Calculus For the Life Sciences: A Modeling Approach

Volume II. Difference and Differential Equations

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Volume II is suitable for students who have studied calculus and are interested in difference and differential equation models in the life sciences.

Volume II consists of Chapter 1 from Volume I, Chapter D - the differential equations that appear in Chapters 3 - 10 of Volume I, and five additional chapters on difference and differential equations.

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Duane Nykamp of the University of Minnesota is writing **Math Insight, http://mathinsight.org**, a collection of web pages and applets designed to shed light on concepts underlying a few topics in mathematics. Some of his pages are based on our material and we point to those pages at critical points in our text.

A review of this text by Steven Deckelman and George Jennings appears in the Mathematical Association of America Digital Library at http://mathdl.maa.org/mathDL/19/?pa=reviews&sa=viewBook&bookId=73094

Contents

Chapter D

Dynamical Equations from Volume I

D.1 From Chapter 3: Mathematical models using the derivative.

The rate of change of a function provides a powerful new way of thinking about models of biological processes.

The changes in biological and physical properties that were measured in discrete packages ($P_{t+1} - P_t$, $A_{t+1} - A_t$, $I_{t+1} - I_t$) in Chapter 1 can be more accurately represented as instantaneous rates of change ($P'(t)$, $A'(t)$, $I'(t)$) using the derivative. The *V. natrigens* populations were measured at 16 minute intervals but were growing continuously. The kidneys filter continuously, not in 5-minute spurts. Here we begin a process of using the derivative to interpret mathematical models that continues through out the book.

D.1.1 Mold growth.

We wrote in Section **??** that the *daily* increase in the area of a mold colony is proportional to the circumference of the colony at the beginning of the day. Alternatively, we might say:

Mathematical Model D.1.1 Mold growth. The rate of increase in the area of the mold colony at time t is proportional to the circumference of the colony at time t.

Letting $A(t)$ be area and $C(t)$ be circumference of the mold colony at time t, we would write

$$A'(t) = k \times C(t)$$

Because $C(t) = 2\sqrt{\pi}\sqrt{A(t)}$ (assuming the colony is circular)

$$A'(t) = k \times 2\sqrt{\pi}\sqrt{A(t)} = K\sqrt{A(t)} \qquad \text{(D.1)}$$

where $K = k \times 2\sqrt{\pi}$.

Equation D.1, $A'(t) = K\sqrt{A(t)}$, is a statement about the function A. We recall also that the area of the colony on day 0 was 4 mm^2. Now we search for a function A such that

$$A(0) = 4 \qquad A'(t) = K\sqrt{A(t)} \qquad t \geq 0 \tag{D.2}$$

Warning: Incoming Lightning Bolt. Methodical ways to search for functions satisfying conditions such as Equations D.2 are described in Chapter 17. At this stage we only write that the function

$$A(t) = \left(\frac{K}{2}\,t + 2\right)^2 \qquad t \geq 0 \qquad \textbf{Bolt Out of Chapter 18.} \tag{D.3}$$

satisfies Equations D.2 and is the only such function. In Exercise D.1.1 you are asked to confirm that $A(t)$ of Equation D.3 is a solution to Equation D.2.

Finally, we use an additional data point, $A(8) = 266$ to find an estimate of K in $A(t) = \left(\frac{K}{2}\,t + 2\right)^2$.

$$\begin{aligned} A(8) &= \left(\frac{K}{2}\,8 + 2\right)^2 \\ 266 &= (4K + 2)^2 \\ K &\doteq 3.58 \end{aligned}$$

Therefore, $A(t) = (\,1.79t + 2)^2$ describes the area of the mold colony for times $0 \leq t \leq 9$. Furthermore, A has a quadratic expression as suggested in Section **??** based on a discrete model. A graph of the original data and A appears in Figure D.1.

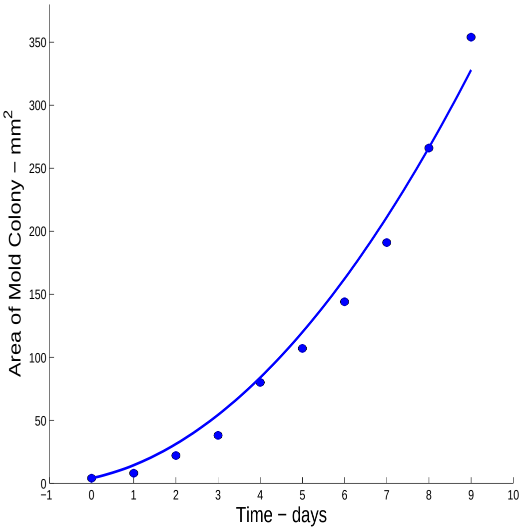

Figure D.1: Graph of $A(t) = (\,1.79t + 2\,)^2$ and the area of a mold colony from Table **??**

D.1.2 Chemical kintetics.

