f'(x_0).$

The slope of the tangent line of the graph f at some point is exactly the value of the first derivative of f at that point.

On Figure 3.3, linked on GeomDer we considered the points $A(x_0,f(x_0))$ and $B(x_0+h,f(x_0))$ on the graph of a function f. Then the slope of the secant line through A and B is equal to

$$k_s=\frac{f(x_0+h)-f(x_0)}{h},$$

while the slope of tangent line of f at the point A,

$$k_t=\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}$$

is equal to the first derivative of f at x_0.

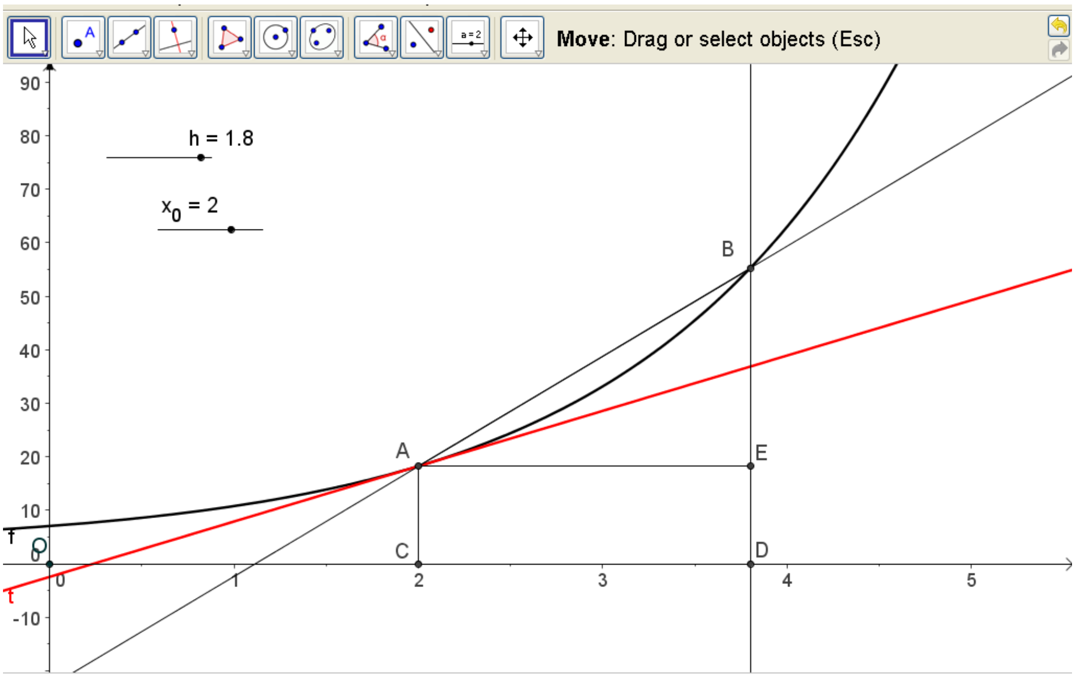


Figure 3.3

On Figure 3.4, linked on GeomInter, we consider the function $f(x) = x^2$, and the points $A(a, f(a))$, and $B(a, f'(a))$, depending on a, which can be changed with the slider. The tangent line t at the point A is constructed and the angle α, between the tangent line and $x-$axes is considered. It can be follows that $\tan\alpha$ is equal to the slope of tangent line at A.

