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Partial differential equations:

Basic facts on PDEs

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Contents

Chapter 1

PDEs Classification

1.1 Classification of Evolutionary Systems

COnsider the following system of quasilinear PDEs

$$\frac{\partial u_i}{\partial t} + \sum_{j=1}^{n} a_{ij} \frac{\partial u_j}{\partial x} + b_i = 0, i = 1, \ldots, n, \tag{1.1}$$

where $u = u(x,t)$, $a_{ij} = a_{ij}(x,t,u(x,t))$, $b_i = b_i(x,t,u(x,t))$. Let us make a linear combination of these equations

$$\sum_{i=1}^{n} l_i \left(\frac{\partial u_i}{\partial t} + \sum_{j=1}^{n} a_{ij} \frac{\partial u_j}{\partial x} \right) + \sum_{i=1}^{n} b_i = 0.$$

We want to find curves $x = x(t)$, $t = t$, such that the above linear combination has the form

$$\sum_{i=1}^{n} l_i \frac{\mathrm{d}\, u_i}{\mathrm{d}\, t} + \sum_{i=1}^{n} b_i = 0.$$

These curves are called *characteristics* (1.1). That is possible if

$$\sum_{j=1}^{n} l_j a_{ji} = l_i \cdot x', i = 1, \ldots, n,$$

because $\frac{\partial u_i}{\partial x} x' = \frac{\partial u_i}{\partial t}$, $i = 1, \ldots, n$. If A denotes the matrix $[a_{ij}]_{i=1,\ldots,n, j=1,\ldots,n}$, then x' is eigenvalue , and $(l_1, \ldots, l_n)$ is its eigenvector

Definition 1. *System (1.1) is called* strictly hyperbolic*, if A has n distinct real eigenvalues. The system is* hyperbolic *if all eigenvectors of A are real and linearly independent. The system is weakly hyperbolic if all eigenvalues of A are real and the above do not hold. If A has no real eigenvalues, the system is elliptic.*

Note that we could solve (locally) initial data problem for (1.1) by the method of characteristics only if it is hyperbolic at least.

Example 1. (a) PDE $u_{tt} - \gamma u_{xx} = 0$ with $v = u_x$, $w = u_t$ becomes

$$\begin{aligned} v_t - w_x &= 0 \\ w_t - \gamma v_x &= 0, \end{aligned}$$

that is hyperbolic for $\gamma > 0$, weakly hyperbolic for $\gamma = 0$ and elliptic for $\gamma < 0$.

(b) Linear Klein-Gordon equation

$$u_{tt} - \gamma^2 u_{xx} + u = 0$$

with $v = u_t + \gamma u_x$ becomes

$$\begin{aligned} v_t - \gamma v_x + u &= 0 \\ u_t + \gamma u_x - v &= 0, \end{aligned}$$

that is strictly hyperbolic. Let us note that the "natural" change of variables: $u_t = w$, $v = u_x$ gives

$$\begin{aligned} u_t - w &= 0 \\ v_t - w_x &= 0 \\ w_t - \gamma^2 v_x + u &= 0, \end{aligned}$$

that is equivalent (for C^2-solutions) to the following equation

$$\frac{\partial}{\partial t}\left(u_{tt} - \gamma^2 u_{xx} + u\right) = 0.$$

(c) The heat equation, $u_t - a^2 u_{xx} = 0$, becomes into the weakly hyperbolic system but with x being the evolutionary parameter that is far from the physical reality.

1.2 Classification of Second Order PDEs

Let the following PDE with two independent variables

$$a u_{xx} + 2b u_{xy} + c u_{yy} = d, \tag{1.2}$$

be given, with a, b, c and d depend on x, y, u and its first derivatives. Our aim is to see when we can find a solution to the equation in a neighbourhood of the curve

$$\omega : x = f(s), y = g(s)$$

with the following data given on the curve ω:

$$u = h(s), u_x = r(s), u_y = t(s).$$

Note that one of the conditions is superfluous since

$$\frac{\mathrm{d}\, u}{\mathrm{d}\, s} = \frac{\partial u}{\partial x}\frac{\mathrm{d}\, x}{\mathrm{d}\, s} + \frac{\partial u}{\partial y}\frac{\mathrm{d}\, y}{\mathrm{d}\, s}$$

that is

$$h'(s) = r(s)f'(s) + t(s)g'(s).$$

For the second order derivatives on ω we have

$$\begin{aligned} r''(s) &= \tfrac{\mathrm{d}\, u_x}{\mathrm{d}\, s} = u_{xx}f'(s) + u_{xy}g'(s) \\ t''(s) &= \tfrac{\mathrm{d}\, u_y}{\mathrm{d}\, s} = u_{xy}f'(s) + u_{yy}g'(s). \end{aligned}$$

Thus, one can always find the second order derivatives of u on ω (uniquely) from these two equations and (1.2) if

$$D_S = \begin{vmatrix} f' & g' & 0 \\ 0 & f' & g' \\ a & 2b & c \end{vmatrix} = ag'^2 - 2bf'g' + cf'^2 \neq 0.$$

A point on ω is called *characteristic* if $D_S = 0$. With the change of variables $g' = \frac{\mathrm{d}\, y}{\mathrm{d}\, s}$ and $f' = \frac{\mathrm{d}\, x}{\mathrm{d}\, s}$ we get so called characteristic equation

$$\frac{\mathrm{d}\, y}{\mathrm{d}\, x} = \frac{b \pm \sqrt{b^2 - ac}}{a} =: \lambda \tag{1.3}$$

(if $a = 0$, then we express the term $\frac{\mathrm{d}\, x}{\mathrm{d}\, y}$.

Definition 2. If $b^2 - ac < 0$, then (1.2) is called *elliptic*, if $b^2 - ac = 0$, it is called *parabolica*, and if $b^2 - ac > 0$, it is called *hyperbolic*.

