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Partial differential equations:

The notion of weak solutions

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CHAPTER 1

Functions

1. Classical function spaces

1.1. Space of differentiable functions. Denote by $\Omega \subset \mathbb{R}^n$ an open set, its closure $\overline{\Omega}$ and boundary $\partial\Omega$.

$\mathcal{C}^k(\Omega)$ is the set of all function $u : \Omega \to \mathbb{R}$ (or $\mathbb{C}$, but all functions in conservation laws are real-valued) with continuous derivatives of order k, $0 \le k \le \infty$.

$\mathcal{C}^k(\overline{\Omega})$ is the set of all functions $u \in \mathcal{C}^k(\Omega)$ such that there exist a function $\phi \in \mathcal{C}^k(\Omega')$, $u \equiv \phi$ on $\overline{\Omega} \subset \Omega'$, where Ω' is an open set.

$\mathcal{C}^k_b(\Omega)$ consists of functions from $\mathcal{C}^k(\Omega)$bounded together with all their derivatives.

It holds:

$$\mathcal{C}^k(\mathbb{R}^n)|_\Omega \subset \mathcal{C}^k(\overline{\Omega}) \subset \mathcal{C}^k(\Omega).$$

If Ω is bounded, then $\mathcal{C}^k(\overline{\Omega}) \subset \mathcal{C}^k_b(\Omega)$.

Denote by $\text{supp} u$, $u : \Omega \to \mathbb{R}$, the complement of the largest open set Ω' such that $u|_{\Omega'} = 0$. The set $\text{supp} u$ is called *support* of the function u. Since $\Omega \subset \mathbb{R}^n$,

$$\text{supp} u = \overline{\{x \in \Omega : \ u(x) \neq 0\}}.$$

Notation $A \subset\subset B$ means that there exists a compact K such that $A \subset K \subset B$.

$\mathcal{C}^k_0(\Omega) = \{u \in \mathcal{C}^k(\Omega) : \ \text{supp} u \subset\subset \Omega\}$.

Elements of $\mathcal{C}^\infty_0(\Omega)$ are called *test functions.*

1.2. L^p-Spaces. A set $A \subset \Omega \subset \mathbb{R}^n$ is of *Lebesgue measure zero*, $\mathcal{L}(A) = 0$, if for each $\varepsilon > 0$ there exists a numerable union $\bigcup_{i\in\mathbb{N}} C_i$ of parallelepipeds $C_i \subset \mathbb{R}^n$ such that

$$\text{mes} \bigcup_{i=1}^{\infty} C_i < \varepsilon$$

(the measure of parallelopiped is the product of its edges lenghts).

In the set of all Lebesgue measurable functions $u : \Omega \to \mathbb{R}$ (all elementary functions and their compositions are Lebesgue measurable, for example) we define the equivalence relation "equal almost everywhere in Ω", $f \sim g$, if

$$\mathcal{L}(\{x : f(x) \neq g(x)\}) = 0.$$

Let $1 \leq p < \infty$. From now on Ω will be an open connected set. "Measurable" stands for Lebesgue measurable.

$$L^p(\Omega) = \{f/\sim : \Omega \to \mathbb{R} : f \text{ is measurable }, \int_\Omega |f(x)|^p \mathrm{dx} < \infty\}.$$

It is Banach space with the norm

$$\|f\|_{L^p(\Omega)} = \Big(\int_\Omega |f(x)|^p \mathrm{dx}\Big)^{1/p}.$$

$L^2(\Omega)$ is Hilbert space, with the product $(f|g)$ defined by

$$(f|g) = \int_\Omega f(x)\overline{g(x)}\mathrm{dx},$$

where $\overline{g(x)}$ stands for complex conjugate of $g(x)$. If we are in the space of real-valued functions (which will usually be the case) then

$$(f|g) = \int_\Omega f(x)g(x)\mathrm{dx}.$$

For $p = \infty$ we have a different definition:

$$\begin{aligned} L^\infty(\Omega) = &\{f/\sim : \Omega \to \mathbb{R} : f \text{ is measurable and there exists} \\ &\text{real } M \text{ such that } |f(x)| \leq M, \text{ for every } x \in \Omega\}. \end{aligned} \tag{1}$$

$L^\infty(\Omega)$ is also Banach space with the norm

$$\|f\|_{L^\infty} = \inf M, \text{ where the constant } M \text{ is from (1).}$$

The most important spaces are L^2-spaces and L^1_{loc}-spaces which are defined by

$$\begin{aligned} L^p_{loc}(\Omega) = &\{f/\sim : \Omega \to \mathbb{R} : f \text{ is measurable and for every } K \subset\subset \Omega \\ &\int_K |f(x)|^p \mathrm{dx} \leq \infty\}. \end{aligned}$$

Functions from L^1_{loc} are called *locally integrable* ones.

Hölder inequality

$$\int_\Omega |u(x)v(x)|\mathrm{dx} \leq \|u\|_{L^p}\|v\|_{L^q}, \ u \in L^p(\Omega), \ v \in L^q(\Omega), \ \frac{1}{p}+\frac{1}{q} = 1. \tag{2}$$

will often be used. The special case $p = q = 2$ is called *Schwartz inequality.*

Corollaries of Hölder inequality:

1.

$$\mathrm{mes}(\Omega)^{-1/p}\|u\|_{L^p} \leq \mathrm{mes}(\Omega)^{-1/q}\|u\|_{L^q},\ u \in L^q(\Omega),\ p \leq q.$$

2.

$$\|u\|_{L^q} \leq \|u\|^{\lambda}_{L^p}\|u\|^{\lambda}_{L^r},\ u \in L^r(\Omega),\ p \leq q \leq r,\ \frac{1}{q} = \frac{\lambda}{p} + \frac{1-\lambda}{r}.$$

3.

$$\int_\Omega u_1...u_m \mathrm{dx} \leq \|u\|_{L^{p_1}}...\|u\|_{L^{p_m}},$$
$$u_i \in L^{p_i}(\Omega),\ i = 1, ..., m,\ \frac{1}{p_1} + ... + \frac{1}{p_m} = 1.$$

2. Weak derivative and weak solution

2.1. Weak derivative. Denote by $|\alpha| = \alpha_1 + ... + \alpha_n$ multiindex $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}_0^n$ and

$$\partial^\alpha f(x) = \frac{\partial^{|\alpha|}}{\partial^{\alpha_1} x_1 ... \partial^{\alpha_n} x_n}.$$

(If $\alpha_i = 0$ for some i, there is no derivative with respect to the variable x_i.)

Definition 1. A function $f \in L^1_{loc}(\Omega)$ has α-th weak derivative, $|\alpha| \leq m$, denoted again by $\partial^\alpha f$, if there exist a function $g \in L^1_{loc}(\Omega)$ such that

$$\int_\Omega f(x)\partial^\alpha \phi(x)\mathrm{dx} = (-1)^{|\alpha|} \int_\Omega g(x)\phi(x)\mathrm{dx},$$

for every $\phi \in \mathcal{C}_0^\infty(\Omega)$. The function g will be called α-th weak derivative for f.

The following theorem is very useable and illustrative.

Theorem 1. *If there exists a weak derivative for a locally integrable function u, then u is almost everywhere differentiable and the weak derivative equals to a strong at the points where it exists.*

2.2. Weak solution of partial differential equations. Notion of a weak solution is not defined in a unique manner. It should be defined to fit a physical problem as much as it can.

