Non-Standard Forms of Teaching Mathematics and Physics: Experimental and Modeling Approach

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Calculus with dynamic geometry

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Continuation to the

Takači, Đ., Takači, A., Takači A., Experiments in Calculus with Geogebra

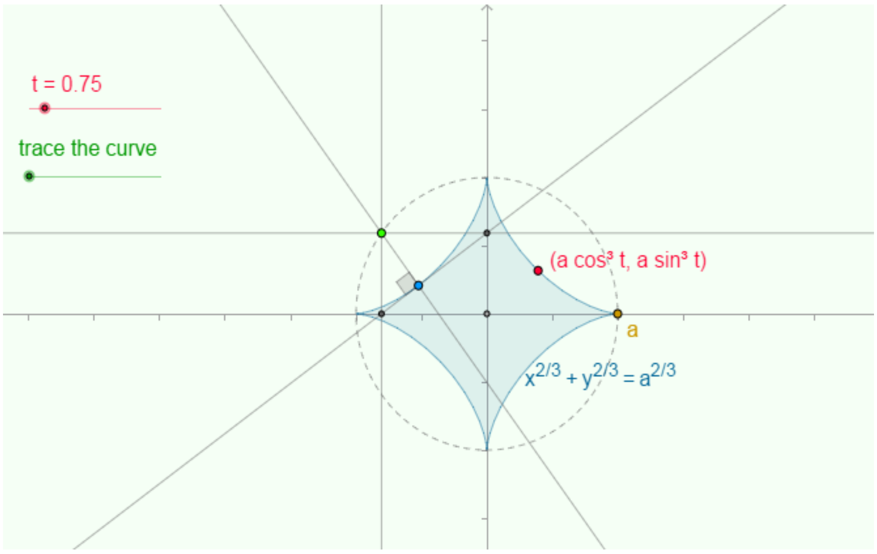

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TABLE OF CONTENTS

Introduction 3
1.Functions 4
1.1 Basic Notion 4
1.2 Definition of Functions 4
1.3 Extreme 5
1.4 Polynomials 6
1.5 Rational Functions 6
1.6 Curves given in parametric forms 7
1.7 Curves given in polar Coordinates 8
2. Limits and Continuity 9
2.1 Sequences 9
2.2 Continuous function 10
2.3 Limits of the function........13
3. Derivative of the function 13
3.1 Tangent line 14
3.2 Differential of the function 15
4. Integral 16
4.1 Area problem 16
References: 16

Introduction

The calculus contents based on dynamic properties of the GeoGebra packages is presented. This teaching material can be used for both by complete beginners, as well as by students that already has gone through calculus course. This teaching material represent the modernization of the following course at University of Novi Sad:

Calculus for physics, chemistry, mathematic, informatics, pharmacy student.

This book is continuation for the book [7] and both are the additions for the book [4]. In this book all pictures are presenred in GeoGebra, but they can be opened in GeoGebra tube. All definitions, and exercises can be found in this book on Serbian language.

1. Functions

1.1 Basic Notions

Definition of Functions

Let A and B be two nonempty sets. By definition, a relation f from A into B is a subset of the direct product $A \times B.$

A relation f is a function which maps the set A into the set B if the following two conditions hold:

- *for every $x \in A$ there exists an element $y \in B$ such that the pair (x, y) is in f;*
- *if the pairs (x, y_1) and (x, y_2) are in f, then necessarily $y_1 = y_2$.*

EXAMPLE 1

Determine the domain D_f for the following functions:

$$a)\ f(x) = (ax^b + c)^d, \quad b)\ g(x) = \log_d(ax^b + c), \quad c)\quad h(x) = da^{x^b + c},$$

for different values a, b, c, d

Solution: The graphs can be drawn by using corresponding sliders.

Domain

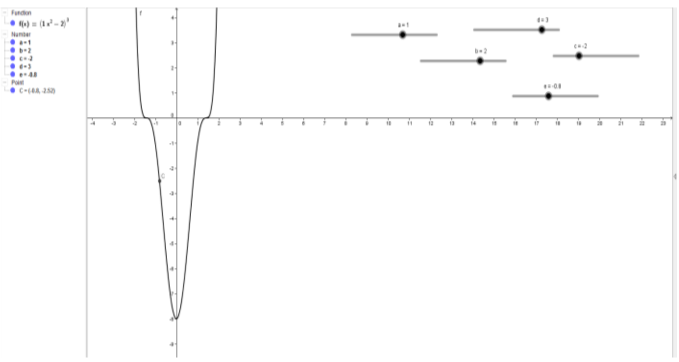

Figure 1.1

You can choose the values for a, b, c, d and discussed the corresponding domains.

Odd and Even Function

Let us suppose that the domain A of a function $f : A \rightarrow B,$ is *symmetric*. Then f is an

- even function, if for every $x \in A$ it holds $f(-x) = f(x);$
- odd function, if for every $x \in A$ it holds $f(-x) = -f(x).$

Geometrically, the graph of an even function is symmetric to the $y-$ axis, while the graph of an odd function is symmetric to the origin.

EXAMPLE 2

Determine 3 sets of parameters in order to make the following functions

$$a)\ f(x)=(ax^b+c)^d,\quad b)\ g(x)=\log_d(ax^b+c),\quad c)\quad h(x)=da^{x^b+c},\quad d)\quad p(x)=a\sin(bx)+c\cos(dx)$$

odd and even. For example: Even function, Odd Function, Symetry, Exercise, Exercise2.