If the curve ω is not given in the explicit form, but $w(x, y) = 0$, then the characteristics are determined by the equation

$$\frac{\mathrm{d}\, w}{\mathrm{d}\, x} + \lambda\frac{\mathrm{d}\, w}{\mathrm{d}\, y} = 0.$$

Let us now consider more-dimensional case. For the sake of simplicity, we assume that the equations are now linear.

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(x)u_{x_i x_j} + \Phi(x, u, u_{x_1}, \dots, u_{x_n}) = 0. \tag{1.4}$$

We assume that the matrix $[a_{ij}]$ is symmetric with C^2 elements. Symmetry is always possible to achieve because $u_{x_i x_j} = u_{x_j x_i}$.

Let $y = y(x)$ be non-degenerate change of variables (i.e. Jacoby determinant $x \to y$ is not zero, $|D_x y| \neq 0$). With

$$\tilde{a}_{lk} = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\frac{\partial y_l}{\partial x_i}\frac{\partial y_k}{\partial x_j},$$

equation (1.4) becomes

$$\sum_{l=1}^{n}\sum_{k=1}^{n}\tilde{a}_{lk}(y)u_{y_l y_k}+\tilde{\Phi}(y,u,u_{y_1},\ldots,u_{y_n})=0.$$

Put $A=[a_{ij}]$ and $\tilde{A}=[a_{ij}]$. The matrix A is symmetric, and their eigenvalues are real and $\tilde{A}=JAJ^*$, where $J^*=D_x y$. Using these matrices, our aim is to transform quadratic form

$$\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}p_i p_j \text{ into another one } \sum_{l=1}^{n}\sum_{k=1}^{n}\tilde{a}_{lk}q_l q_k.$$

If we use the well known facts from linear algebra, we can always achieve that the second quadratic form be

$$\sum_{l=1}^{r}q_l^2-\sum_{l=r+1}^{m}q_l^2, m\leqslant n.$$

The shape of this form determines a type of PDE.

Definition 3. If

1. $m=n$ and $r=m$ or $r=0$, PDE (1.4) is elliptic.
2. $m=n$ and $1\leqslant r\leqslant n-1$, the PDE is ultrahyperbolic. If $r=1$ or $r=n-1$, the PDE is hyperbolic.
3. $m<n$, the PDE is ultraparabolic. If $m=n-1$ or $r=1$ or $r=n-1$, the PDE is parabolic.

1.2.1 Canonical forms of PDEs with two independent variables

Now, we will reduce PDE (1.2) into a simpler form, so called *canonical form* using the characteristics.

- Hyperbolic PDE. There exists two real values $\lambda_{1,2}$ for the right-hand side of (1.3). Let $\xi(x,y)=c_1$ and $\eta(x,y)=c_2$ be solutions of (1.3) such that $\xi_y\neq 0$ and $\eta_y\neq 0$. Then

$$|D_{(x,y)}(\xi,\eta)|=\frac{2\sqrt{b^2-ac}}{a}\xi_y\eta_y\neq 0,$$

and the change of variables $(x,y)\mapsto(\xi,\eta)$ is non-singular. Then we get

$$u_{\xi\eta}=\tilde{\phi}(\xi,\eta,u,u_\xi,u_\eta).$$

- Parabolic PDE. There exists only one value for λ and only one real solution to (1.3), given by $\xi(x,y) = c_1$, $\xi_y \neq 0$ (or $\xi_x \neq 0$, when we proceed with interchanged values of x and y bellow). The second variable we can chose arbitrary, say $\eta = x$, when we have

$$|D_{(x,y)}(\xi, \eta)| = -\xi_y \neq 0,$$

and the change is non-singular. We get in that case

$$u_{\xi\xi} = \tilde{\phi}(\xi, \eta, u, u_\xi, u_\eta) \text{ or } u_{\eta\eta} = \tilde{\phi}(\xi, \eta, u, u_\xi, u_\eta).$$

- Elliptic PDE. Now, λ is not real. Denote by λ_1 and λ_2 complex valued right-hand side of (1.3). Let ω be a complex valued solution to that equation

$$\omega_x + \lambda_1 \omega_y = 0,$$

such that $\omega_y \neq 0$. Put

$$\xi = \frac{\omega + \bar{\omega}}{2} \text{ and } \eta = \frac{\omega - \bar{\omega}}{2}.$$

(Note that $\bar{\omega}$ satisfies $\bar{\omega}_x + \lambda_1 \bar{\omega}_y = 0$.) Next,

$$|D_{(x,y)}(\xi, \eta)| = |D_{(\omega,\bar{\omega})}(\xi, \eta)| \cdot |D_{(x,y)}(\omega, \bar{\omega})| = \frac{-\sqrt{b^2 - ac}}{ia} \omega_y \bar{\omega}_y \neq 0.$$

That is, the variables change $(x, y) \mapsto (\xi, \eta)$ is non-singular. With that change we get

$$u_{\xi\xi} + u_{\eta\eta} = \tilde{\phi}(\xi, \eta, u, u_\xi, u_\eta).$$

Chapter 2

Second Order Hyperbolic PDEs

2.1 One-dimensional wave equation

2.1.1 Cauchy (initial data) problem

Solutions will be classical ones in the following three chapters, i.e. they will belong to C^m, where m is an order of a PDE.

Let

$$u_{tt} - c^2 u_{xx} = 0, c > 0.$$

With $\xi = x + ct$, $\eta = x - ct$ it becomes $u_{\xi\eta} = 0$, with a solution

$$u = p(\xi) + q(\eta) = p(x + ct) + q(x - ct),$$

where $p, q \in C^2$ are arbitrary functions.