First, we shall give the definition for first order systems. Later on, the definition will be easily adopted to an equation of higher order.

DEFINITION 2. A system of first order partial differential equation is in *divergence form* if it can be written as

$$\partial_t a_0(t,x,u) + \partial_{x_1} a_1(t,x,u) + ... + \partial_{x_n} a_n(t,x,u) = b(t,x,u), \tag{3}$$

where $u = u(t, x_1, ..., x_n)$ is a vector valued function. Suppose that u satisfies initial condition $u(x,0) = u_0(x)$. It is said that

$$u \in \left(L^1_{loc}([0,T) \times \Omega)\right)^n$$

is *weak solution* to system (3) with the above given initial data if

$$\begin{aligned} &\int_0^t \int_\Omega \partial_t \phi(t,x) a_0(t,x,u) + \partial_{x_1} \phi(t,x) a_1(t,x,u) + ... \\ &+ \partial_{x_n} \phi(t,x) a_n(t,x,u) \mathrm{dxdt} + \int_\Omega u_0(x) \mathrm{dx} \\ = &\int_0^t \int_\Omega b(t,x,u) \phi(t,x) \mathrm{dxdt}, \end{aligned} \tag{4}$$

for every $\phi \in \mathcal{C}_0^\infty((-\infty,\infty) \times \Omega)$.

As one can see, vector valued function u is not necessary differentiable and the name "weak solution" comes from that fact.

Also, it is easy to check, using integration by parts, that every $\mathcal{C}^1$-solution of (3) also satisfies (4), i.e. it is weak solution, too.

For practical reasons we shall use the following simpler (and weaker) condition instead of (4):

$$\begin{aligned} &\int_0^t \int_\Omega \partial_t \phi(t,x) a_0(t,x,u) \partial_{x_1} \phi(t,x) a_1(t,x,u) + ... \\ &+ \partial_{x_n} \phi(t,x) a_n(t,x,u) \mathrm{dxdt} \\ = &\int_0^t \int_\Omega b(t,x,u) \phi(t,x) \mathrm{dxdt} \\ &\lim_{t \to 0} u(t,x) = u_0 \text{ almost everywhere in } \Omega, \end{aligned} \tag{5}$$

for every $\phi \in \mathcal{C}_0^\infty((0,\infty) \times \Omega)$. Note that now ϕ is defined on a smaller domain, i.e. equals zero on the x-axe ($t = 0$).

REMARK 1. If a system is not given in the divergence form, then a definition of a weak solution is much more difficult to give and more specific. That was out of scope here.

Systems where t is distinguished variable are called *evolution systems* (or systems "written in evolution form").

3. Distribution spaces

In this section we shall present a simplified version of distribution theory. We shall use convergence in vector spaces and not topology.

Mapping from a vector space over some field into that field (usually $\mathbb{R}$ or $\mathbb{C}$) is called *functional*.

Let us introduce a convergence in the set $\mathcal{C}_0^\infty(\Omega)$.

Definition 3. A sequence $\{\phi_j\} \subset \mathcal{C}_0^\infty(\Omega)$ converge to zero as $j \to \infty$ if

- There exists a compact $K \subset\subset \Omega$ such that $\text{supp}\phi_j \subset K$, for every $j \in \mathbb{N}$.
- For each $\alpha \in \mathbb{N}_0^n$, $\|\partial^\alpha \phi\|_{L^\infty(\Omega)} \to 0$, as $j \to \infty$.

This convergence is denoted by $\xrightarrow{\mathcal{D}}$.

The set $\mathcal{C}_0^\infty(\Omega)$ with the convergence defined in this way will be denoted by $\mathcal{D}(\Omega)$. Elements of this space will be called test functions.

Definition 4. Linear continuous functional S with the domain $\mathcal{D}(\Omega)$ is called *distribution*. Its acting on the test function ϕ is denoted by $\langle S, \phi\rangle$.

Continuity is understood in the means of convergence: S is continuous if for each sequence of test functions $\{\phi_j\}_j$ converging to zero as $j \to \infty$ it holds $\langle S, \phi_j\rangle \to 0$, as $j \to \infty$.

Vector space of distributions is denoted by $\mathcal{D}'(\Omega)$.

Now, we shall give some important examples of distributions. The first one shows how locally integrable function can be treated as distributions and the second one is an example of distribution which can not be treated as a usual function.

Example 1. Let $f \in L^1_{loc}(\Omega)$ and ϕ be a test function. Then mapping from $\mathcal{D}$ into $\mathbb{R}$ defined by

$$S_f: \ \langle S_f, \phi\rangle = \int_\Omega f(x)\phi(x)\mathrm{dx}$$

define a distribution: Functional S_f is obviously linear and

$$|\langle S_f, \phi\rangle| \le \|\phi\|_{L^\infty(\Omega)} \int_{\text{supp}\phi} |f(x)|\mathrm{dx}.$$

That means that if a sequence $\{\phi_j\}$ converges to zero in $\mathcal{D}$, then $\langle S_f, \phi_j\rangle \to 0$ as $j \to \infty$, i.e. S_f is a distribution.

Example 2. Let $a \in \Omega$. Relation

$$\langle \delta_a, \phi \rangle = \phi(a)$$

defines *Dirac delta distribution* at the point a. If $a = 0$, then we write just δ instead δ_0.

4. Properties and operations with distributions

(1) For a sequence of distributions $\{S_j\} \subset \mathcal{D}'(\Omega)$ is said to *converge to zero* if

$$\langle S_j, \phi \rangle \to 0, \text{ as } j \to \infty,$$

for every $\phi \in \mathcal{D}(\Omega)$. Convergence in the distribution space is denoted by $\overset{\mathcal{D}'}{\to}$. (In distribution theory this convergence is called "weak"). Convergence to zero is enough since the distribution space is a vector one: $S_j \to T$, $T \in \mathcal{D}'(\Omega)$ if and only if

$$\langle S_j - T, \phi \rangle \to 0, \text{ as } j \to \infty,$$

for every test function ϕ.

(2) $S \in \mathcal{D}'(\Omega)$ is zero on $\omega \subset \Omega$ if

$$\langle S, \phi \rangle = 0,$$

for every test function ϕ with a support in ω.

Definition 5. *Support of a distribution* $S \in \mathcal{D}'(\Omega)$, supp$S$, is a complement of the maximum open set where $S = 0$ (i.e. set of point in Ω which do not have a neighborhoods ω where $S = 0$).

Definition 6. $\mathcal{E}'(\Omega)$ is the *space of distributions with compact support.*

Example 3. supp$\delta = \{0\}$, because for each $x \in \Omega$, $x \neq 0$, there exists its neighborhoods ω not containing zero and there exist a test function ϕ with a support in ω. That means

$$\langle \delta, \phi \rangle = \phi(0) = 0.$$

Definition 7. *Distributional derivative* S of order $\alpha \in \mathbb{N}_0^n$ is defined by

$$\langle \partial^\alpha S, \phi \rangle := (-1)^{|\alpha|} \langle S, \partial^\alpha \phi \rangle, \text{ for every } \phi \in \mathcal{D}(\Omega).$$

Since $\partial^\alpha \phi$ is also in $\mathcal{D}(\Omega)$, one can see that the definition makes sense, i.e. each distribution has a derivative of every order. That fact is the main reason why distributions are so important.