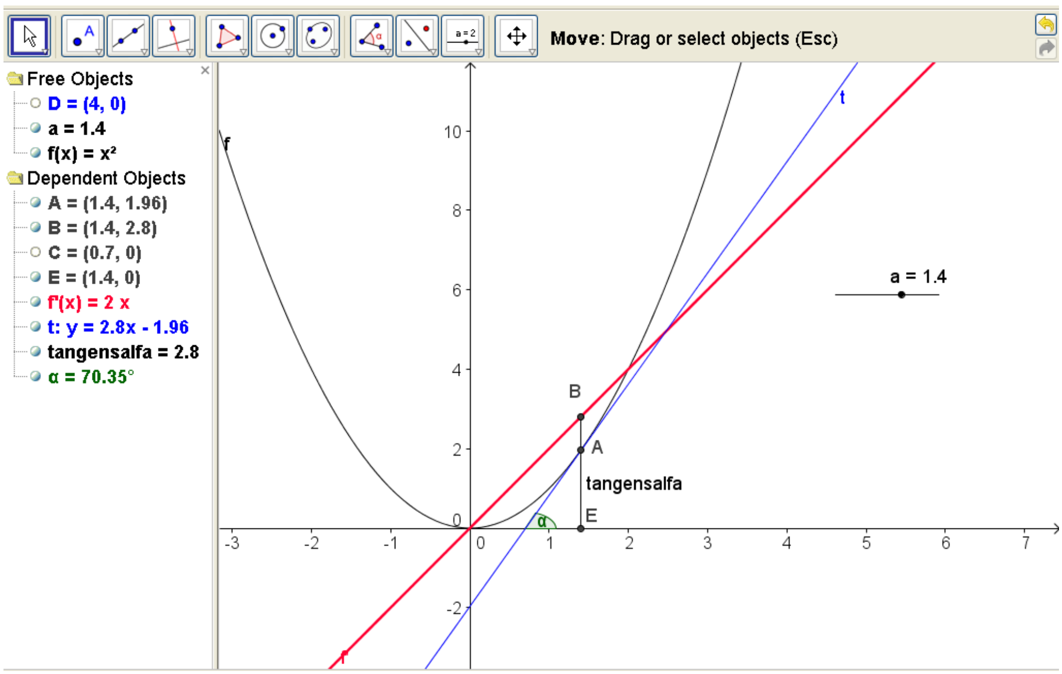


Figure 3.4

Differential of the function

A function $f:(a,b)\to R$ is **differentiable at the point** x_0, if its increment Δy at the point $x_0 \in (a,b)$ can be written in the form

$$\Delta f = f(x_0+h) - f(x_0) = D\cdot h + r(h)\cdot h,$$

for some number D (independent from h), and it holds $\lim_{h\to 0} r(h) = 0$.

On Figure 3.5, linked on diferencijal, the function $f:R\to R$, points $A(x_0, f(x_0)$ and $B(x_0+h, f(x_0+h))$ are consider with the sliders x_0, and h. The points $A_1(x_0,0)$ and $B_1((x_0+h),0)$ are the projections of A and B respectively onto the x-axis. Also, the point $G(x_0+h, f(x_0))$, and the point F, the intersection of the tangent line and vertical line parallel to $y-$axes through the point G. Since it holds

$$\tan\alpha = \frac{FG}{AG},\quad f'(x_0) = \frac{FG}{h}\quad,$$

$$\text{i.e., } FG = f'(x_0)h \text{ and } dy = f'(x_0)dx$$

it follows that FG is the geometric interpretation of the differential of the function f at the point A. In Figure 1 we took $x_0 = 2$, and $h = 1.7$.