Theorem 1. *Let $f \in C^2(\mathbf{R})$ and $g \in C^1(\mathbf{R})$. be given. Then the Cauchy problem*

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= 0 \\ u|_{t=0} &= f(x) \\ u_t|_{t=0} &= g(x) \end{aligned}$$

has a unique classical solution given by so called D'Alambert formula

$$u(x,t) = \frac{1}{2}(f(x + ct) + f(x - ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)\,\mathrm{d}\,y.$$

Proof. As we already saw, the general solution to the homogeneous wave equation is given by

$$u(x,t) = p(x + ct) + q(x - ct). \tag{2.1}$$

We will find appropriate p and q that satisfies the initial data. The first condition is

$$p(x) + q(x) = f(x). \tag{2.2}$$

Using $\xi = x + ct$, $\eta = x - ct$ we get that for $t = 0$ the following condition

$$\begin{aligned}\frac{\partial u}{\partial t}|_{t=0} &= \left(\tfrac{\partial p}{\partial \xi}\tfrac{\partial \xi}{\partial t} + \tfrac{\partial p}{\partial \eta}\tfrac{\partial \eta}{\partial t} + \tfrac{\partial q}{\partial \xi}\tfrac{\partial \xi}{\partial t} + \tfrac{\partial q}{\partial \eta}\tfrac{\partial \eta}{\partial t}\right)|_{t=0} \\ &= cp'(x) - cq'(x) = g(x).\end{aligned}$$

Differentiating (2.2)

$$p'(x) + q'(x) = f'(x),$$

and using the previous equation we have

$$p' = \frac{cf' + g}{2c}, q' = \frac{cf' - g}{2c},$$

i.e.

$$\begin{aligned}p(x) &= \tfrac{1}{2}\left(f(x) + \tfrac{1}{c}\textstyle\int_0^x g(y)\,\mathrm{d}\,y\right) + c_1 \\ q(x) &= \tfrac{1}{2}\left(f(x) - \tfrac{1}{c}\textstyle\int_0^x g(y)\,\mathrm{d}\,y\right) + c_2,\end{aligned}$$

and from (2.1),

$$u(x,t) = \frac{1}{2}(f(x+ct) + f(x-ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)\,\mathrm{d}\,y,$$

because $u(x,0) = f(x)$ implies $c_1 + c_2 = 0$. □

Note that $\|f - f_1\|_{L^\infty} < \varepsilon$ and $\|g - g_1\|_{L^\infty} < \varepsilon$, then, if v denotes a solution to initial data problem with f_1 instead f and g_1 instead g, we have

$$\begin{aligned}|u - v| \leqslant\ & \tfrac{1}{2}|f(x+ct) - f_1(x+ct)| + \tfrac{1}{2}|f(x-ct) - f_1(x-ct)| \\ & + \tfrac{1}{2c}\textstyle\int_{x-ct}^{x+ct} |g(y) - g_1(y)|\,\mathrm{d}\,s.\end{aligned}$$

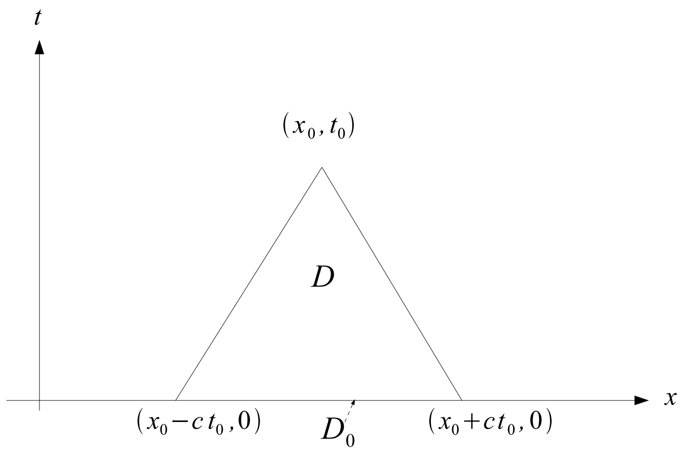
For every $t > 0$ we have

$$\|u(\cdot,t) - v(\cdot,t)\|_{L^\infty} \leqslant \varepsilon + \frac{1}{2c}\varepsilon \cdot 2tc = \varepsilon(1+t).$$

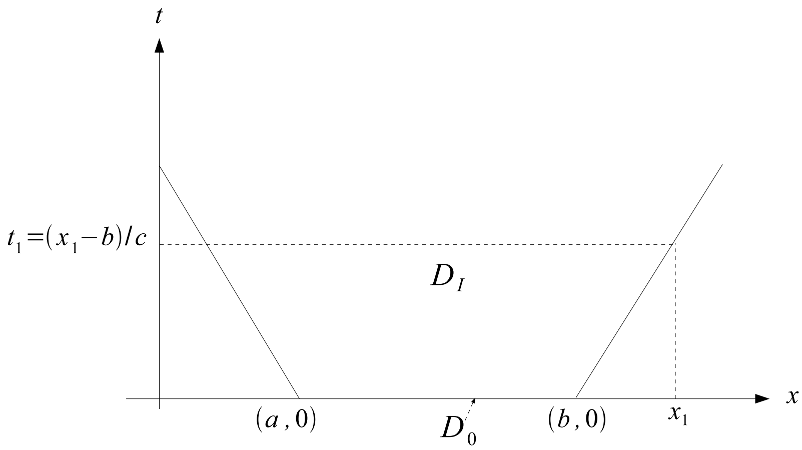
That means that the above Cauchy problem is *well posed* in L^∞-topology (in the Hadamard sense): It has a unique solution that depends continuously on the initial data.

Let us draw characteristics (these are lines with slopes equal $\pm c$) from a point (x_0, t_0) towards x-axes ("*backward characteristics*").

On the basis of D'Alambert formula one can see that$u(x_0, t_0)$ depends only on value of the initial data atD_0, so D is called *domain of dependence* for the point (x_0, t_0). Let us note that if some point (x_1, t_1) belongs to D, its domain of dependence is a subset of D.



Let us now take an interval $I = [a, b]$ and from the points a and b draw characteristics. All the characteristics emanating from any point in I lies between them. The area defined in such a way will be denoted by D_I and called the *domain of influence* of interval I. As the slopes of the characteristics are $\pm c$, each disturbance initially placed in the interval I (i.e. $t = 0$) will reach a point $x_1 > b$ in time equals $t_1 = (x_1 - b)/c$, i.e. it propagates with the speed c. That property for a PDE is called *finite propagation speed* and represents one of the most important properties shared by hyperbolic equations.