LEMMA 1. *Differentiation is a continuous operation in the distribution space.*

EXAMPLE 4. We can easily calculate each derivative of the delta distribution,

$$\langle \partial^\alpha \delta, \phi \rangle = (-1)^{|\alpha|} \langle \delta, \partial^\alpha \phi \rangle = (-1)^{|\alpha|} \partial^\alpha \phi(0).$$

One can easily verify the following. If $g \in L^1_{loc}(\Omega)$ is α-th weak derivative of $f \in L^1_{loc}(\Omega)$, then $S_g = \partial^\alpha S_f$, where S_f (or S_g) is the distributional image of f (or g).

EXAMPLE 5. Define Heaviside function

$$H(x) = \begin{cases} 0, & x < 0 \\ 1, & x > 0. \end{cases}$$

Since H is locally integrable function we can identify it with a distribution defined on $\mathbb{R}$. We will show that its derivative is the delta distribution. Let ϕ be an arbitrary test function on $\mathbb{R}$. Then

$$\langle H', \phi \rangle = -\langle H, \phi' \rangle = -\int_0^\infty \phi'(x) \mathrm{dx} = \phi(0) = \langle \delta, \phi \rangle.$$

If $W^k(\Omega)$ stands for the space of locally integrable functions on Ω having all derivatives of order less or equal to k, then

$$\mathcal{C}^k(\Omega) \subset W^k(\Omega) \subset \mathcal{D}'(\Omega).$$

(Here, function is identified with its image in the space of distributions.) If $f \in \mathcal{C}^\infty(\Omega)$, then we can define its product with a distribution S, $T = Sf$, in the following way

$$\langle T, \phi \rangle := \langle S, f\phi \rangle, \ \phi \in \mathcal{D}(\Omega).$$

But, there is no general definition of the product if f is not smooth. This is the main disadvantage of distributions.

At the end of the paper, we shall give a possibility to overcome that fact by introducing Colombeau-type generalized function spaces.

5. Sobolev spaces

5.1. Definitions. Let $m \in \mathbb{N}_0$, $p \geq 1$ and Ω be an open subset of $\mathbb{R}^n$. Denote by $W^k(\Omega)$ the vector space of all locally integrable functions on Ω which has all weak derivatives of order less or equal to k. We are in position to define its subspaces which will have the advantage to be normed (space $W^k(\Omega)$ is only locally convex, with topology defined by a sequence of seminorms).

Definition 8. Sobolev space $H^{m,p}(\Omega)$ is the set of functions $u \in W^m(\Omega)$, such that

$$\partial^\alpha u \in L^p(\Omega),$$

for every $\alpha \in \mathbb{N}_0^n$, $|\alpha| \leq m$.

Norma is given by

$$\|u\|_{H^{m,p}(\Omega)} = \|u\|_{m,p,\Omega} := \Big(\int_\Omega \sum_{|\alpha|\leq m} |\partial^\alpha u(x)|^p \mathrm{dx} \Big)^{1/p}.$$

An equivalent norm to the above one is given by

$$\|u\|'_{H^{m,p}(\Omega)} = \sum_{|\alpha|\leq m} \|\partial^\alpha u\|_{L^p(\Omega)}.$$

In the sequel we shall not distinguish them by notation, i.e. each of these two norms will be denoted by $\|u\|_{H^{m,p}(\Omega)}$.

If $p = 2$, then we shall omit that number in the superscript.

Theorem 2. *For each $m \in \mathbb{N}_0$, $H^m(\Omega)$ is Hilbert space with the product*

$$(u|v) = \int_\Omega u(x)v(x)\mathrm{dx} + \sum_{i=1}^m \int_\Omega \nabla^m u(x) \nabla^m v(x) \mathrm{dx}. \tag{6}$$

If $p \geq 1$, then $H^{m,p}(\Omega)$ is only Banach space.

Denote by $\langle \xi \rangle = \sqrt{1+\xi^2}$. Then, the usual norm of u in $H^m(\mathbb{R}^n)$ is equivalent with the following one

$$\|u\|''_{H^m} := \sup_{\xi \in \mathbb{R}^n} \sum_{j=0}^m \|\langle \xi \rangle \hat{u}\|_{L^2(\Omega)}.$$

We are in position to define $H^s(\mathbb{R}^n)$, $s \in \mathbb{R}$ now: $u \in H^s(\mathbb{R}^n)$ if and only if $\|u\|''_{H^m} < 0$.

The following spaces are important for defining value of an element in Sobolev space on the boundary in a simple way.

Definition 9. $H_0^{m,p}(\Omega)$ is the closure of $\mathcal{C}_0^\infty(\Omega)$ in a norm $H^{m,p}(\Omega)$. $v \in H_0^{m,p}(\Omega)$ means that there exists a sequence $\{\phi_j\} \subset \mathcal{C}_0^\infty(\Omega)$ such that

$$v_j \overset{H^{m,p}}{\to} v, \ j \to \infty.$$

For $u \in H^{m,p}(\Omega)$, boundary condition $u|_{\partial\Omega} = 0$ in a weak sense means that $u \in H_0^{m,p}(\Omega)$. If $v \in H^{m,p}(\Omega)$, then $u = v$ on $\partial\Omega$ if and only if $u - v \in H_0^{m,p}(\Omega)$.

5.2. Imbedding theorems. We shall give only few of the large number of these important theorems.

Definition 10. Banach space B_1 is *continuously imbedded* in a Banach space B_2, $B_1 \to B_2$, if there exists bounded linear injection from B_1 into B_2.

Theorem 3. *For an open set* $\Omega \subset \mathbb{R}^n$ *it holds*

$$H^{m,p}(\Omega) \to L^q(\Omega),\ mp > n,\ p \leq q \leq \infty$$
$$H^{m,p}(\Omega) \to L^q(\Omega),\ mp = n,\ p \leq q < \infty$$
$$H^{m,p}(\Omega) \to \mathcal{C}_b^0(\Omega),\ mp > n$$

Theorem 4. *Let* Ω *be bounded and posses* conus property*: For each* $x \in \Omega$ *there exists a conus of the height* h *with the edge at* x *completely lying in* Ω*. Then*

$$H^{m,p}(\Omega) \to L^q(\Omega),\ p \leq q \leq np/(n - mp)$$
$$H^{m+j,1}(\Omega) \to \mathcal{C}_b^j(\Omega),\ mp > n$$

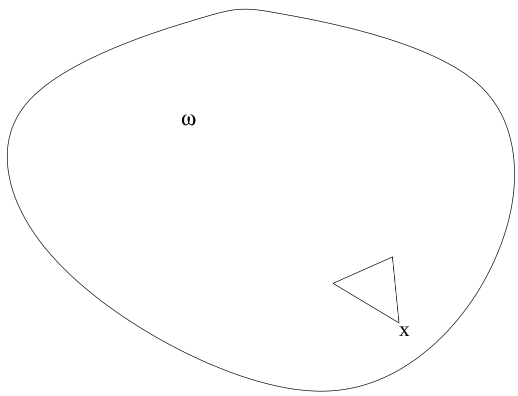

Figure 1. A set with a conus property

CHAPTER 2

Physical examples

1. Introduction

There is no precise definition of wave, but one can describe it as a signal traveling from one place to another one with clearly visible speed.