Extreme

A function $f: A \to B$ is monotonically increasing (resp. monotonically decreasing) on the set $X \subset A$ if for every pair of elements x_1 and x_2 from the set X it holds

$$x_1<x_2 \Rightarrow f(x_1)<f(x_2) \quad x_1<x_2 \Rightarrow f(x_1)>f(x_2)).$$

A function $f: A \to B$ has a local maximum (resp. local minimum**)** in the point $x_0 \in A$ if there exists a number $\varepsilon > 0$ such that

$$(\forall x \in (x_0-\varepsilon, x_0+\varepsilon) \cap A) \quad f(x) \le f(x_0)\ (f(x) \ge f(x_0)).$$

A function $f: A \to B$ has a global maximum (resp. global minimum) in the point $x_0 \in A$ if

$$(\forall x \in A)\ \ f(x) \le f(x_0)\ \ (\ \ f(x) \ge f(x_0)).$$

EXAMPLE 3

Determine 3 sets of parameters in order to make the following functions

$$a)\ f(x)=(ax^b+c)^d,\quad b)\ g(x)=\log_d(ax^b+c),\quad c)\quad h(x)=da^{x^b+c},\quad d)\quad p(x)=a\sin(bx)+c\cos(dx)$$

have exstreme values.

See Extreme values, Extreme values2

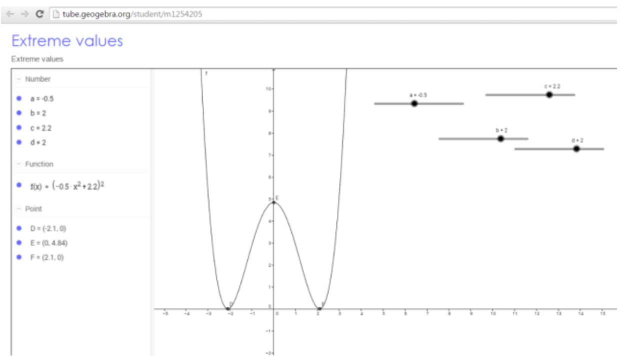

Figure 1.2

Monotonicity of linear function can be followed by link Monotonocity-Lin.

Also, monotonicity can be followed from: Defin, ExampleMon.

Periodic Function

A number $\tau \neq 0$ is called the period of the function $f : A \to B$ if for all $x \in A$ the points $x+\tau$ and $x-\tau$ are also in A and it holds

$$(\forall x \in A)\ \ f(x+\tau) = f(x).$$

The smallest positive period, if it exists, is called the basic period of the function f. Clearly, if we know the basic period T of a function, then it is enough to draw its graph on any set $X \subset A$ of the length T.

EXAMPLE 4

Can you determine 3 sets of parameters in order to make the following functions

$a)\ f(x) = (ax^b + c)^d, \quad b)\ g(x) = \log_d(ax^b + c), \quad c)\ \ h(x) = da^{x^b+c}, \quad d)\ \ p(x) = a\sin(bx) + c\cos(dx)$

periodic. See also: Example-sin, Example1, Example2.

1.2 Polynomials

The function

$$P_n(x) = \ a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0, x \in \mathbf{R}, \ \ (\ \ x \in \mathbf{C}),$$

where the coefficients $a_j,\ j = 0,1,\ldots,n,$ are real numbers, is called polynomial of degree $n \in \mathbf{N},$ if $a_n \neq 0.$

By definition, the constant function is a polynomial of degree zero.

EXAMPLE 5

Can you determine the sets of parameters in order to make the following functions

$r(x) = (ax^b + c)^d,$ as

a)	$f(x) = x^2,$	$g(x) = x^4,$	$h(x) = x^6;$
b)	$f(x) = -x^2,$	$g(x) = -x^4$	$h(x) = -x^6;$
c)	$f(x) = x,$	$g(x) = x^3,$	$h(x) = x^5;$
e)	$f(x) = -x,$	$g(x) = -x^3,$	$h(x) = -x^5.$

The sign of linear function cab visualized as The sgn of linFunc.

1.3 Rational Functions

The **rational function** is the quotient of functions

$$R(x) = \frac{P_n(x)}{Q_m(x)}, \qquad Q_m(x) \neq 0,$$

where $P_n(x)$ and $Q_m(x)$ are polynomials of degree n and m.

EXAMPLE 6

Can you determine the sets of parameters in order to make the following functions $r(x) = (ax^b + c)^d$, as

$$\text{a)}\quad f(x) = \tfrac{1}{x}, \quad g(x) = \tfrac{1}{x^3}, \quad h(x) = \tfrac{1}{x^5};$$
$$\text{b)}\quad k(x) = \tfrac{1}{x^2}, \quad l(x) = \tfrac{1}{x^4}, \quad m(x) = \tfrac{1}{x^6}.$$

The tests for students can be find on: Exercises.

1.4 Curves given in parametric forms

Cycloid: On Figure 1.3 the Cycloid, linked by Cycloid, $x = a(t - \sin t)$, $y = a(1 - \cos t)$, is drawn by using the point $A(a(t - \sin t), a(1 - \cos t))$ and two sliders a, t enabling their changes.