We shall take $c = 1$ in the sequel (one can just change the variable $t \mapsto ct$.

Theorem 2. *Let $F \in C^2(\mathbf{R}^2)$, $f \in C^2(\mathbf{R})$ and $g \in C^1(\mathbf{R})$. Then there is a* *classical solution to the Cauchy problem*

$$\begin{aligned} u_{tt} - u_{xx} &= F(x,t) \\ u|_{t=0} &= f(x) \\ u_t|_{t=0} &= g(x) \end{aligned} \tag{2.3}$$

given by

$$u(x,t) = \frac{1}{2}(f(x+t) + f(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} g(y)\,\mathrm{d}\,y + \int\int_{D_{(x,t)}} F(y,s)\,\mathrm{d}\,y\,\mathrm{d}\,s,$$

where $D_{(x,t)}$ is the domain of dependence for (x,t) (see Figure 2.1.1).

Proof. Denote by D the area $D_{(x,t)}$ and by C its boundary (positively oriented), $C = C_0 \cup C_1 \cup C_2$, where

$$\begin{aligned} C_0 &= \{(y,0) : y \in [x-t, x+t]\} \\ C_1 &= \{(y,s) : s \in [0,t], y = x+t-s\} \\ C_2 &= \{(y,s) : s \in [0,t], y = x-t+s\}. \end{aligned}$$

Integrating the given PDE from (2.3) over D,

$$I := \int\int_D (u_{tt} - u_{xx})\,\mathrm{d}\,y\,\mathrm{d}\,s = \int\int_D F(y,s)\,\mathrm{d}\,y\,\mathrm{d}\,s.$$

By Green's Theorem we have

$$I = -\int_C u_t\,\mathrm{d}\,y + u_x\,\mathrm{d}\,s.$$

Calculating the line integrals over C,

$$\int_{C_0} u_t\,\mathrm{d}\,y + u_x\,\mathrm{d}\,s = \int_{x-t}^{x+t} u_t\,\mathrm{d}\,y = \int_{x-t}^{x+t} g(y)\,\mathrm{d}\,y.$$

At C_1 we have $\mathrm{d}\,y = -\mathrm{d}\,s$, so

$$\begin{aligned} \int_{C_1} u_t\,\mathrm{d}\,y + u_x\,\mathrm{d}\,s = &\quad -\textstyle\int_{C_1} u_t\,\mathrm{d}\,s + u_x\,\mathrm{d}\,y = -\int_{C_1}\mathrm{d}\,u = u(x+t,0) - u(x,t) \\ = &\quad f(x+t) - u(x,t). \end{aligned}$$

At C_2 we have $\mathrm{d}\,y = \mathrm{d}\,s$, and

$$\begin{aligned} \int_{C_2} u_t\,\mathrm{d}\,y + u_x\,\mathrm{d}\,s = &\quad \textstyle\int_{C_2} u_t\,\mathrm{d}\,s + u_x\,\mathrm{d}\,y = \int_{C_1}\mathrm{d}\,u = u(x-t,0) - u(x,t) \\ = &\quad f(x-t) - u(x,t). \end{aligned}$$

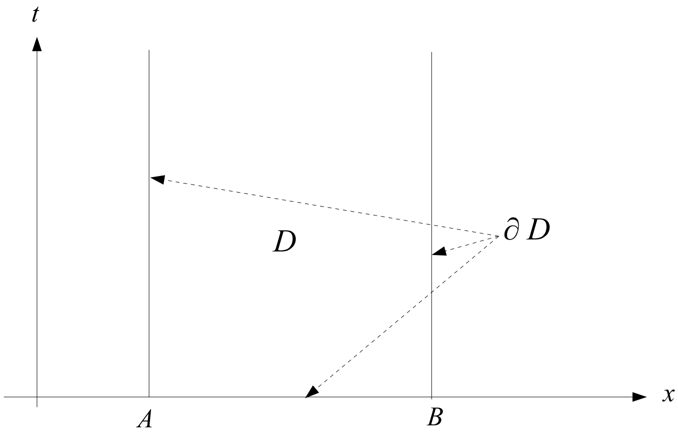
Adding all these integrals, we get

$$2u(x,t) - f(x+t) - f(x-t) - \int_{x-t}^{x+t} g(y)\,\mathrm{d}\,y = \int\int_D F(y,s)\,\mathrm{d}\,y\,\mathrm{d}\,s,$$

and that proves the theorem. □

2.1.2 Mixed problem

Now, we are interested in solution of wave equation in some bounded interval, $x \in [A, B]$. Because of that we have to prescribe boundary conditions in the points $x = A$ and $x = B$ for $t > 0$.



Let us consider the following problem

$$\begin{array}{ll} u_{tt} - u_{xx} = \varphi(x,t), & A < x < B, t > 0 \\ u(x,0) = f(x), u_t(x,0) = g(x), & A < x < B \\ u(A,0) = a(t) \text{ or } u_x(A,0) = a(t), & t > 0 \\ u(B,0) = b(t) \text{ or } u_x(B,0) = b(t), & t > 0. \end{array} \tag{2.4}$$

Here, we assume the *compatibility condition* $a(0) = f(A)$ and $b(0) = f(B)$. If we have Von Neumann's conditions, then the compatibility conditions are $a(0) = f'(A)$, $b(0) = f'(B)$. Denote by D the area $\{(x,t) : x \in (A,B), t > 0\}$, and by ∂D its boundary. As before, let us look at the case $c \equiv 1$.

Theorem 3. *There exists at most one solution $u \in C^2(D) \cap C^0(\partial D)$ to mixed problem (2.4).*

Proof. The assertion will be proved if we show that the only solution of (2.4) that satisfies homogenous initial and boundary data is the trivial one ($u \equiv 0$). That follows from the fact that our problem is linear one. In order to prove that, we will use so called *energy integral*

$$E(t) := \frac{1}{2}\int_A^B u_x^2(x,t) + u_t^2(x,t)\,\mathrm{d}\,x.$$

Differentiating $E(t)$ we get

$$\begin{aligned} \frac{\mathrm{d}\,E(t)}{\mathrm{d}\,t} &= \int_A^B (u_x u_{xt} + u_t u_{tt})\,\mathrm{d}\,x = \int_A^B (u_x u_{xt} + u_t u_{xx})\,\mathrm{d}\,x \\ &= \int_A^B \frac{\partial}{\partial x}(u_x u_t)\,\mathrm{d}\,x = u_x u_t|_{x=A}^{x=B} = 0. \end{aligned}$$

We have used that $u|_{x=A} = 0$ implies $u_x|_{x=A} = 0$ and the same for the second boundary condition.