The signal can be any disturbance, like some kind of maxima or change of some quantity.

We shall define two kinds of waves.

DEFINITION 11. (i) Hyperbolic waves. They are solution to hyperbolic equations.
(ii) Dispersive waves. They are solutions to some equation(s) of the form

$$\varphi = a\psi(kx - \omega t), \tag{7}$$

where its *frequency* ω is some real function of *wave number* k, and $\omega(k)$ is defined via a system of partial or ordinary differential or integral equations.

Phase velocity is $\dfrac{\omega(k)}{k}$. The wave is *dispersive* if $\omega'(k)$ is not a constant, i.e. $\omega''(k) \neq 0$.

Group velocity, defined by

$$c(k) = \frac{\mathrm{d}\omega}{\mathrm{d}k},$$

is especially important for observation of wave propagation.

There exist partial differential equations belonging to both of the groups. One of these is Klein-Gordon equation

$$u_{xx} - u_x + u = 0$$

. It is hyperbolic equation with solutions of form (7), with $\omega^2 = k^2 + 1$. But that group is relatively small.

2. Kinetic waves

In a lot of physical problems a disturbance in a material, or in a state of a medium can arise. So we shall describe the basic building blocks, density $\rho(x,t)$, flux $q(x,t)$ and the flow velocity:

$$v(x,t) = \frac{q(x,t)}{\rho(x,t)} = \frac{\text{flux}}{\text{density}}.$$

as well as relations between them.

Homogeneous relation between ρ and q is the simplest one: $q = Q(\rho)$. Denote $c(\rho) = Q'(\rho)$. The above equation now reads

$$\rho_t + q_x = 0 \tag{8}$$

i.e.

$$\rho_t + c(\rho)\rho_x = 0 \tag{9}$$

if ρ and q are regular enough.

First, let us note that the characteristics for (9) are given by the following ordinary differential equations

$$\gamma : \frac{\mathrm{dx}}{\mathrm{dt}} = c(\rho).$$

Since we are dealing with a conservation law (right-hand side of (8) equals zero), the curves given by γ are straight lines, i.e. speed of a wave, $c(\rho)$, is constant.

3. Traffic flow

Obviously, the flow velocity

$$v(\rho) = \frac{q(\rho)}{\rho}$$

is obviously decreasing function with respect to ρ which take values from a maximum one, at $\rho = 0$, to zero, as $\rho \to \rho_y$. Here ρ_y is maximal car density at a road (cars touches one another). The flux q is therefore a convex function (see Fig. 1) and has a maximal value q_m for some density ρ_m, while $q(0) = q(\rho_y) = 0$.

After observations made in Lincoln tunnel, New York, experimental data for one right of way are: $\rho_y \approx 225\frac{\text{verichles}}{\text{mile}}$, $\rho_m \approx 80\frac{\text{verichles}}{\text{hour}}$. (The maximum flow for the above data could be obtained for car speeds $q_m \approx 20\frac{\text{milja}}{\text{sat}}$).

A rough model for more than one right of way can be obtained by multiplying the above values with their multiplicity.

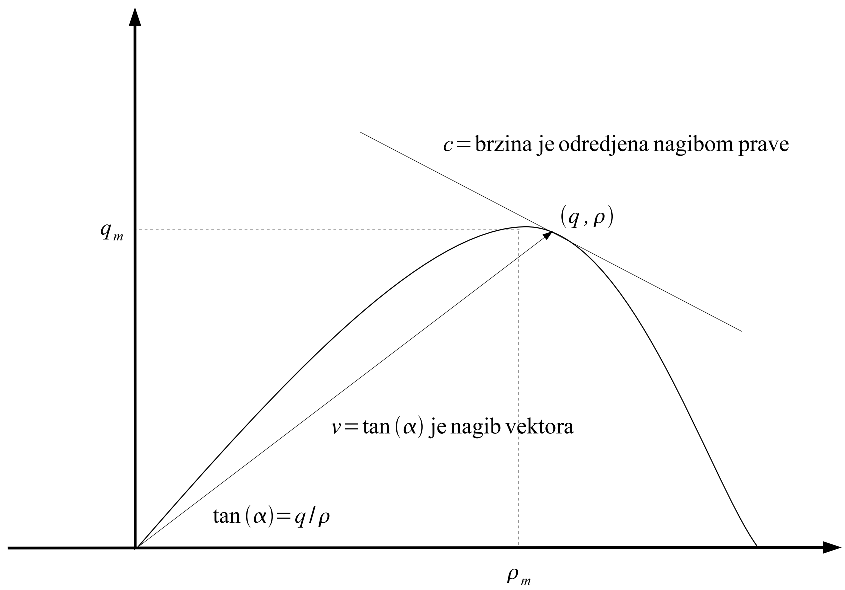

FIGURE 1. Flux function for traffic flow model

Suppose that q depends only on ρ, $q := Q(\rho)$. Speed of waves is given by

$$c(\rho) = Q'(\rho) = v(\rho) + \rho v'(\rho).$$

Since $v'(\rho) < 0$, it is less than flow velocity. It means that drivers can see a disturbance ahead.

In this particular case, speed of waves c is speed of cars, and a flow velocity is an average velocity of motion of a road relative to all of cars. Let us note that $c > 0$ for $\rho < \rho_m$ (cars are moving faster than average if density is small) and $c < 0$ for $\rho > \rho_m$ (opposite case: high density of the cars has lower speed than average).

Greenberg's model for the above tunnel are calculated in the following way: $Q(\rho) = a\rho \log \frac{\rho_j}{\rho}$, $a = 17.2\frac{m}{h}$, $\rho_j = 228\frac{v}{m}$. $\rho_m = 83\frac{v}{m}$, $\rho_m = 1430\frac{v}{h}$. Logarithmic function definitely does not approximate states in a neighborhoods of the point $\rho = 0$ in a good way, but this is practically not interesting case, anyway.

A solution to the above problem is just illustrated in the Fig. 2.

4. Sedimentation in a river, chemical reactions

This model describes exchange processes between river-bed and fluid in the river, i.e. sedimentation transport, more precisely.

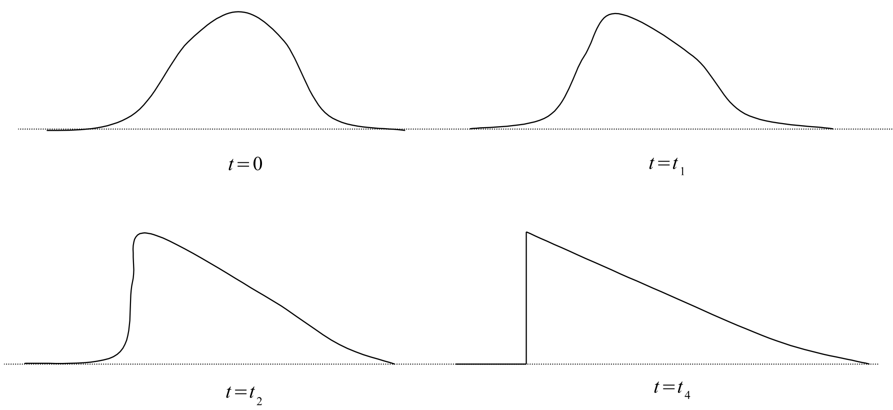

FIGURE 2. Density of cars

Denote
ρ_1 . . . . density of a fluid
ρ_0 . . . . density of a solid material.
Then, the density is given by

$$\rho = \rho_0 + \rho_1$$

and flux by $q = u\rho_1$, where u is a fluid speed. Conservation of mass law is given by

$$(\rho_1 + \rho_0)_t + u\rho_{1x} = 0,$$

with the supposition that fluid speed is a constant.