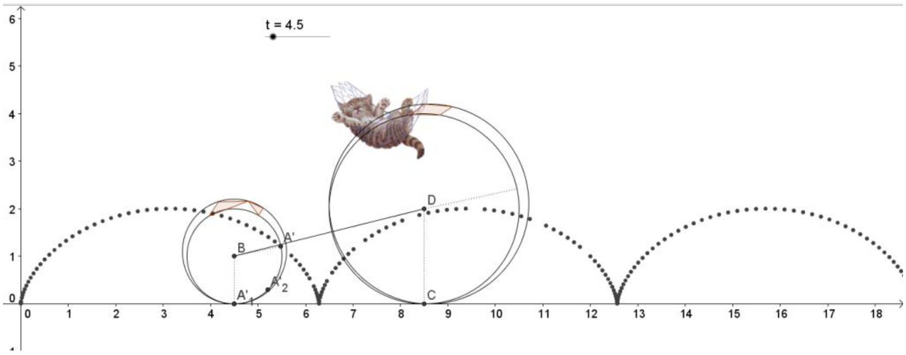

Figure 1.3

Astroid: On Figure 1.4 the Astroid, linked by Astroid , $x = a\cos^3 t$, $y = a\sin^3 t$, is drawn

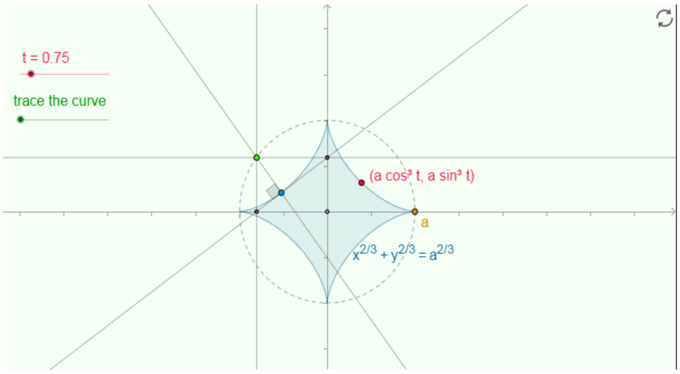

Figure 1.4

Descates curve: On Figure 1.5 the Decartes leaves, linked on DecLeav , $x = \dfrac{at}{1+t^3}$, $y = \dfrac{at^2}{1+t^3}$. is drawn

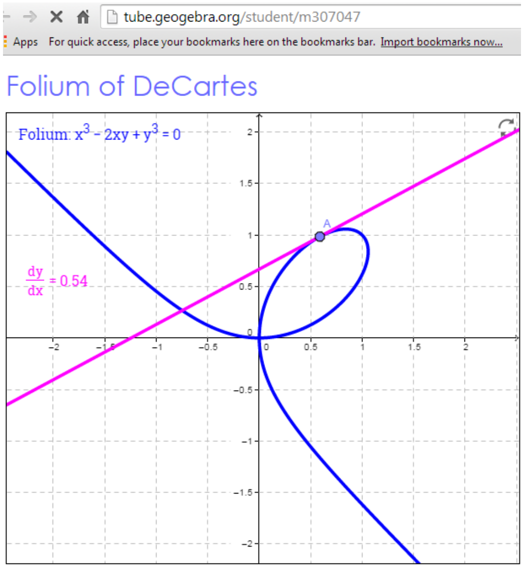

Figure 1.5

1.5 Curves given in polar Coordinates

On Figure 1.6 Lemniscata Bernoulli, linked on BernLemnis , $r = a^2 \cos(2t)$ is drawn.

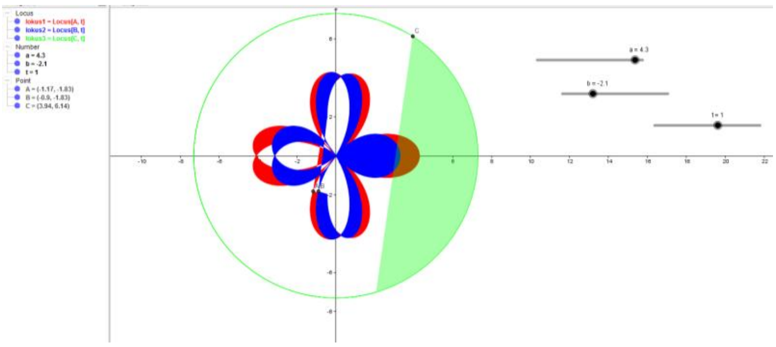

Figure 1.6

The graph of the curve given by polar coordinates can drawn by using Polar equation grapher.

On Figure 1.7 Cardioid : $r = a(1 + \cos t)$ linked on Cardioid, and Cardioid2, Cardioid3 is drawn.