Thus, $E(t) \equiv \text{const}$, and since the homogenous initial data imply $E(0) = 0$ and $u(x,0) \equiv 0$, we have $E(t) \equiv 0$, and $u \equiv \text{const}$. □

For the construction of solution of (2.4) for $\phi \equiv 0$ we will use the following interesting lemma.

Lemma 1. *Let A, B, C and D vertices of a rectangle whose sides are characteristic lines of a homogenous wave equation $u_{tt} - u_{xx} = 0$. A function $u = u(x,y)$ is a classical solution to that equation if and only if*

$$u(A) + u(C) = u(B) + u(D),$$

for every rerctangle $ABCD$.

Proof. Suppose $u \in C^2(\mathbf{R}^2)$ solves the homogenous wave equation. Then

$$u(x,t) = p(x+t) + q(x-t),$$

for any pair of functions $p, q \in C^2(\mathbf{R})$. Let $A(x+k, t+h)$ for some $k > 0$ and $h > 0$. Then the other coordinates of the vertices are $B(x-h, t-k)$, $C(x-k, t-h)$, $D(x+h, t+k)$. Substitution of these values into the above expression for u gives

$$\begin{aligned} u(A) + u(C) &= p(x+k+t+h) + q(x+k-t-h) \\ &= p(x-k+t-h) + q(x-k-t+h) = u(B) + u(D). \end{aligned}$$

Opposite, let u satisfies the above difference equation for any k and h. Put $h = 0$ divide the whole equation with k^2,

$$\frac{u(x+k,t) + u(x-k,t) - 2u(x,t)}{k^2} = \frac{u(x,t-k) + u(x,t+k) - 2u(x,t)}{k^2},$$

Taylor expansion of u around the point (x,t) gives

$$\begin{aligned} u(x \pm k, t) &= u(x,t) \pm u_x(x,t)k + \tfrac{1}{2}u_{xx}(x,t)k^2 + k^2\mathcal{O}(k), \\ u(x, t \pm k) &= u(x,t) \pm u_t(x,t)k + \tfrac{1}{2}u_{tt}(x,t)k^2 + k^2\mathcal{O}(k), \quad k \to 0. \end{aligned}$$

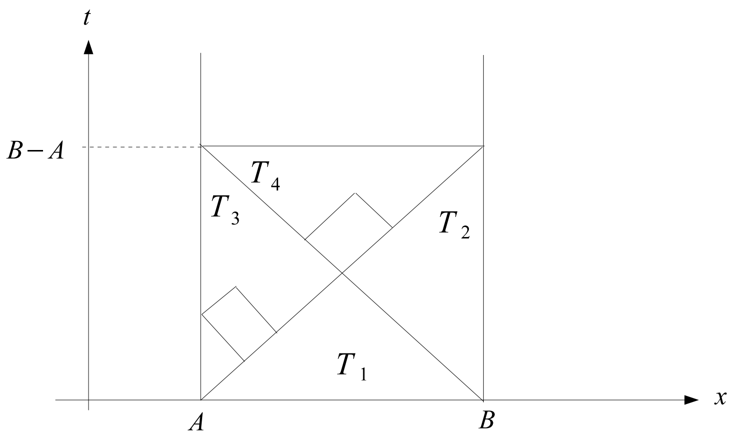
Substituting these terms into the above expressions and letting $k \to 0$, we get

$$u_{tt} - u_{xx} = \mathcal{O}(k), \ k \to 0.$$

□

We will use this lemma for construction of a solution to

$$\begin{aligned} u_{tt} - u_{xx} &= 0, & A < x < B, t > 0 \\ u(x,0) = f(x), u_t(x,0) &= g(x), & A < x < B \\ u(A,0) = a(t), u(B,0) &= b(t), & t > 0. \end{aligned}$$



Let us draw the characteristics lines emanating from the points A and B inside the domain $\{(x,t) : A < x < B, t > 0\}$. That defines the following triangles

$$\begin{aligned} T_1 &= (A,0)(B,0)\left(\frac{A+B}{2}, \frac{A+B}{2} - A\right), \\ T_2 &= \left(\frac{A+B}{2}, \frac{A+B}{2} - A\right)(B,0)(B, B-A), \\ T_3 &= (A,0)\left(\frac{A+B}{2}, \frac{A+B}{2} - A\right)(A, B-A), \\ T_4 &= (A, B-A)\left(\frac{A+B}{2}, \frac{A+B}{2} - A\right)(B, B-A). \end{aligned}$$

In T_1, a solution is given by D'Alambert formula (T_1 is a domain of dependence). Each point in T_2 and T_3 is a vertex of a rectangle that satisfies the conditions in the previous lemma. One of other vertices lies at the boundary of T_1 and final two are at $x = A$ or $x = B$ Similarly, each point in T_4 is a vertex of a rectangle with other three vertices lying at the boundaries of T_2 and T_3. So, the previous lemma gives the value at that point in T_4. One just continue the procedure but now starting from the line $\{(x,t) : A \leqslant x \leqslant B, t = B - A\}$ as far as one wants.

2.1.3 Energy Integral

Now, we return to the Cauchy problem and want to prove uniqueness of a solution.