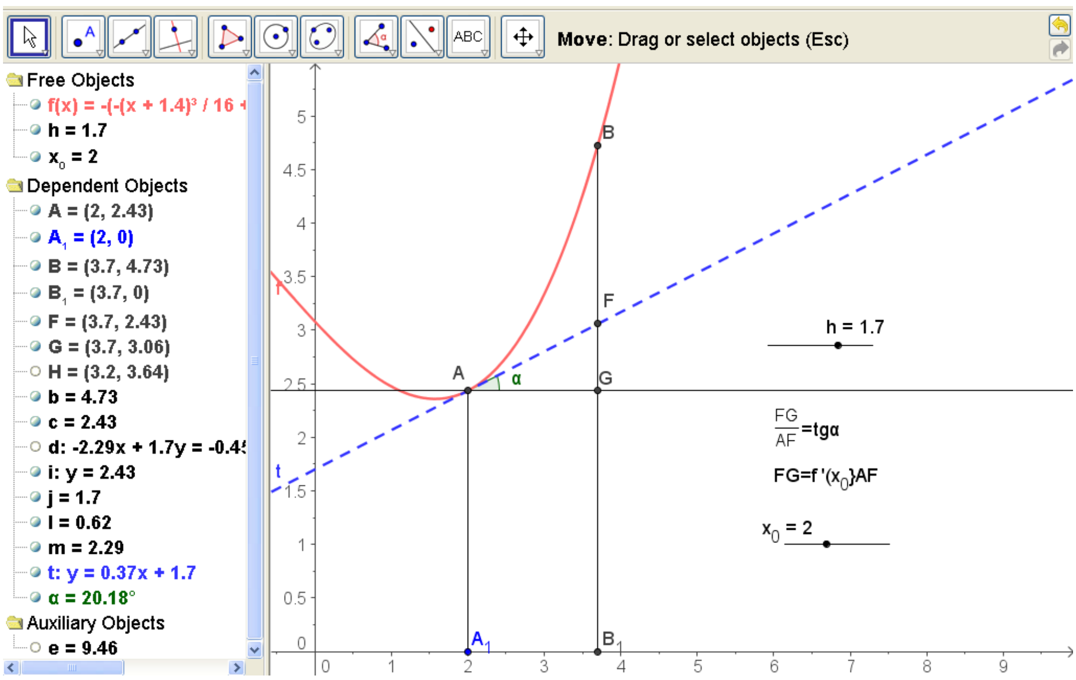


Figure 3.5

Application of derivatives - Graph of Functions

In following the graphs of considered functions are drawn on corresponding figure and on the links to GeoGebra files. Besides, determine:

- The domain of the function f .
- Is it odd or even function.
- The zeros of the function f .
- The first derivative of the function f .
  - The critical points of f .
  - The monotonicity of function f .
- The second derivative of function f.
  - The extremes of f.
  - The concavity of f.
  - The points of inflection of f .
- The asymptotes of f
  - vertical asymptotes.
  - horizontal asymptotes.
  - slanted asymptotes.

In following the graphs of considered functions are drawn in black color, their first derivative in red color, and second derivative in green color.

EXAMPLE 12

The graph of the function $f(x) = x^4 - x^3$, is linked on Graph1 and drawn on Figure 3.6