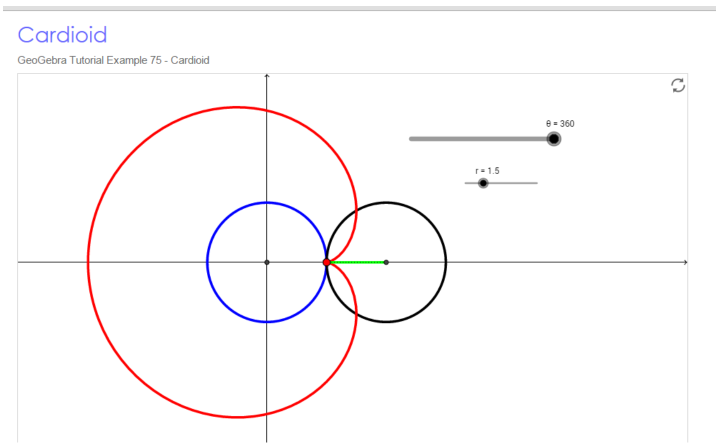

Figure 1.7

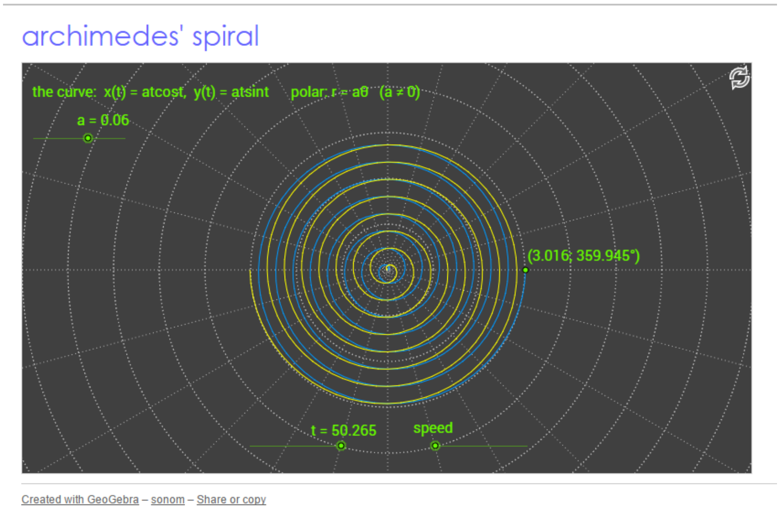

Figure 1.8

On Figure 1.8 Arhimed Spiral linked on, Spiral and AsBisectrce, Spiral2: $r = at$, , is drawn.

Fibnacci golden spiral is presented on Fibonacci spiral, Fibonacci spiral (Figure 1.9), and Fibonacci-GeoGeba, and Golden spiral

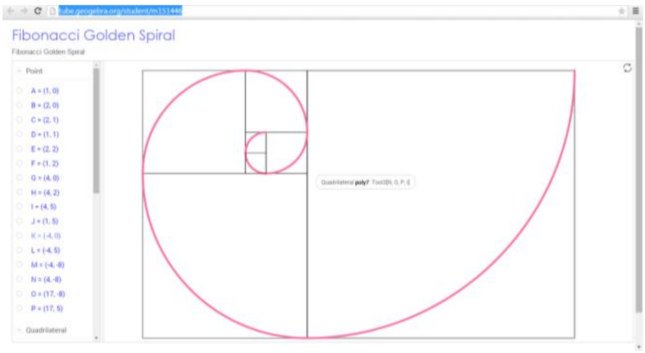

Figure 1.9

Also, we have Family rose curve, and Curves in polar coordinates,

2. Limits and Continuity

2.1 Sequences

A **sequence** is a function $a : \mathbf{N} \to \mathbf{R}$. It is usual to write

$$a_n := a(n),\ n \in \mathbf{N} \quad a = (a_n)_{n \in \mathbf{N}}.$$

In package *Geogebra* the sequences can be visualized by using sliders, and animations.

On Figure 2.1 (linked on Seq) we drew the graph of the sequence $a_n = \frac{1}{n}$, by using slider n, and the point $A(n, \frac{1}{n})$, with the trace on. In fact the point A has the $A(n, f(n))$, coordinates meaning that one can change the function f and the sequences is changed also.

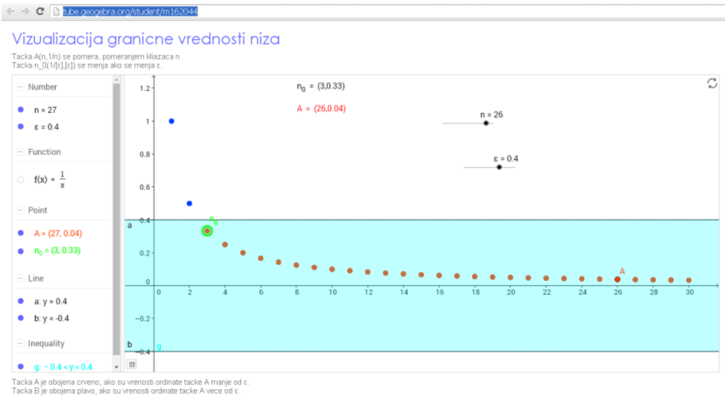

Figure 2.1

The examples of different sequences are given Example.

The definition of the limit of the sequence is also visualized on the picture 2.1 and link DefLim

Definition of limit of the sequence $\lim_{n\to\infty} f_n = L$ iff for every $\varepsilon > 0$, there exists a $n_0(\varepsilon)$, such that for every $n \in N$, $n > n_0$, it holds $|f_n - L| < \varepsilon$.

In DefLim the function can be changed and the corresponding points are obtained but $n_0(\varepsilon)$, has to determined. Be careful and do it.