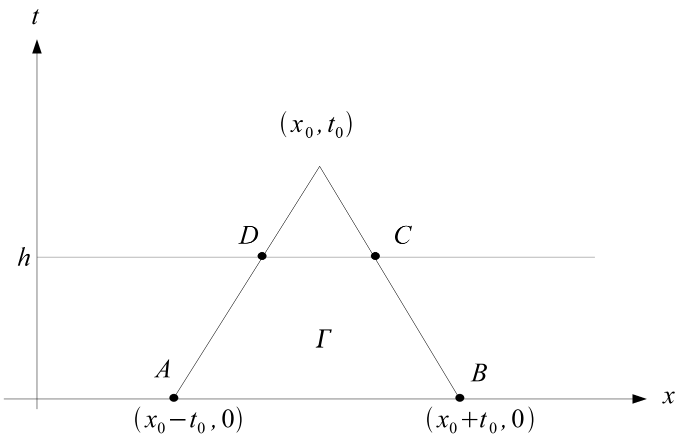
Theorem 4. *Let $F \in C^2(\mathbf{R}^2)$, $f \in C^2(\mathbf{R})$ and $g \in C^1(\mathbf{R})$. Then the Cauchy problem (2.3) has an unique solution in the space $C^2(\mathbf{R}^2)$.*

Proof. We will give the proof only for $t > 0$. The proof for lower half-plane is the same. Due to linearity, it would be enough to prove that a solution to

$$\begin{aligned} u_{tt} - u_{xx} = 0, &\qquad (x,t) \in \mathbf{R} \times \mathbf{R}^+ \\ u|_{t=0} = 0, u_t|_{t=0} = 0, &\quad x \in \mathbf{R} \end{aligned} \tag{2.5}$$

is identically equal zero.

Let $(x_0, t_0) \in \mathbf{R} \times \mathbf{R}^+$ and let D_{x_0,t_0} be its domain of dependence. Denote by Γ the trapezoid $ABCD$ obtained by the intersection of the line $t = h > 0$ and D_{x_0,t_0}, where one of vertices is given by $A(x_0 - t_0, 0)$. (See figure. 2.1.3)



Multiplying the equation in (2.5) with $-2u_t$, we get

$$0 = -2u_t(u_{tt} - u_{xx}) = -(u_x^2 + u_t^2)_t + 2(u_x u_t)_x.$$

Integration of this expression over Γ and use of the Green's formula imply

$$0 = \int_{\partial\Gamma} ((u_x^2 + u_t^2)t_\nu - 2(u_x u_t)x_\nu)\, \mathrm{d}\, s,$$

where t_ν and x_ν are components of outer normal on $\partial\Gamma$. At lines AD and BC we have $t_\nu = 1/\sqrt{2}$ and $x_\nu = \pm 1/\sqrt{2}$, at AB we have $t_\nu = -1$, $x_\nu = 0$, and at CD we have $t_\nu = 1$, $x_\nu = 0$. That means

$$\begin{aligned} 0 = \ &\textstyle\int_{AB} -(u_x^2 + u_t^2)\, \mathrm{d}\, x + \int_{CD}(u_x^2 + u_t^2)\, \mathrm{d}\, x \\ &+ \textstyle\int_{BC \bigcup DA} \frac{1}{t_\nu}(u_x t_\nu - u_t x_\nu)^2\, \mathrm{d}\, s. \end{aligned}$$

Using the non-negativity of the last term we have

$$\int_{AB} (u_x^2 + u_t^2)\, \mathrm{d}\, x \geqslant \int_{CD} (u_x^2 + u_t^2)\, \mathrm{d}\, x.$$

Since $u = u_t = 0$ at $t = 0$, one can see that

$$\int_{CD} (u_x^2(x,h) + u_t^2(x,h)) \, \mathrm{d}\, x = 0,$$

for every h, i.e. $u \equiv \text{const}$, and the initial data implies $u \equiv 0$, $\square$

Chapter 3

Parabolic Second Order PDEs

Parabolic equations arises in describing diffusion processes, or time-irreversible processes: Mathematically – these equations are not invariant for the variable exchange $t \mapsto -t$. One could say that we can only guess the future and not the past. They also has an additional property: A solution is usually more regular than initial data.

The simplest (but also the most important) example is so called *heat equation*

$$Hu \equiv u_t - k^2 \Delta u = 0, k \in \mathbf{R}. \tag{3.1}$$

It is a good model for heat transmission (u denotes a temperature in that case) when a material is good heat transmitter.

Let us consider the cylindric area

$$D = \Omega \times (0, T), T \leqslant \infty, \Omega \subset \mathbf{R}^n,$$

where Ω is an open, bounded set. Denote by D' the closure of D, $\bar{D}$, without points where $t = T$,

$$D' = \bar{\Omega} \times \{t = 0\} \cup \partial\Omega \times [0, T].$$

The following two theorems, *maximum principles* are very important. Their use is in uniqueness proof (as we will see) and in obtaining certain a priori estimates (out of scope of this lecture notes). One could say that it is an analogue of energy integral for the wave (and other hyperbolic) equation.

Theorem 5. *Let $u \in C(\bar{D}) \cap C^2(D)$ be a solution of (3.1). Then* $\max u$ *and* $\min u$ *are not reached inside D.*

Proof. We will prove the theorem only for a maximum. A proof for minimum is the same, or one can use this proof and change the dependent variable $u \mapsto -u$. Denote $M = \max_{D'} u$. For $0 < \varepsilon \ll 1$ define

$$v(x,t) := u(x,t) + \varepsilon |x|^2.$$

Then

$$Hv = -2nk^2\varepsilon < 0.$$

Let $\bar{t} < \infty$ and $\bar{t} \leqslant T$ be given. A maximum of v cannot be reached in the set $\Omega \times (0, \bar{t})$, because in that case we would have

$$v_t \geqslant 0 \text{ and } \Delta v \leqslant 0 \text{ for } \varepsilon \text{ small enough}$$

and

$$Hv = v_t - k^2 \Delta v \geqslant 0,$$

and that contradicts $Hv < 0$. Also, a maximum cannot be reached at the point with time coordinate $t = \bar{t}$, because in that case we would have $v_t \geqslant 0$ (function is non-decreasing with respect to t up to the boundary), and the same argument follows: $Hv \geqslant 0$. Since v being continuous on compact set $\Omega \times [0, \bar{t}]$), it reaches its maximum on $\bar{\Omega} \times [0, \bar{t}] \cap D'$, i.e. in a point where $u \leqslant M$. That implies

$$v < M + \varepsilon \max_{\bar{\Omega}} |x|^2,$$

and for ε small enough u cannot reach a maximum out of D'. Since $\bar{t}$ is arbitrary, the assertion follows. □

Corollary 1. *The mixed problem*

$$\begin{aligned} Hu &= f \quad \text{na}\, D \\ u &= g \quad \text{na}\, D' \end{aligned}$$

with $f \in C^2(D)$ and $g \in C(D')$ has at most one solution in the space $C(\bar{D}) \cap C^2(D)$.