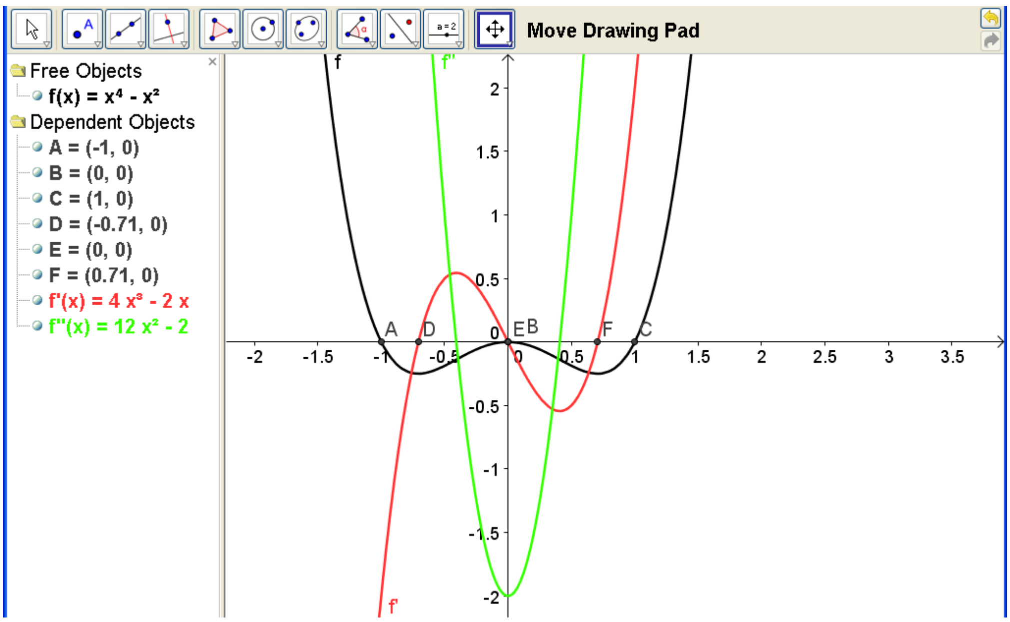


Figure 3.6

- What is the difference between the points $A, C,$ and the point B, as they represent the zeroes of the function?
- Read from the graphs the connection of the function f, and it first derivative f' in the sence of monotonicity and extremes.
- Read from the graphs the connection of the function f', and it first derivative f'' in the sence of monotonicity and extremes.
- Read from the graphs the concavity of the function and its inflection point.
- Where are the asymptotes?

EXAMPLE 13

The graph of the function $f(x) = x^6 - 12x^4 + 48x^2 - 189$ is linked on Graph2 and drawn on Figure 3.7.

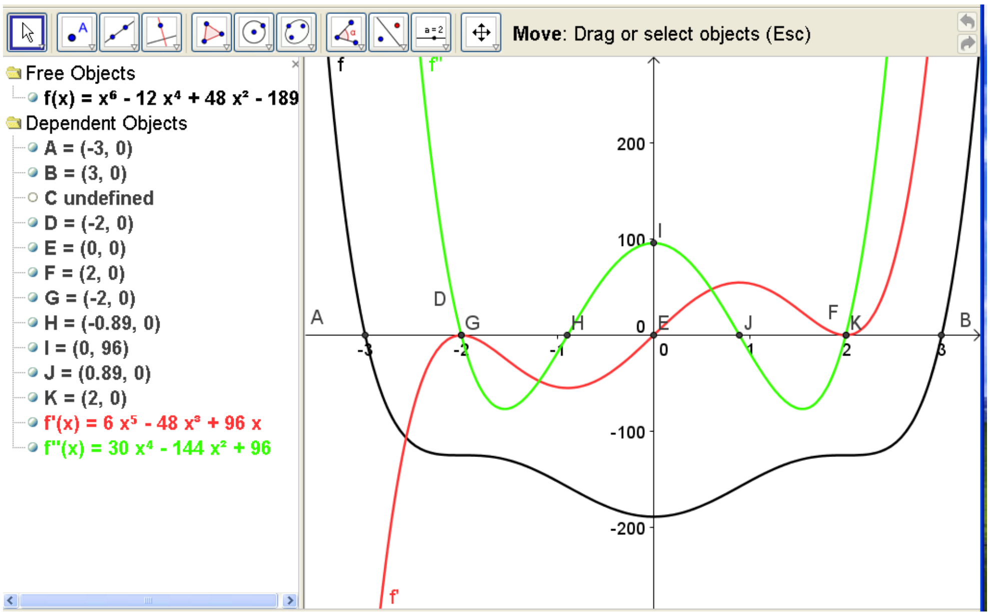


Figure 3.7

- The function is polynomial of 6th power, but it has only 2 zeroes. Explain.
- Read from the graphs the connection of the function f, and it first derivative f' in the sence of monotonicity and extremes. Add extremes by using GeoGebra.
- Read from the graphs the connection of the function f', and it first derivative f'' in the sence of monotonicity and extremes.
- Read from the graphs the concavity of the function and its inflection point.
- Determine the zeroes of the derivative and explain graphically.
- Where are the asymptotes?

EXAMPLE 14

The graph of the function $f(x) = \dfrac{x}{(x-2)^2}$ is linked on [Graph3]() and drawn on Figure 3.8.