2.2 Continuous function

Let us consider the functions

$$f(x) = \begin{cases} \dfrac{x^2-1}{x-1}, & x \neq 1, \\ 2, & x = 1 \end{cases} \qquad g(x) = \begin{cases} \dfrac{2\sin(x-1)}{(x-1)}, & x \neq 1, \\ 2, & x = 1 \end{cases} \qquad h(x) = \begin{cases} \dfrac{2e^{x-1}}{(x-1)}, & x \neq 1 \\ 2, & x = 1 \end{cases}$$

The links introCont IntroCont1 (Figure 2.2) are shown the graphs of the functions $f, g,$ and h on the intervals $(0.9999887, 1{,}0000189)$, and $(-.4, 5.5)$, respectively. The given functions are defined at the point $x = 1$. On The points A, C and D belong to the graphs of corresponding functions. The color of the points corresponds to the color of the graph. By moving the points A, C and D on the graphs and at a moment the will be coincidence with the point $B(1,2)$ belonging to each graph.

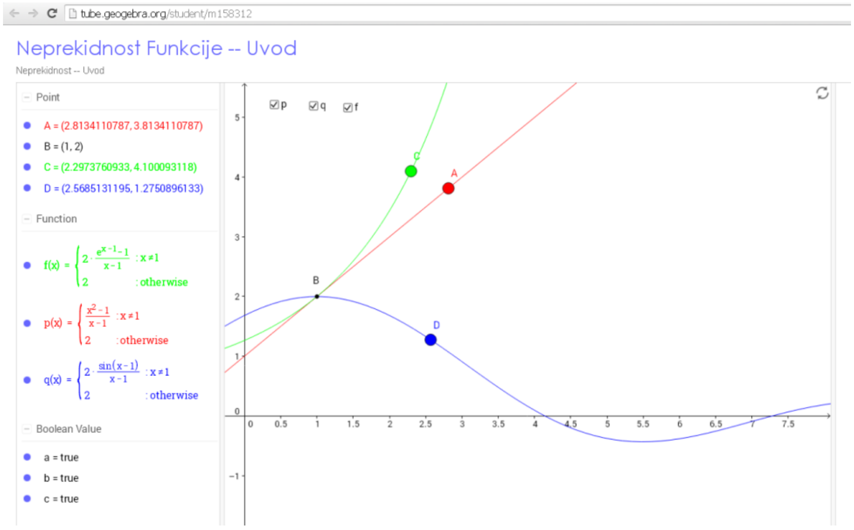

Figure 2.2

Definition of continuity (the visualization is shown on Figure 2.3, and linked on DefinitionCont). A function $f\ :\ A \subset \mathbf{R} \to \mathbf{R}.$ is continuous at a point $x_0 \in A$ iff for every $\varepsilon > 0,$ there exists a $\delta > 0,\ \delta = \delta(\varepsilon),$ such that for every $x \in A$ it holds:

$$0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \varepsilon.$$

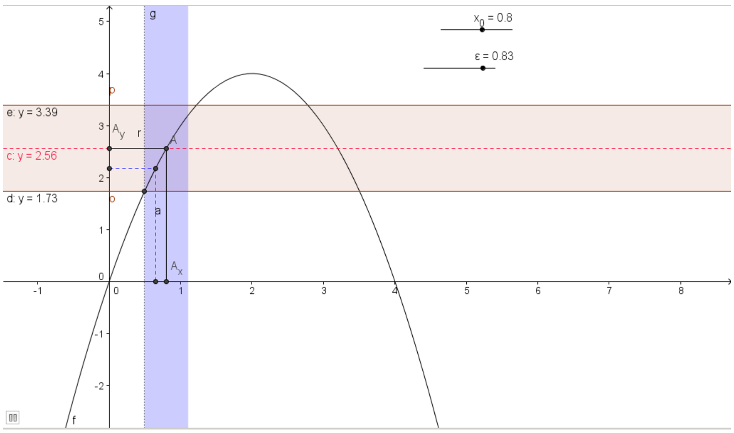

Figure 2.3

If the point $x_0 \in A$ is an accumulation point of the set $A,$ then the following two definitions can also be used:

Definition Heine (linked on Heine), (Figure 2.4). A function $f\ :\ A \subset \mathbf{R} \to \mathbf{R}$ is continuous at a point $x_0 \in A,$ where x_0 is an accumulation point of the domain $A,$ if for every sequence $(x_n)_{n\in\mathbf{N}}$ of elements from A it holds that

$$\lim_{n\to\infty} f(x_n) = f(x_0), \quad if \quad \lim_{n\to\infty} x_n = x_0.$$