The proof easily follows by using linearity of the problem and previous theorem.

The maximum principle holds true for an unbounded area too. We shall take

$$D = \mathbf{R}^n \times (0,T), 0 < T \leqslant \infty.$$

Theorem 6. *Let u be a solution of (3.1), $u \in C(\bar{D}) \cap C^2(D)$. Let*

$$M = \sup_{(x,t)\in \bar{D}} u(x,t)$$

and

$$N = \sup_{x \in \mathbf{R}^n} u(x,0).$$

Then $M = N$ if $M < \infty$.

Proof. For $0 < \varepsilon \ll 1$ define

$$v(x,t) = u(x,t) - \varepsilon(2nt + |x|^2).$$

It is easy to see that $Hv = 0$. Suppose that $M < \infty$ and $M > N$.

Then

$$v(x,0) = u(x,0) - \varepsilon|x|^2 \leqslant u(x,0) \leqslant N,$$

for every x. If

$$|x|^2 \geqslant \frac{M-N}{\varepsilon} \text{ and } 0 \leqslant t \leqslant T,$$

then

$$v(x,t) = u(x,t) - \varepsilon(2tn + |x|^2) \leqslant M - \varepsilon|x|^2 \leqslant N, \tag{3.2}$$

for ε small enough. Since $M < \infty$, the area

$$\Omega := \left\{ x : |x|^2 < \frac{M-N}{\varepsilon} \right\}$$

is bounded with respect to x-variable and we can use the previous theorem. Thus,

$$v(x,t) \leqslant N \text{ for } x \in \Omega,$$

because $v(x,0) \leqslant N$, and (3.2) implies $v(x,0) \leqslant N$ for $|x|^2 = \frac{M-N}{\varepsilon}$.

These two estimates, one for $x \in \Omega$ and one for $x \notin \Omega$, give

$$v(x,t) \leqslant N, (x,t) \in \mathbf{R}^n \times [0,T],$$

since ε may be as small as needed.

Thus,

$$u(x,t) = v(x,t) + \varepsilon(2nt + |x|^2) \leqslant N + \varepsilon(2nt + |x|^2),$$

fr every $(x,t) \in D$. Let us fix (x,t) and let $\varepsilon \to 0$. Then $u(x,t) \leqslant N$ for every $(x,t) \in D$, that contradicts the assumption $M > N$. □

Corollary 2. *The Cauchy problem*

$$\begin{array}{ll} Hu = f & \textit{in } D, \\ u(x,0) = g(x), & x \in \mathbf{R}^n \end{array}$$

has at most one bounded solution in $u \in C^2(D) \cap C_b(\bar{D})$.

Let us note that the above corollary really depends on the boundedness condition:

For $n = 1$, unbounded function

$$u(x,t) = \sum_{k=0}^{\infty} \frac{1}{(2k)!} x^{2k} \frac{\mathrm{d}^k}{\mathrm{d}\, t^k} e^{-\frac{1}{t^2}}$$

solves $Hu = 0$, $u(x,0) = 0$, but $u \equiv 0$ also solves the same problem.

Theorem 7. *Let $\varphi(x)$ be continuous and uniformly bounded function from $\mathbf{R}^n$ u $\mathbf{R}$. Then*

$$u(x,t) := \int_{\mathbf{R}^n} (4k\pi t)^{-n/2} \exp\left(\frac{-2|z-x|}{4kt}\right) \varphi(z) \,\mathrm{d}\, z$$

is a unique bounded solution to Cauchy problem

$$Hu = 0, u(x,0) = \varphi(x).$$

It is an analytic function for every $x \in \mathbf{C}^n$, $t \in \mathbf{C}$, $\mathrm{Re}(z) > 0$.

The proof is straightforward. If the initial data belongs to $L^2(\mathbf{R}^n)$, the above solution can be calculated by using Fourier transform with respect to x and solving an ODE with respect to t.

Chapter 4

The Second Order Elliptic PDEs

4.1 Introduction

Here, Ω will denote open, bounded and connected subset of $\mathbf{R}^n$. Let L be a partial differential operator. We will look at the following possibilities for boundary problems.

Dirichlet problem (I boundary problem). We look for a solution $u \in C^2(\Omega) \cap C(\bar{\Omega})$ of $Lu = f$ in Ω, with $u|_{\partial\Omega} = g$.

Neumann problem (II boundary problem). $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$, $\frac{\partial u}{\partial \nu}|_{\partial\Omega} = g$.

III boundary problem (Robin problem). $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$, $\frac{\partial u}{\partial \nu}|_{\partial\Omega} + au|_{\partial\Omega} = g$.

Model problem is the Laplace equation

$$Lu \equiv \Delta u = f,$$

. Other elliptic PDEs can be treated similarly)hat is not completely the case with hyperbolic and parabolic equations).