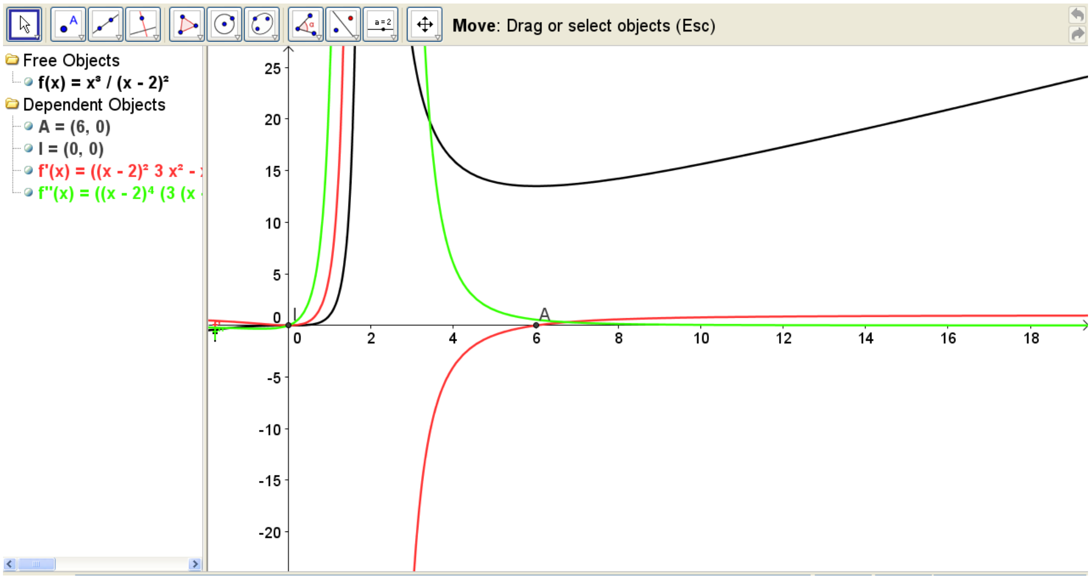


Figure 3.8

- Explain the zero of the function, of the first and second derivative, the point $I(0,0)$.
- Read from the graphs the connection of the function f, and it first derivative f' in the sence of monotonicity and extremes.
- Read from the graphs the connection of the function f', and it first derivative f'' in the sence of monotonicity and extremes.
- Read from the graphs the concavity of the function and its inflection point.
- The vertical asymptotes are?
- The slanted asymptotes are?

EXAMPLE 15

The graph of the function $f(x) = e^{1/x}$ is linked on **Graph4** and drawn on Figure 3.9.