Chemists use the *rate of change in the amount of product* from a chemical reaction as a measure of the *reaction rate.* You will compute some rates of chemical reactions from discrete data of chemical concentration *vs* time.

Chemical reactions in which one combination of chemicals changes to another are fundamental to the study of chemistry. It is important to know how rapidly the reactions occur, and to know what factors affect the rate of reaction. A reaction may occur rapidly as in an explosive mixture of chemicals or slowly, as when iron oxidizes on cars in a junk yard. Temperature and concentration of reactants often affect the reaction rate; other chemicals called catalysts may increase reaction rates; in many biological processes, there are enzymes that regulate the rate of a reaction. Consider

$$\mathrm{A} \longrightarrow \mathrm{B}$$

to represent (part of) a reaction in which a reactant, A, changes to a product, B. The rate of the reaction may be measured as the rate of disappearance of A or the rate of appearance of B.

Butyl chloride.

When butyl chloride, C_4H_9Cl, is placed in water, the products are butyl alcohol and hydrochloric acid. The reaction is

$$\mathrm{C_4H_9Cl + H_2O \longrightarrow C_4H_9OH + H^+ + Cl^-}$$

As it takes one molecule of C_4H_9Cl to produce one atom of Cl^-, the rate at which butyl chloride disappears is the same as the rate at which hydrochloric acids appears. The presence of Cl^- may be measured by the conductivity of the solution. Two students measured the conductivity of a solution after butyl chloride was added to water, and obtained the results shown in Figure D.2. The conductivity probe was calibrated with 8.56 mmol NaCl, and conductivity in the butyl chloride experiment was converted to mmol Cl^-. The experiment began with butyl chloride being added to water to yield 9.6 mmol butyl chloride.

The average rate of change over the time interval [30,40]

$$m_{30,40} = \frac{1.089 - 0.762}{40 - 30} = 0.0327,$$

and the average rate of change over the time interval [40,50]

$$m_{40,50} = \frac{1.413 - 1.089}{50 - 40} = 0.0324.$$

both approximate the reaction rate. A better estimate is the average of these two numbers, $\frac{0.0327+0.0324}{2} = 0.03255$. We only use the average when the backward and forward time increments, -10 seconds and +10 seconds, are of the same magnitude. The average can be computed without computing either of the backward or forward average rates, as

$$m_{30,50} = \frac{1.413 - 0.762}{50 - 30} = 0.03255$$

Time sec	Cl^- mmol	Time sec	Cl^- mmol
0	0.014	80	2.267
10	0.188	90	2.486
20	0.456	100	2.683
30	0.762	110	2.861
40	1.089	120	3.010
50	1.413	130	3.144
60	1.728	140	3.255
70	2.015	150	3.355

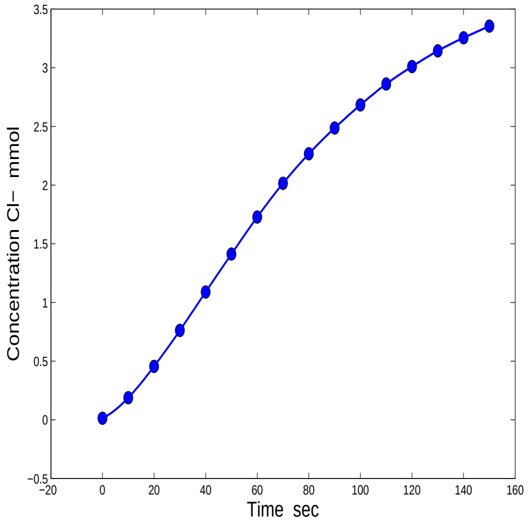

Figure D.2: Conductivity of a water and butyl chloride solution at times after butyl chloride was added to water.

In the case of $t = 150$, knowledge of the forward time increment is not available, and we use the backward time increment only.

$$m_{140,150} = \frac{3.355 - 3.255}{150 - 140} = 0.01$$

Explore D.1.1 Estimate the reaction rate at time $t = 80$ seconds. ■

Example D.1.1 It is useful to plot the reaction rate *vs* the concentration of Cl^- as shown in Figure D.3. The computed reaction rates for times $t = 0,\ 10,\ \cdots\ 40$ are less than we expected. At these times, the butyl chloride concentrations are highest and we expect the reaction rates to also be highest. Indeed, we expect the rate of the reaction to be proportional to the butyl chloride concentration. If so then the relation between reaction rate and Cl^- concentration should be linear, as in the parts corresponding to times $t = 60,\ 70,\ \cdots\ 160$ s. The line in Figure D.3 has equation $y = 0.0524 - 0.0127x$. We can not explain the low rate of appearance of Cl^- at this time.