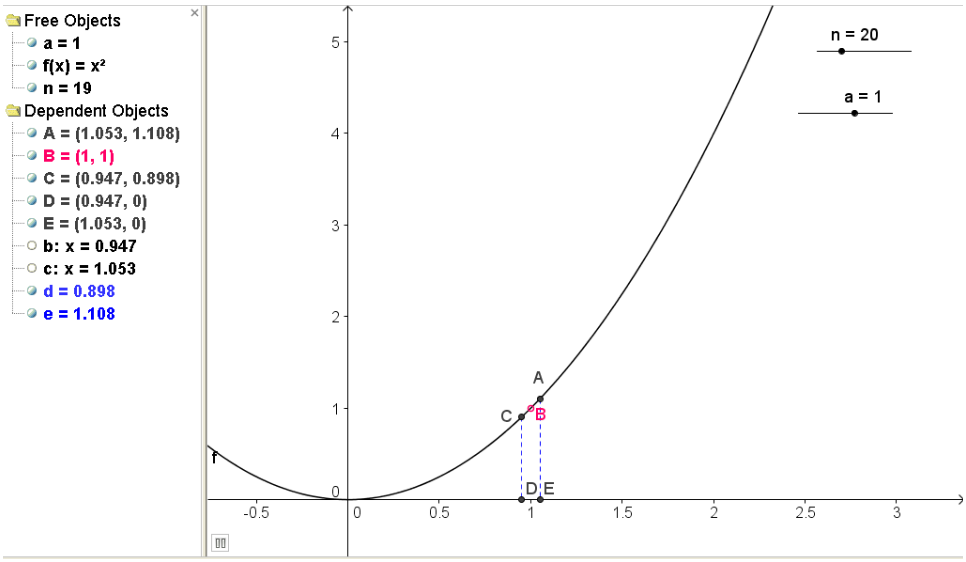

Figure 2.4

2.3 Limits of the function

The introduction to the limits of function can be done similarly Intr-Limits, (Figure 2.5). Where is the main difference?

The limits of the function $f(x)=\frac{x^2-1}{x-1}$ is analyzed at Limits.

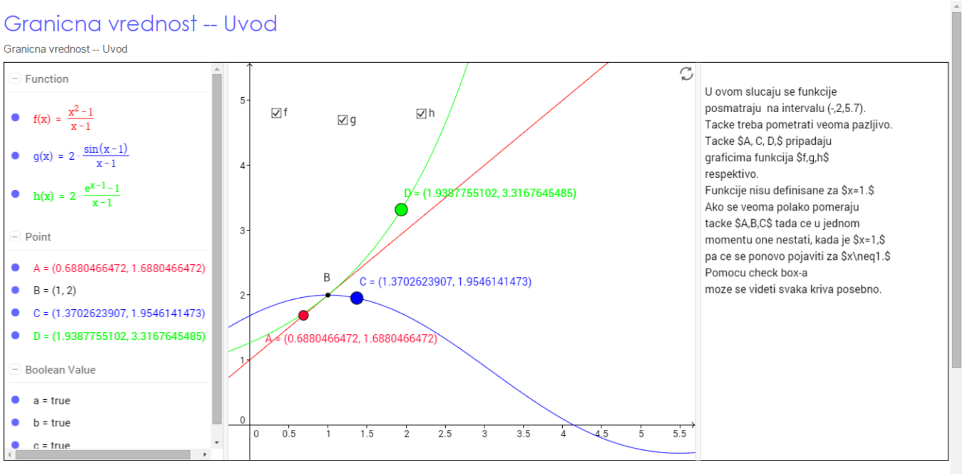

Figure 2.5

The limits of the functions can be found on the following links: Exercise1, Exercise2, Exercise3, Exercise4,

Left and limits of the functions can be found on the following links: ExerciseLD2, ExerciseLD3, ExerciseLD4, Exercise5,

3. Derivative of the function

Let f be a real function defined on an open interval (a,b) and let $x_0 \in (a,b)$. Then the following limit

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

(provided it exists) is called the first derivative of f at the point x_0.

The number h in is called the increment of the independent variable x at the point x_0, while the difference $f(x_0+h) - f(x_0)$ is called the increment of the dependent variable at the point x_0.

On Figure 3.1, linked on Definition Derivative , we consider the function $f(x) = x^2$, and the points $A(a, f(a))$, and $B\left(a, \dfrac{f(a+h)-g(a)}{h}\right)$, depending on a, and h, which can be changed with the sliders. If we fixed a, and change h, then we are so close to value of the first derivative at the point a.

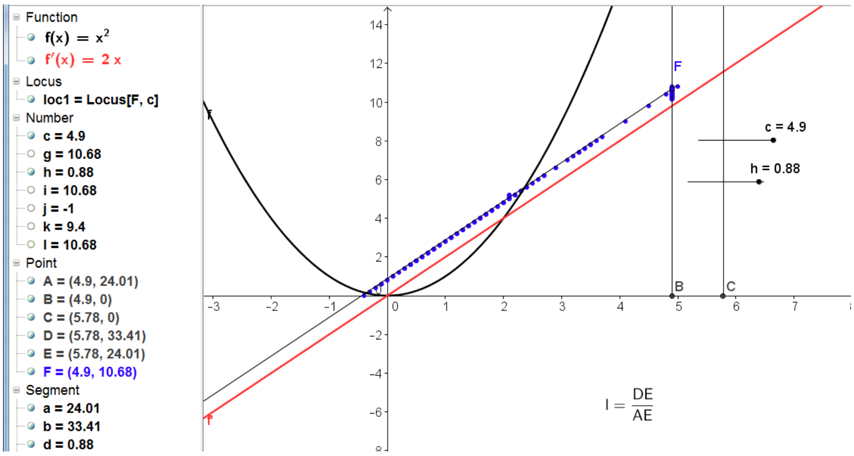

Figure 3.1

On Figure 3.2, linked on Derivative of the polynomial, we consider the function the polynomil $f(x) = (ax^b + c)^d$, $b, d \in N$, and the function $g(a,h) = \dfrac{f(x+h)-g(x)}{h}$ of two variable representing differential quotation, depending on h, and x. If we move h, with „animation on” , we obtain the lines as close to the red line, the graph of first derivative as h.