Reaction between these two materials are given by

$$\frac{\partial \rho_0}{\partial t} = k_1(A_1 - \rho_0)\rho_1 - k_2\rho_0(A_2 - \rho_1),$$

where k_1 and k_2 are coefficients depending on a reaction speed, and A_1, A_2 are constants depending on material specifications (both solid and fluid ones).

Let us take a special case, so called quasi-equilibrium, when changes of solid material density due chemical reactions are neglected, i.e.

$$\frac{\partial \rho_0}{\partial t} = 0.$$

We shall also suppose that space-time position is negligible, i.e.

$$\rho_0 = r(\rho_1).$$

Then we have the following system

$$\rho_{1t}(1 + r'(\rho_1)) + u\rho_{1x} = 0$$

i.e.

$$\rho_{1t} + \frac{u}{1 + r'(\rho_1)}\rho_{1x} = 0.$$

In some models, one can take

$$r(\rho_1) = \frac{k_1 A_1 \rho_1}{k_2 B + (k_1 - k_2)\rho_1}.$$

Equation which describes waves in this case follows from law of mass conservation. In general, flux is given by $q = \rho u$, where $u \neq$ const, so we need one more equation (for speed u).

5. Shallow water equations

Let us fix some notation:

ρ . . . . height of water level – its depth ($\approx$ density)
u . . . . speed of water flow

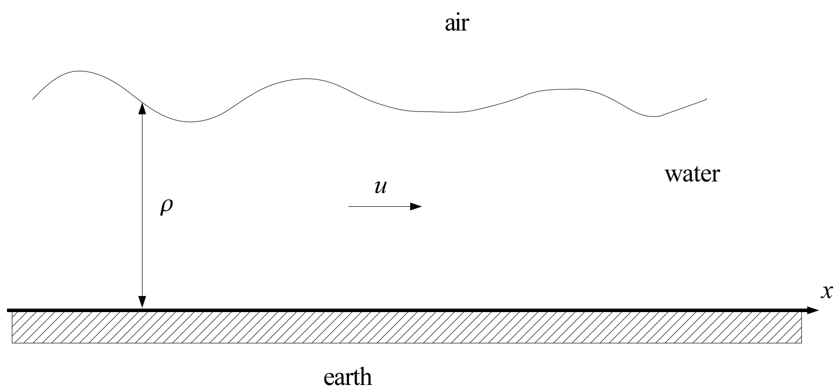

FIGURE 3. Shallow water

This model is used for description of river flow when depth is not so big (in the later case one can safely take that the depth equals infinity). It can be also used for flood, sea near beach, channel flow, avalanche,...

Basic assumption in this model is that a fluid is incompressible and homogenous (forming of "waves", moving of a water visible on its surface, is possible). Bottom of a river is not necessary flat, but for a flat one equations are homogenous – flux is independent of space-time coordinates. That eases finding global solutions to a system.

Mass conservation law gives

$$\rho_t + (\rho u)_x = 0. \tag{10}$$

In order to solve the above equation we shall introduce new partial differential equation involving the speed u and Newton's second law:

$$(mu)^{\cdot} = f \text{ ("force = impuls change per time").}$$

Take a space interval $[x_1, x_2]$ during a time interval $[t_1, t_2]$. Then

$$\int_{x_1}^{x_2} \rho(x,t_2)u(x,t_2)\mathrm{dx} - \int_{x_1}^{x_2} \rho(x,t_1)u(x,t_1)\mathrm{dx}$$
$$= \int_{t_1}^{t_2} \big(\rho(x_1,t)u^2(x_1,t) - \rho(x_2,t)u^2(x_2,t)\big)\mathrm{dt}$$
$$+ \int_{t_1}^{t_2} \big(p(x_1,t) - p(x_2,t)\big)\mathrm{dt}$$

"impuls change per time = kinetic energy + force due to preassure"

Contraction of a time-space interval: $t_1, t_2 \longrightarrow t$ and $x_1, x_2 \longrightarrow x$ for some pint (x,t), gives the following PDE

$$(\rho u)_t + (\rho u^2)_x + p_x = 0. \tag{11}$$

The pressure in the above equation is the hydraulic pressure. One gets (we shall assume that density of water equals 1)

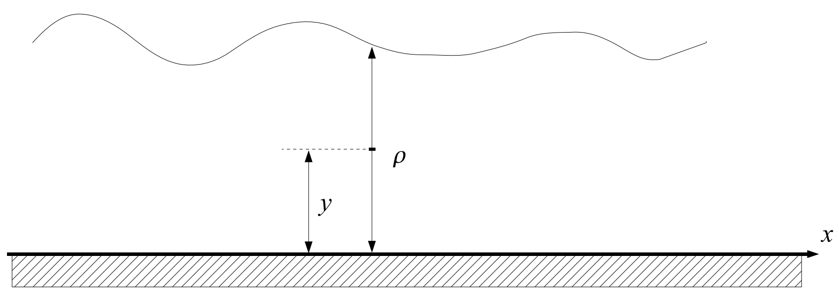

FIGURE 4. Hydraulic pressure

$$\pi(y) = g(\rho - y) \text{ . . . . hydraulic preassure,}$$

where g is the universal gravitational constant (see Fig. 4), and

$$p = \int_0^\rho \pi(y)\mathrm{dy} = \int_0^\rho g(\rho - y)\mathrm{dy} = g\frac{\rho^2}{2}.$$

Substituting this relation into (10) and (11) gives

$$\begin{aligned} &\rho_t + (\rho u)_x = 0 \\ &(\rho u)_t + \big(\rho u^2 + g\frac{\rho^2}{2}\big)_x = 0. \end{aligned} \tag{12}$$

Let us differentiate the second equation in the above system assuming enough regularity of solutions:

$$\rho_t u + \rho u_t + 2\rho u u_x + \rho_x u^2 + g\rho\rho_x = 0.$$

Then substitute ρ_t from the first equation in the modified second equation. After that procedure we get

$$u_t + u u_x + g\rho\rho_x = 0,$$

and finally the system becomes

$$\begin{aligned} &\rho_t + (\rho u)_x = 0 \\ &u_t + \big(\frac{u^2}{2} + g\rho\big)_x = 0. \end{aligned} \tag{13}$$

If solutions are not necessarily differentiable, one substitute $\omega = \rho u$ (ω is a flux) into system (12) so we get a different one

$$\begin{aligned} &\rho_t + \omega_x = 0 \\ &\omega_t + \big(\frac{\omega^2}{\rho} + g\frac{\rho^2}{2}\big)_x = 0. \end{aligned} \tag{14}$$

In subsequent sections one will see that systems (13) and (14) are not equivalent in practice (concerning weak solutions) due to the use of differentiation.