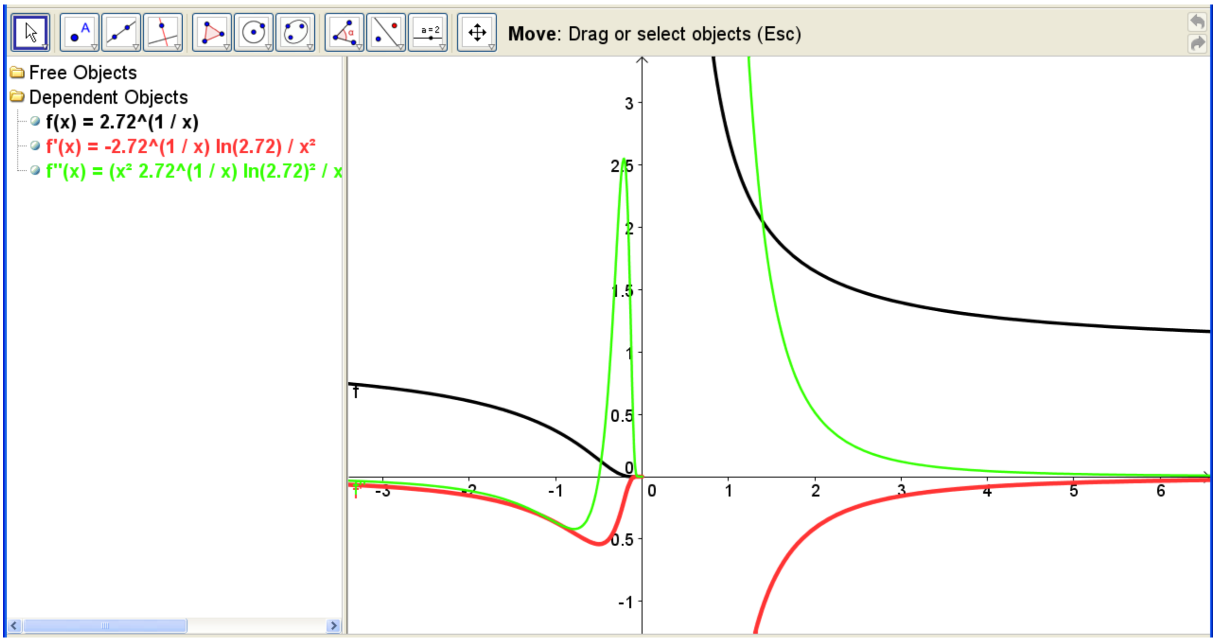


Figure 3.9

EXAMPLE 16

The graph of the function $f(x) = xe^{1/x}$ is linked on Graph5 and drawn on Figure 40.

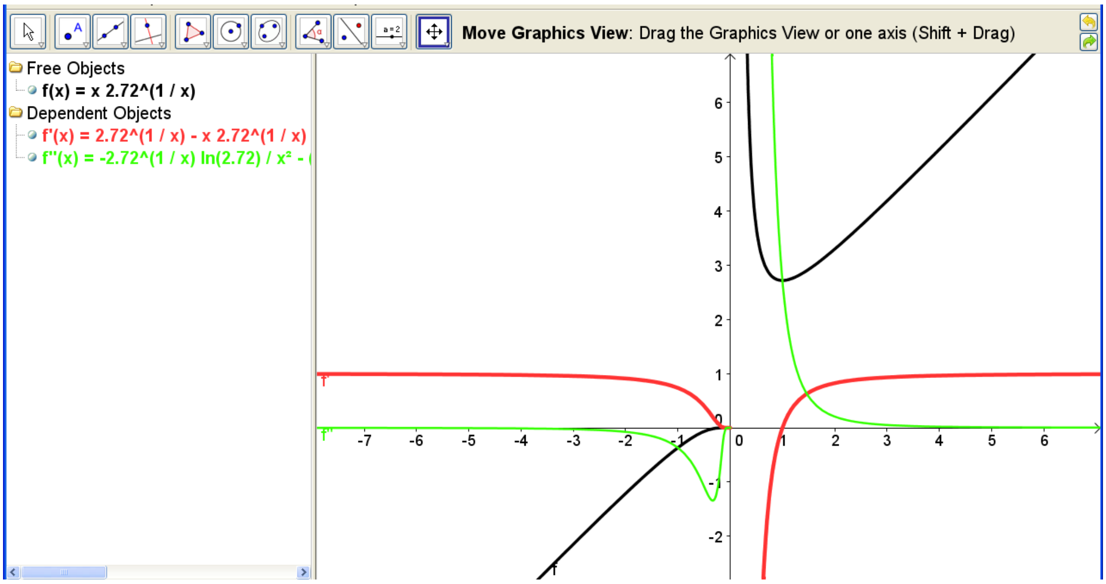


Figure 40

EXAMPLE 17

The graph of the function $f(x) = \sin^3 x + \cos^3 x$ is linked on **GraphS** and drawn on Figure 4.11.