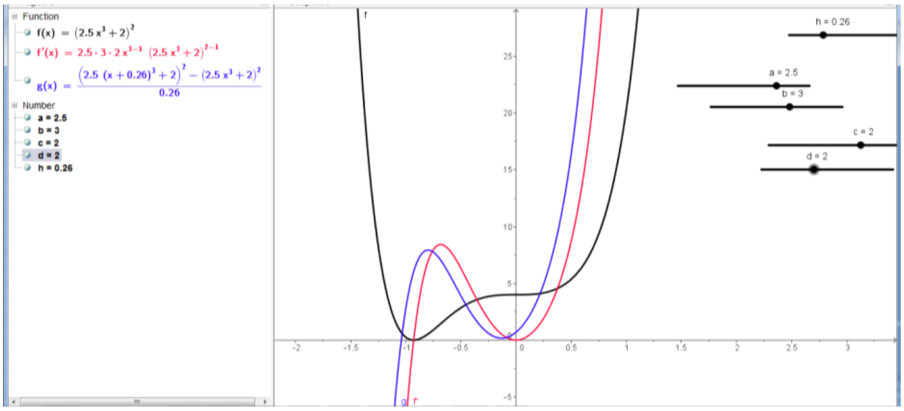

Figure 3.2

Tangent line

If a function $f\ :(a,b)\to\mathbf{R}$ has a first derivative at the point $x_0\in(a,b)$, then the line

$$y-y_0=f'(x_0)(x-x_0),$$

where $y_0=f(x_0)$, is the tangent line of the graph of the function f at the point $T(x_0,f(x_0))$.

If it holds $f'(x_0)\neq 0$, the line

$$y-y_0=-\frac{1}{f'(x_0)}(x-x_0)$$

is the perpendicular line of the graph of the function f at the point $T(x_0,f(x_0))$.

If a function f has a first derivative at a point x_0, and $0\leq\alpha<\pi$ is the angle between the tangent line at the point x_0 and the positive direction of the $x-$ axis, then it holds

$\tan\alpha=f'(x_0)$.

The slope of the tangent line of the graph f at some point is exactly the value of the first derivative of f at that point.

On Figure 3.3, linked on GeomDer we considered the points $A(x_0,f(x_0))$ and $B(x_0+h,f(x_0+h))$ on the graph of a function f. Then the slope of the secant line through A and B is equal to

$$k_s=\frac{f(x_0+h)-f(x_0)}{h},$$

while the slope of tangent line of f at the point A,

$$k_t = \lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}$$

is equal to the first derivative of f at x_0.

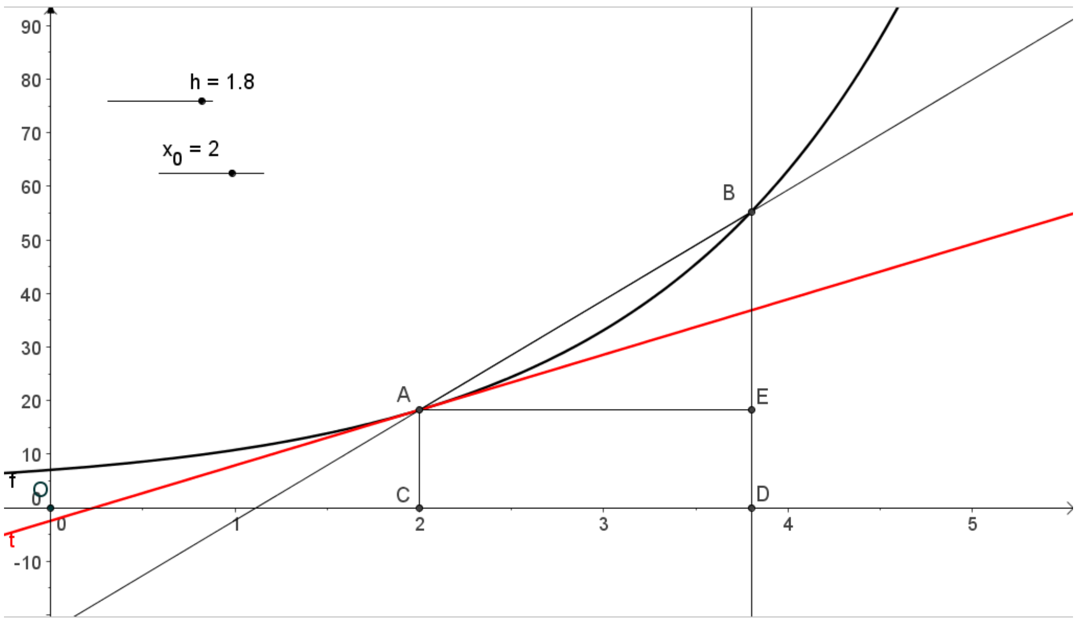

Figure 3.3

The similar visualization can be followed on the following links:

secant and tangent line, Vusualiztion of the derivative, From secand to tangent

Differential of the function

A function $f:(a,b)\to R$ is **differentiable at the point** x_0, if its increment Δy at the point $x_0 \in (a,b)$ can be written in the form

$$\Delta f = f(x_0+h)-f(x_0) = D\cdot h + r(h)\cdot h,$$

for some number D (independent from h), and it holds $\lim_{h\to 0} r(h) = 0$.