6. Gas dynamics (viscous)

We shall use the following notation:

ρ . . . . gas density
u . . . . gas velocity (gas molecule speed)
σ . . . . pressure (force/area)

As before, we have the following system of conservation laws

$$\begin{aligned} &\rho_t + (\rho u)_x = 0 \\ &(\rho u)_t + (\rho u^2)_x - \sigma_x = 0. \end{aligned}$$

The following relation holds in general:

$$\sigma = -p + \nu u_x,$$

where p is a pressure of a gas without moving, and ν is a viscosity ($\ll 1$) (see Fig. 5).

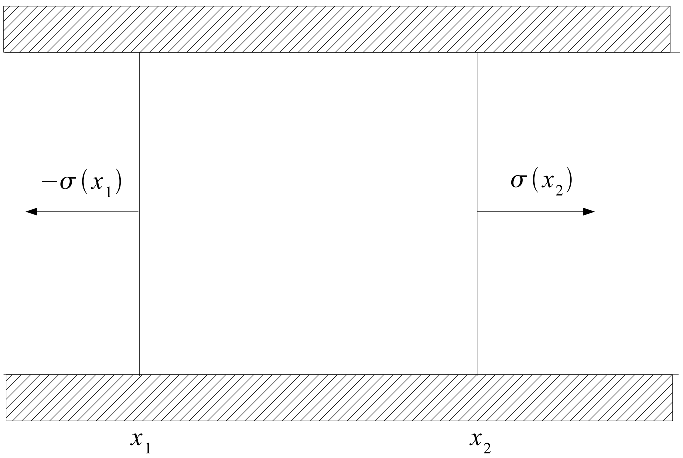

FIGURE 5. Pressure in share gas

Thus,

$$\rho_t + (\rho u)_x = 0$$
$$(\rho u)_t + (\rho u^2)_x + p_x = \nu u_{xx}$$

holds for viscous fluids. For gases it holds $\nu \to 0$, so one can often take $\nu \equiv 0$.

6.1. Thermodynamically effects with gases. We shall continue to use notation from the previous section. Let $p = p(\rho, S)$, where new independent variable S stands for entropy.

In order to close the system we need an extra equation. For adiabatic case one can take

$$S_t + uS_x = 0,$$

for example.

For an isotropic, ideal gas, one takes

$$S \equiv \text{const}, \ \nu \equiv 0.$$

Now, the viscous case is modeled by

$$\rho_t + (\rho u)_x = 0$$
$$(\rho u)_t + (\rho u^2)_x + (p(\rho))_x = 0$$
$$p(\rho) = \kappa\rho^\gamma,\ 1 < \gamma < 3,\ \gamma = 1 + 2/n,$$

where κ stands for universal gas constant, and n is a number of atoms in gas molecule.

Let us note that for constant density, $\rho = \rho_0 \in \mathbb{R}$, there is no changes in pressure and speed of the gas – no gas movements.

In more than one space dimensions we have well known Navier-Stokes equation

$$\rho_t + \operatorname{div}(\rho\vec{u}) = 0$$
$$(\rho u)_t + \vec{u}\cdot\operatorname{grad}(\rho\vec{u}) + (\rho\vec{u})\cdot\operatorname{div}\vec{u} + \operatorname{grad}p = 0$$
$$(\text{or } = \nu\triangle\vec{u} \text{ for viscous fluids}).$$

7. Hyperbolic conservation law

Let $u \in C^1(\mathbb{R}\times[0,\infty))$ be a solution to the following partial differential equation

$$\begin{aligned} u_t + (f(u))_x &= 0 \\ u(x,0) &= u_0(x). \end{aligned} \tag{15}$$

Take $\varphi \in C_0^1(\mathbb{R}\times[0,\infty))$, i.e. smooth function such that its support intersected by $\mathbb{R}\times[0,\infty)$ is compact.

Then

$$\begin{aligned} 0 &= \int_0^\infty\int_{-\infty}^\infty (u_t(x,t) + (f(u))_x\varphi(x,t)\mathrm{dtdx} \\ &= -\int_0^\infty\int_{-\infty}^\infty f(u)\varphi_x\mathrm{dtdx} + \int_{-\infty}^\infty u(x,t)\varphi(x,t)\mathrm{dx}\Big|_{t=0}^{t=\infty} \\ &\quad - \int_0^\infty\int_{-\infty}^\infty u\varphi_t\mathrm{dxdt} \\ &= -\int_0^\infty\int_{-\infty}^\infty (u\varphi_t + f(u)\varphi_x)\mathrm{dxdt} - \int_{-\infty}^\infty u_0(x)\varphi(x,0)\mathrm{dx}. \end{aligned}$$

The above calculation inspired the following definition of weak solution for (15).

Definition 12. $u \in L^\infty(\mathbb{R}\times(0,\infty))$ (u is bounded function up to a set of Lebesgue measure zero) is called *weak solution* of (15) if

$$\int_0^\infty\int_{-\infty}^\infty (u\varphi_t + f(u)\varphi_x)\mathrm{dxdt} + \int_{-\infty}^\infty u_0(x)\varphi(x,0)\mathrm{dx} = 0,$$

for every $\varphi \in C_0^1(\mathbb{R} \times [0, \infty))$

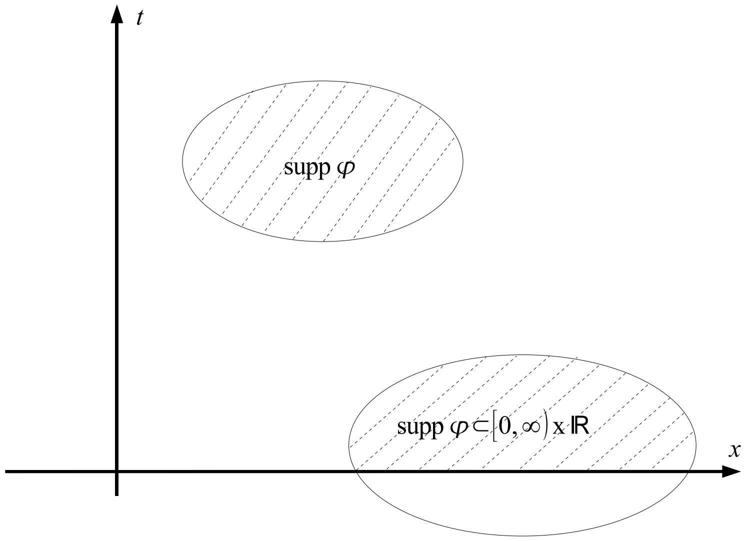

FIGURE 6. Supports of test functions in halfplane

REMARK 2. (1) All classical solutions are also weak.
(2) If u is a weak solution, then u is also a distributive solution.
(3) If $u \in C^1(\mathbb{R} \times [0, \infty))$ is a weak solution, then it is a classical, too.

If we do not say differently, "solution" will mean weak solution from now on.

In a few steps we shall find necessary conditions for existence of piecewise differentiable weak solution to some conservation law.