Definition 4. Function u we call *harmonic (subharmonic, superharmonic)* if

$$\Delta u = 0 (\Delta u \geqslant 0, \Delta u \leqslant 0).$$

In this chapter we will use frequently Green's Theorem

$$\int_\Omega \Delta u \,\mathrm{d}\, x = \int_{\partial\Omega} \nabla u \cdot \nu \,\mathrm{d}\, S = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} \,\mathrm{d}\, S. \tag{4.1}$$

Theorem 8. *Let $u \in C^2(\Omega)$ satisfies*

$$\Delta u = 0 (\Delta u \geqslant 0, \Delta u \leqslant 0) \text{ in } \Omega.$$

Then ifor any ball $B = B_R(y) \Subset \Omega$ (of radius R and center in y) it holds

$$u(y) = (\leqslant, \geqslant) \frac{1}{n\omega_n R^{n-1}} \int_{\partial B} u \,\mathrm{d}\, S \tag{4.2}$$

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$$u(y) = (\leqslant, \geqslant) \frac{1}{\omega_n R^n} \int_B u \,\mathrm{d}\, x, \tag{4.3}$$

where $\omega_n = \frac{2\pi^{n/2}}{n\Gamma(n/2)}$ is the volume of the unit ball in $\mathbf{R}^n$.

Proof. Let $\rho \in (0, R)$. From (4.1) for $B_\rho = B_\rho(y)$, we have

$$\int_{\partial B_\rho} \frac{\partial u}{\partial \nu} \,\mathrm{d}\, S = \int_{B_\rho} \Delta u \,\mathrm{d}\, x = (\geqslant, \leqslant) 0.$$

Putting $r = |x-y|$, $\omega = \frac{x-y}{r}$, $u(x) = u(y + rw)$, we get

$$\begin{aligned}
\int_{\partial B_\rho} \frac{\partial u}{\partial \nu} \,\mathrm{d}\, S &= \int_{\partial B_\rho} \frac{\partial u}{\partial \nu} = \rho^{n-1} \int_{|\omega|=1} \frac{\partial u}{\partial \nu}(y + \rho\omega) \,\mathrm{d}\, \omega \\
&= \rho^{n-1} \int_{|\omega|=1} \frac{\mathrm{d}\, u}{\mathrm{d}\, \rho}(y + \rho\omega) \,\mathrm{d}\, \omega = \rho^{n-1} \frac{\mathrm{d}}{\mathrm{d}\, \rho} \int_{|\omega|=1} u(y + \rho\omega) \,\mathrm{d}\, \omega \\
&=: \rho^{n-1} \frac{\mathrm{d}}{\mathrm{d}\, \rho} g(\rho) = (\geqslant, \leqslant) 0.
\end{aligned}$$

That implies

$$g(\rho) = \rho^{1-n} \int_{\partial B_\rho} u \,\mathrm{d}\, S = (\leqslant, \geqslant) g(R) + R^{1-n} \int_{\partial B_R} u \,\mathrm{d}\, S.$$

for every $\rho < R$. Since

$$g(0) = \lim_{\rho \to 0} \rho^{1-n} \int_{\partial B_\rho} u \,\mathrm{d}\, S = n\omega_n u(y),$$

we get (4.2). Relation (4.3) can be obtained as follows.

$$n\omega_n \rho^{n-1} u(y) = (\leqslant, \geqslant) \int_{\partial B_\rho} u \,\mathrm{d}\, S, \rho \leqslant R$$

summing this term as $\rho \in [p, R]$ □

4.2 Maximum Principle

Theorem 9. *Let $\Delta u \geqslant 0$ (or $\Delta u \leqslant 0$) in Ω. Suppose that there exists a point $y \in \Omega$ such that*

$$u(y) = \sup_\Omega u(\text{ or } u(y) = \inf_\Omega u).$$

Then $u \equiv \mathrm{const}$. Specially, non-constant harmonic function does not have either minimum not maximum in the interior of Ω.

Proof. Let $\Delta u \geqslant 0$ in Ω and $M = \sup_\Omega u$. Define

$$\Omega_M = \{x \in \Omega : u(x) = M\}.$$

Using the hypothesis, $\Omega_M \neq \emptyset$, and since u is continuous Ω_M is a closed set (inverse of the closed set $\{M\} \subset \mathbf{R}$).

Let z be an arbitrary point in Ω_M. Using (4.3) in the ball $B = B_R(z) \Subset \Omega$ for the function $u - M$ (that is also subharmonic: $\Delta(u - M) \geqslant 0$),

$$0 = u(z) - M \leqslant \frac{1}{\omega_n R^n} \int_B (u - M) \, \mathrm{d}\, x \leqslant 0.$$

That implies $u - M \equiv 0$ in B. So, $B \subset \Omega_M$ is a neighbourhood of the point z, and Ω_M is an open set then. Since Ω is a connected set, we have $\Omega = \Omega_M$, i.e. $u \equiv M$ in Ω.

A proof for superharmonic case follows after the change of the dependent variable $u \mapsto -u$. □

Immediately we have the following

Theorem 10. *Let Ω be bounded, open, and connected set, $u \in C^2(\Omega) \cap C(\bar{\Omega})$ and $\Delta u \geqslant 0$ (or $\Delta u \leqslant 0$) in Ω. Then*

$$\sup_\Omega u = \sup_{\partial\Omega} u(\text{ or } \inf_\Omega u = \inf_{\partial\Omega} u).$$

Specially, for harmonic u,

$$\inf_{\partial\Omega} u \leqslant u(x) \leqslant \sup_{\partial\Omega} u, x \in \Omega.$$

And finally,

Theorem 11. *a) Let $u, v \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfy*

$$\Delta u = \Delta v \text{ in } \Omega, u = v \text{ at } \partial\Omega.$$

Then $u = v$ in Ω.

b) If u and v are harmonic and subharmonic, resp., and $u = v$ at $\partial\Omega$, then $v \leqslant u$ in Ω.

Proof. a) Put $w = u - v$. Then

$$\Delta w = 0 \text{ in } \Omega, w = 0 \text{ at } \partial\Omega.$$

By the maximum principle, $w = 0$ in Ω.

b) Directly from the same theorem. □

The following corollary od the maximum principle we will leave without a proof.

Theorem 12. *(***Harnack inequality***) Let u be non-negative harmonic function in Ω. Then for any open, connected $\Omega' \Subset \Omega \subset \mathbf{R}^n$ there exists a constant $C = C(n, \Omega', \Omega)$ such that*

$$\sup_{\Omega'} u \leqslant C \inf_{\Omega'} u.$$