On Figure 3.4, linked on differential function $f:R\to R$, points $A(x_0, f(x_0))$ and $B(x_0+h, f(x_0+h))$ are consider with the sliders x_0, and h. The points $A_1(x_0,0)$ and $B_1((x_0+h),0)$ are the projections of A and B respectively onto the x-axis. Also, the point $G(x_0+h, f(x_0))$, and the point F, the intersection of the tangent line and vertical line parallel to $y-$axes through the point G. Since it holds

$$\tan\alpha = \frac{FG}{AG},\quad f'(x_0) = \frac{FG}{h}\text{, i.e., } FG = f'(x_0)h \text{ and } dy = f'(x_0)\,dx$$

it follows that FG is the geometric interpretation of the differential of the function f at the point A. In Figure 1 we took $x_0 = 0.9$, and $h = 1.6$.

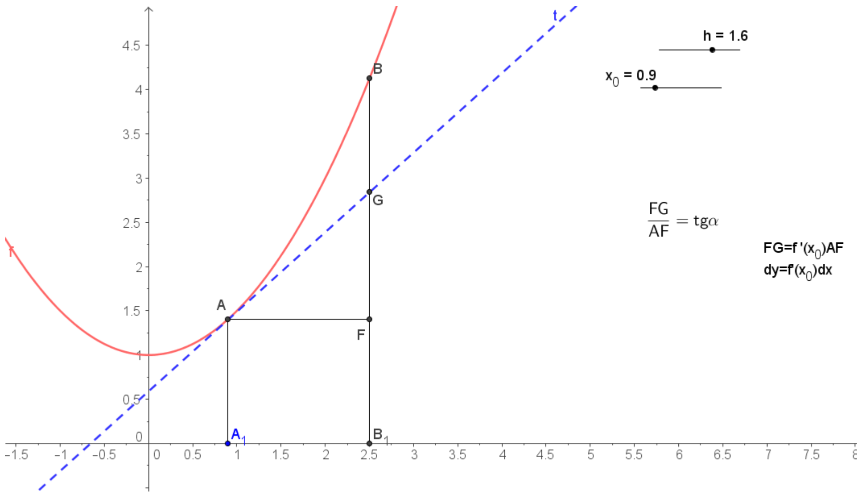

Figure 3.4

Chain rule for derivative is visualized as follows: Chain rule**.**

Interesting exercises can be found on the following links:

Exercise1, Exercise2, Exercise 3, Exercise4, Exercise5, Exercise6, Exercise7, Exercise8, Exercise9, Exercise10, exercise11,

4. INTEGRAL

Area problem

EXAMPLE 7

Let us consider the function $f(x) = x^2$. Determine the area between the graph of the function f, the interval $[0, a]$, $a > 0$, and the lines determined by lines $x = 0$, and $x = a$.

Solution: First we divide the interval $[0, a]$, $a > 0$, on n subintervals and calculate the sum of the area, of rectangular determined by the points

$$\left(\frac{a}{n}(i-1),0\right),\ \left(\frac{a}{n}(i-1),\ f(\frac{a}{n}(i-1))\right),\ \left(\frac{a}{n}i,\ f(\frac{a}{n}(i-1))\right),\ \left(\frac{a}{n}i,0\right),\quad i = 1,\dots,n$$

called lower sum and dented by P_L and the upper sum, P_U is calculated for the points

$$\left(\frac{a}{n}(i-1),0\right),\ \left(\frac{a}{n}(i-1),\ f(\frac{a}{n}i)\right),\ \left(\frac{a}{n}i,\ f(\frac{a}{n}i)\right),\ \left(\frac{a}{n}i,0\right),\qquad i = 1,\dots,n.$$

Using link Area it can be followed that the numbers P_L, P_U are closing to the number P, denoted the area are we asked for.

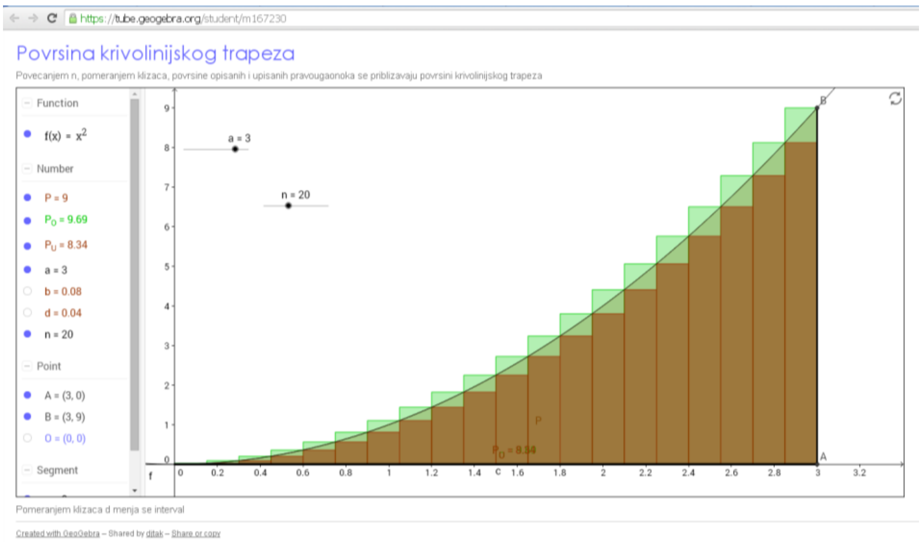

Figure 4.2

The following exercise can be followed Exercise1, Exercise2.

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