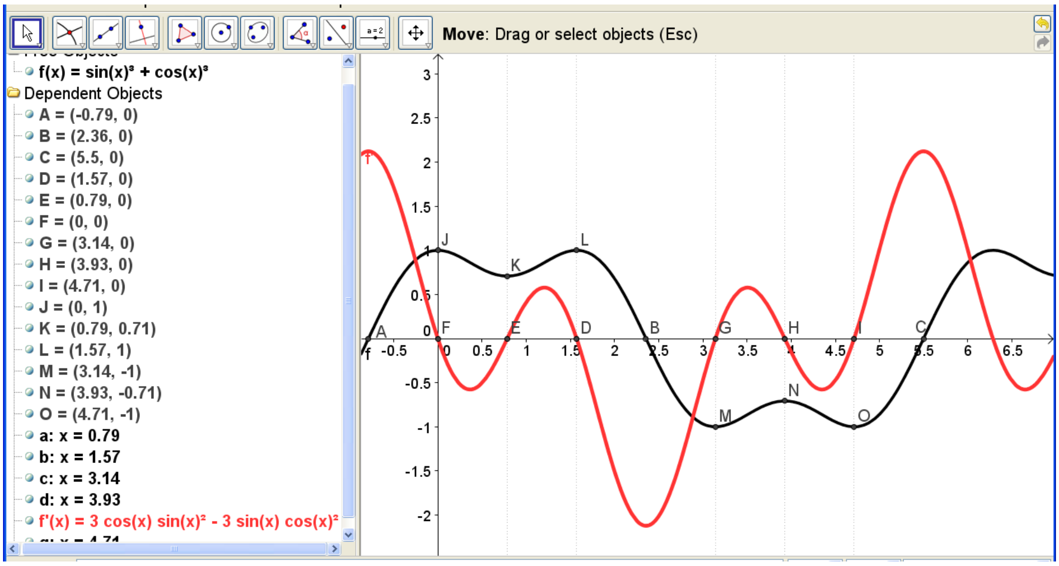


Figure 4.11

4. Integral

Area problem

EXAMPLE 18

Let us consider the function $f(x) = x^2$. Determine the area between the graph of the function f, the interval $[0, a]$, $a > 0$, and the lines determined by lines $x = 0$, and $x = a$.

SOLUTION

First we divide the interval $[0, a]$, $a > 0$, on n subintervals and calculate the sum of the area, of rectangular determined by the points

$$\left(\frac{a}{n}(i-1), 0\right), \left(\frac{a}{n}(i-1), f(\frac{a}{n}(i-1))\right), \left(\frac{a}{n}i, f(\frac{a}{n}(i-1))\right), \left(\frac{a}{n}i, 0\right), \quad i = 1, \ldots, n$$

called lower sum and dented by P_L.

On Figure 4.1, (linked on LowerSum), two slider are introduced a, and n and P_L is calculate for $n = 4$, and $a = 1.6$.