THEOREM 5. *Necessary and sufficient condition that*

$$u(x,t) = \begin{cases} u_l(x,t), & x < \gamma(t),\ t \geq 0 \\ u_d(x,t), & x > \gamma(t),\ t \geq 0, \end{cases}$$

where u_l and u_d are C^1 solutions on their domains, be a weak solution to (15) is

$$\dot{\gamma} = \frac{f(u_d) - f(u_l)}{u_d - u_l} =: \frac{[f(u)]_\gamma}{[u]_\gamma}. \tag{16}$$

Proof. The proof will be given in few steps.
1. Let

$$u(x,t)=\begin{cases} u_l(x,t), & x<\gamma(t),\ t\geq 0\\ u_d(x,t), & x>\gamma(t),\ t\geq 0,\end{cases}$$

where u_l and u_d are defined above, be a weak solution to (15). Then

$$\int_0^\infty\int_{-\infty}^\infty (u\varphi_t+f(u)\varphi_x)\mathrm{dxdt}+\int_{-\infty}^\infty u(x,0)\varphi(x,0)\mathrm{dx}=0,$$

for every $\varphi\in(\mathbb{R}\times[0,\infty))$.

Also $(u_l)_t+f(u_l)_x=0$ for $x<\gamma(t)$ and $t>0$ as well as $(u_d)_t+f(u_d)_x=0$ for $x>\gamma(t)$ and $t>0$.

That is consequence of the fact that

$$\begin{aligned}0&=\int\int u_l\varphi_t+f(u_l)\varphi_x\mathrm{dxdt}\\ &=-\int\int(u_l)_t\varphi+(f(u_l))_x\varphi\mathrm{dxdt},\end{aligned}$$

for every φ, supp$\varphi\subset\{(x,t):\ x<\gamma(t),\ t>0\}$ and C^1-function u_l. And since φ is arbitrary, we have

$$(u_l)_t+(f(u_l))_x=0.$$

The same arguments hold for u_d, too.

2.

$$\begin{aligned}&\int_0^\infty\int_{-\infty}^\infty(u\varphi_t+f(u)\varphi_x)\mathrm{dxdt}+\int_{-\infty}^\infty u_0(x)\varphi(x,0)\mathrm{dx}\\ =&\int_0^\infty\int_{-\infty}^{\gamma(t)}(u_l\varphi_t+f(u_l)\varphi_x)\mathrm{dxdt}+\int_0^\infty\int_{\gamma(t)}^\infty(u_d\varphi_t+f(u_d)\varphi_x)\mathrm{dxdt}\\ &+\int_{-\infty}^\infty u_0(x)\varphi(x,0)\mathrm{dx}.\end{aligned}$$

3. Let us calculate the first integral from above. It holds

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{\gamma(t)}u_l\varphi\mathrm{dx}\\ =&\dot\gamma(t)u_l(\gamma(t),t)\varphi(\gamma(t),t)+\int_{-\infty}^{\gamma(t)}((u_l)_t\varphi+u_l\varphi_t)\mathrm{dx}.\end{aligned}$$

That implies

$$\int_0^\infty \int_{-\infty}^{\gamma(t)} u_l \varphi_t \mathrm{dxdt} = -\int_0^\infty \int_{-\infty}^{\gamma(t)} (u_l)_t \varphi \mathrm{dxdt}$$
$$-\int_0^\infty \dot{\gamma}(t) u_l(\gamma(t), t)\varphi(\gamma(t), t)\mathrm{dt} + \int_0^\infty \frac{\mathrm{d}}{\mathrm{d}t} \int_{-\infty}^{\gamma(t)} u_l \varphi \mathrm{dxdt}.$$

On the other hand,

$$\int_0^\infty \int_{-\infty}^{\gamma(t)} f(u_l)\varphi_x \mathrm{dxdt} = -\int_0^\infty \int_{-\infty}^{\gamma(t)} f(u_l)_x \varphi \mathrm{dxdt}$$
$$+\int_0^\infty f(u_l(\gamma(t), t))\varphi(\gamma(t), t))\mathrm{dt}$$

Adding these terms and using the fact that u_l is a solution of PDE on the left-hand side of the curve $(\gamma(t), t)$, one gets the following

$$\int_0^\infty (f(u_l) - \dot{\gamma} u_l)\varphi \mathrm{dt} + \int_0^\infty \frac{d}{\mathrm{dt}} \int_{-\infty}^{\gamma(t)} u_l \varphi \mathrm{dxdt}$$

as a value of that integral.

4. Analogously, concerning the right-hand side, one can see that the second integral equals

$$-\int_0^\infty (f(u_d) - \dot{\gamma} u_d)\varphi \mathrm{dt} + \int_0^\infty \frac{d}{\mathrm{dt}} \int_{\gamma(t)}^\infty u_d \varphi \mathrm{dxdt}.$$

5. After adding all the above integrals one gets

$$0 = \int_0^\infty (f(u_l) - f(u_d) - (u_l - u_d)\dot{\gamma})\varphi \mathrm{dt}$$
$$+ \int_0^\infty \frac{\mathrm{d}}{\mathrm{dt}} \int_{-\infty}^\infty u\varphi \mathrm{dxdt} + \int_{-\infty}^\infty u_0(x)\varphi(x, 0)\mathrm{dx},$$

and

$$\int_{-\infty}^\infty u(x, t)\varphi(x, t)\mathrm{dx}\Big|_{t=0}^{t=\infty} = -\int_{-\infty}^\infty u_0(x)\varphi(x, 0)\mathrm{dx}.$$

That is true if

$$\dot{\gamma} = \frac{f(u_d) - f(u_l)}{u_d - u_l} =: \frac{[f(u)]_\gamma}{[u]_\gamma}.$$

Obviously the above condition is sufficient. The proof is complete.

Condition (16) is called *Rankine-Hugoniot* (RH) condition.

EXAMPLE 6. Consider the following Riemann problem

$$u_t + \left(\frac{u^2}{2}\right)_x = 0 \tag{17}$$
$$u_0 = \begin{cases} u_l \in \mathbb{R}, & x < 0 \\ u_d \in \mathbb{R}, & x > 0. \end{cases}$$

Since u_l and u_d are constants, there exist two trivial solutions of (17) out of the discontinuity curve, and RH-condition gives

$$\dot\gamma(t) = \frac{u_d^2 - u_l^2}{2(u_d - u_l)} = \frac{u_d + u_l}{2},$$

i.e. $\dot\gamma(t) = ct$, $c = \frac{u_l+u_d}{2}$ and (see Fig. 7)

$$u(x,t) = \begin{cases} u_l, & x < ct \\ u_d, & x > ct, \end{cases} \tag{18}$$

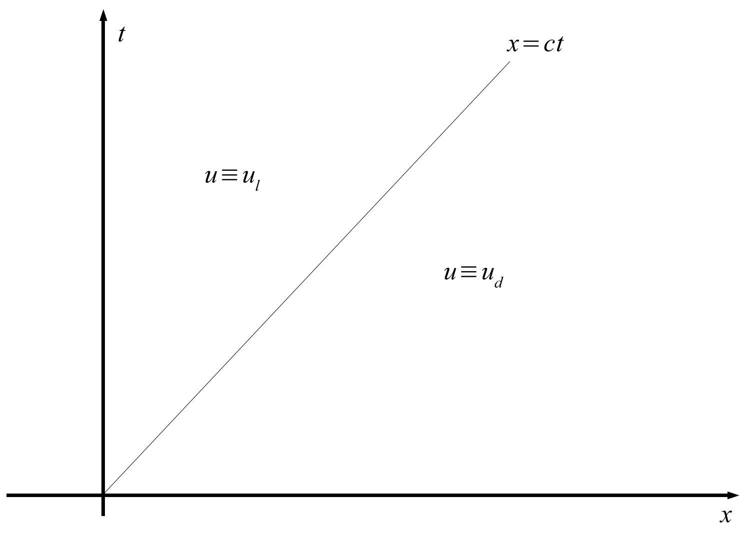

FIGURE 7. Shock wave

If $u_l < u_d$, then except the above solution there exist also the following solutions (Fig. 8):

$$u(x,t) = \begin{cases} u_l, & x < u_l t \\ \frac{x}{t}, & u_l t \le x \le u_d t \\ u_d, & x > u_d t \end{cases} \tag{19}$$

or, (Fig. 9))