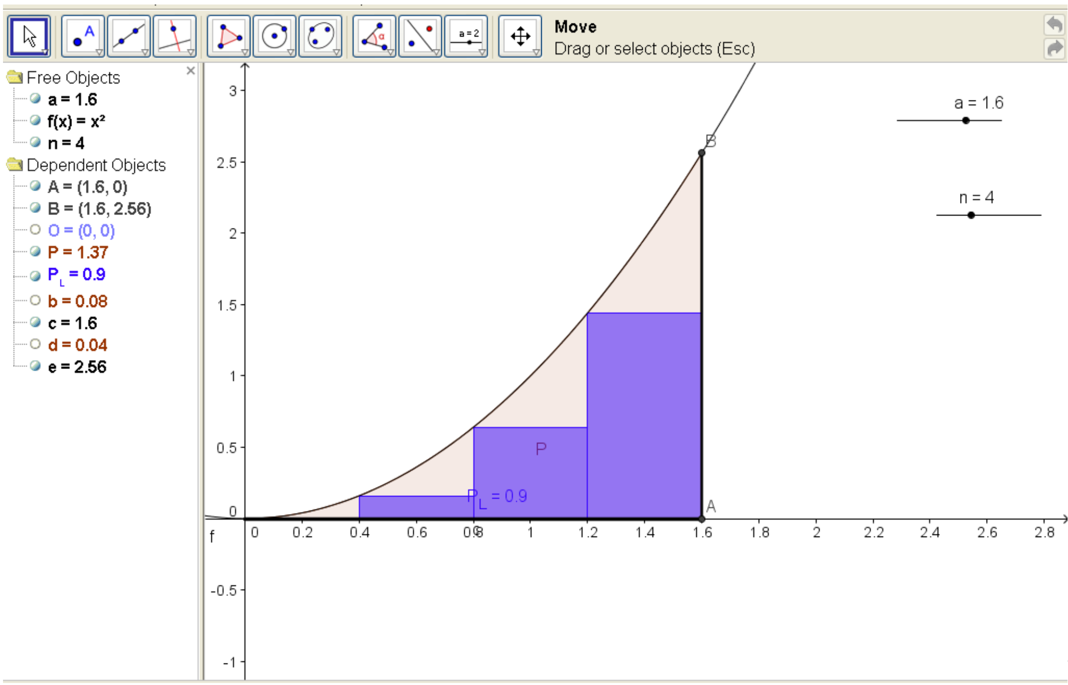


Figure Figure

4.1

On Figure 4.2, (linked on UperSum), two slider are introduced a, and n and P_U, the upper sum, calculated for the points

$$\left(\frac{a}{n}(i-1),0\right),\quad \left(\frac{a}{n}(i-1),\, f(\frac{a}{n}i)\right),\quad \left(\frac{a}{n}i,\, f(\frac{a}{n}i)\right),\quad \left(\frac{a}{n}i,0\right),\qquad i=1,\ldots,n,$$

calculate for $n=10$, and $a=2.5$.

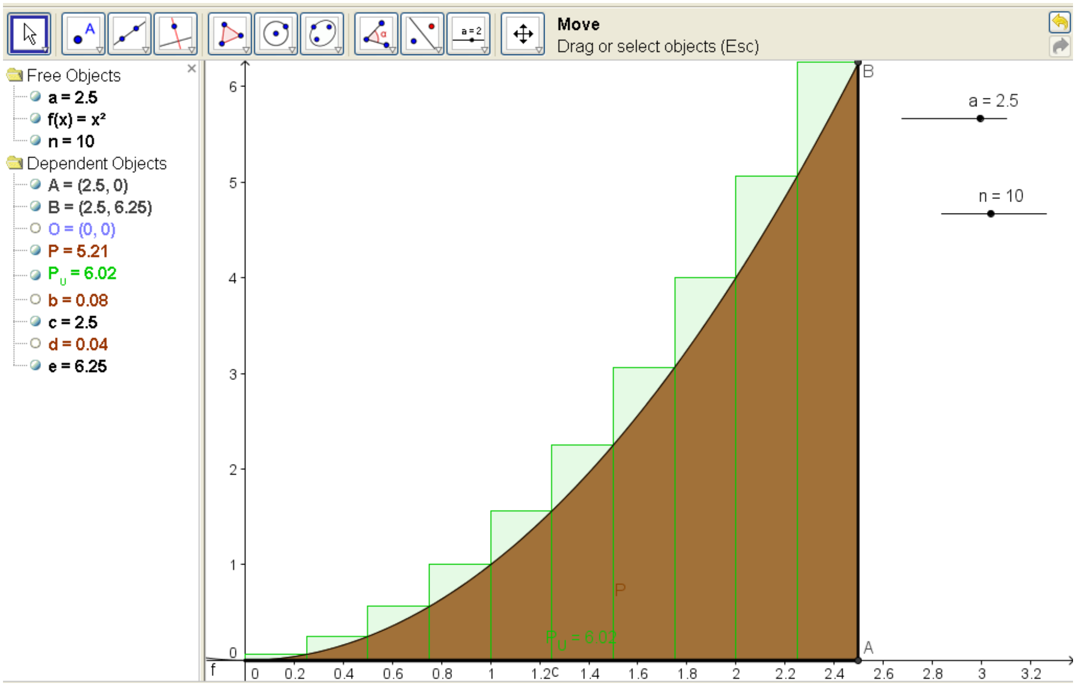


Figure 4.2

From both applets it can be followed that the numbers P_L, P_U are closing to the number P, denoted the area are we asked for.

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