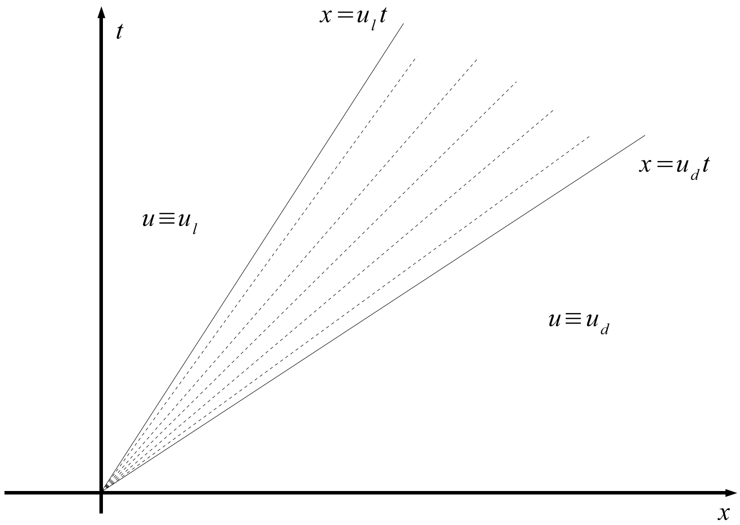

FIGURE 8. Rarefaction wave

$$u(x,t) = \begin{cases} u_l, & x < u_l t \\ \frac{x}{f}, & u_l t \le x \le at \\ a, & at \le x \le \frac{a+u_d}{2} t \\ u_d, & x \ge \frac{a+u_d}{2} t, \end{cases} \tag{20}$$

for some $a \in (u_l, u_d)$.

One can see that there is no uniqueness of solution in the case $u_l < u_d$. That problem (finding admissible or so called “entropy” solutions) will be approached later on.

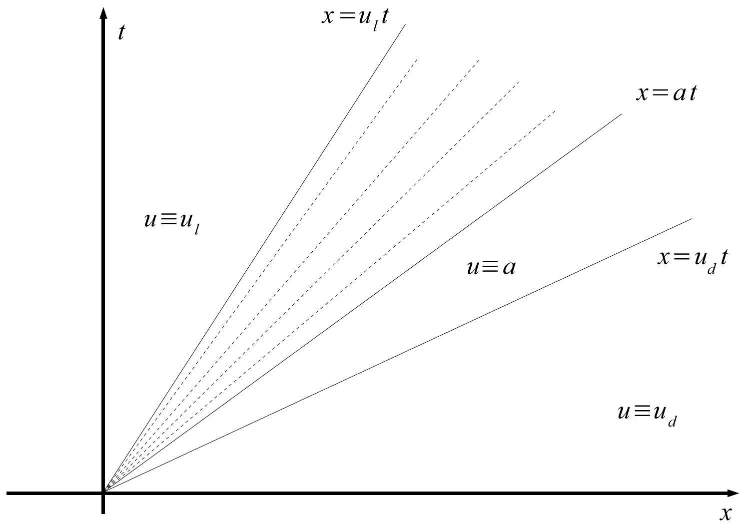

FIGURE 9. Non-entropic weak solution

EXAMPLE 7. Let us multiply partial differential equation(17) by u and transfer it into divergence form

$$\begin{aligned} &u_t + uu_x = 0 \quad / \cdot u \\ &uu_t + u^2 u_x = 0 \\ &\left(\frac{1}{2}u^2\right)_t + \left(\frac{1}{3}u^3\right)_x = 0. \end{aligned}$$

After nonlinear change of variables $\frac{1}{2}u^2 \mapsto v$, one gets the following conservation law

$$\begin{aligned} &v_t + \left(\frac{2\sqrt{2}}{3}v^{3/2}\right)_x = 0 \\ &v\Big|_{t=0} = \begin{cases} v_l = \frac{1}{2}u_l^2, & x < 0 \\ v_d = \frac{1}{2}u_d^2, & x > 0. \end{cases} \end{aligned}$$

RH-conditions give the following speed of shock wave c and the discontinuity line is $\gamma = ct$:

$$\begin{aligned} \dot{\gamma}(t) &= \frac{[\frac{3}{2}v^{3/2}]}{[v]} = \frac{\frac{2\sqrt{2}}{3}\frac{1}{2}(u_d^2)^{3/2} - \frac{2\sqrt{2}}{3}\frac{1}{2}(u_l^2)^{3/2}}{\frac{1}{2}(u_d^2 - u_l^2)} \\ &= \frac{\frac{1}{3}(u_d^3 - u_l^3)}{\frac{1}{2}(u_d^2 - u_l^2)} \neq \frac{u_l + u_d}{2} \text{ in general.} \end{aligned}$$

(For example, for $u_l = 1$, $u_d = 0$ one has $\frac{\frac{1}{3}}{\frac{1}{2}} \neq \frac{1}{2}$.)

This was an "unpleasant" example, because simple but nonlinear transformations of variables do not preserve solutions.

Because of that a precise interpretation of a physical model is of the crucial importance.

Solution of equation (15) of the form $u(x,t) = \tilde{u}(\frac{x}{t})$ is called *self-similar solution.* Now we shall try to find such a solution of (15) in a simple way, just by substituting a function of this form into the equation. After the differentiation we have

$$-\frac{x}{t^2}\tilde{u}'\Big(\frac{x}{t}\Big) + f'\Big(\tilde{u}\Big(\frac{x}{t}\Big)\Big)\frac{1}{t}\tilde{u}'\Big(\frac{x}{t}\Big) = 0$$

after multiplication of the equation with

t and the substitution $\frac{x}{t} \mapsto y$ one gets ODE

$$\tilde{u}'(y)(f'(\tilde{u}(y)) - y) = 0$$

After neglecting constant, so called trivial solutions ($\tilde{u}' \neq 0$), one can see that solution is given by the implicit relation

$$f'(\tilde{u}) = y, \text{ ie. } \tilde{u}(y) = f'^{-1}(y),$$

if f' is bijection (locally).

One can interpret the initial data in the following way:

$$u(x,0) = \begin{cases} u_l, & x < 0 \\ u_d, & x > 0 \end{cases} \Longrightarrow \tilde{u}(+\infty) = u_d,\ \tilde{u}(-\infty) = u_l. \tag{21}$$

If $f'' > 0$ (f is convex), then f' is an increasing function and solution $\tilde{u}$ to the equation satisfying (21) exists if $u_l < u_d$. Such solution is called *centered rarefaction wave* (the initial data has a singularity at zero).

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