The reaction is not quite so simple as represented, for if butyl alcohol is placed in hydrochloric acid, butylchloride and water are produced. You may see from the data that the molarity of Cl^- is tapering off and indeed later measurements showed a maximum Cl^- concentration of 3.9 mmol. If all of the butyl chloride decomposed, the maximum Cl^- concentration would be 9.6, the same as the initial concentration of butyl chloride. There is a reverse reaction and the total reaction may be represented

$$\mathrm{C_4H_9Cl(aq) + H_2O(l)} \quad \underset{\mathrm{k_2}}{\overset{\mathrm{k_1}}{\rightleftarrows}} \quad \mathrm{C_4H_9OH(aq) + H^+ + Cl^-(aq)}$$

The numbers, k_1 and k_2, are called rate constants of the reaction. The number k_1 is the negative of the slope of the line computed in Example D.1.1 (that is, 0.0127).

Time sec	Cl^- mmol	Reaction Rate mmol/sec	Time sec	Cl^- mmol	Reaction Rate mmol/sec
0	0.014	0.0175	80	2.267	0.0235
10	0.188	0.0221	90	2.486	0.0208
20	0.456	0.0287	100	2.683	0.0188
30	0.762	0.0317	110	2.861	0.0164
40	1.089	0.0326	120	3.010	0.0141
50	1.413	0.0319	130	3.144	0.0122
60	1.728	0.0301	140	3.255	0.0105
70	2.015	0.0270	150	3.355	0.010

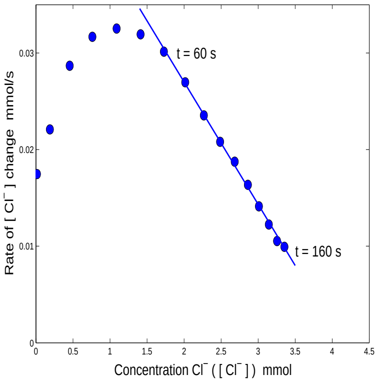

Figure D.3: Rate at which CL^- accumulates as a function of CL^- concentration after butyl chloride is added to water.

Exercises for Section D.1 Mathematical models using the derivative.

Exercise D.1.1 Show that $A(t)$ of Equation D.3,

$$A(t) = \left(\frac{K}{2}t + 2\right)^2 \qquad t \geq 0,$$

satisfies Equation D.2,

$$A(0) = 4 \qquad A'(t) = K\sqrt{A(t)} \qquad t \geq 0$$

You will need to compute $A'(t)$ and to do so expand

$$A(t) = \left(\frac{K}{2}t + 2\right)^2 \quad \text{to} \quad A(t) = \frac{K^2}{4}t^2 + Kt + 4$$

and show that

$$A'(t) = K\left(\frac{K}{2}t + 2\right) = K\sqrt{A(t)}.$$

Exercise D.1.2 Write a derivative equation that describes the following model of mold growth.

> **Mathematical Model.** A mold colony is growing in a circular pattern. The radius of the colony is increasing at a constant rate.

Exercise D.1.3 Write a derivative equation that describes the following model of light depletion below the surface of a lake.

> **Mathematical Model.** The rate at which light intensity decreases at any depth is proportional to the intensity at that depth.

Exercise D.1.4 Write a derivative equation that describes the following model of penicillin clearance.

> **Mathematical Model.** The rate at which the kidneys remove penicillin is proportional to the concentration of penicillin.

Exercise D.1.5 Data from Michael Blaber of Florida State University College of Medicine[1] for the butyl chloride experiment

$$C_4H_9Cl(aq) + H_2O(l) \rightarrow C_4H_9OH(aq) + HCl(aq)$$

are shown in Table D.1.5. These are more nearly what one would expect from this experiment.

a. Graph the data.

b. Estimate the rate of change of the concentration of C_4H_9Cl for each of the times shown.

c. Draw a graph of the rate of reaction versus concentration of C_4H_9Cl.

Table for Exercise D.1.5 Data for Ex. D.1.5.

Time (sec)	0	50	100	150	200	300	400	500	800
C[$_4H_9Cl$] (M)	0.1	0.0905	0.0820	0.0741	0.0671	0.0549	0.0448	0.0368	0.0200

Exercise D.1.6 Data from Purdue University[2] for the decrease of the titration marker phenolphthalein (Hln) in the presence of excess base are shown in Table D.1.6. The data show the concentration of phenolphthalein that was initially at 0.005 M in a solution with 0.61 M OH^- ion.

a. Graph the data.

b. Estimate the rate of change of the concentration of phenolphthalein (Hln) for each of the times shown.

c. Draw a graph of the rate of reaction versus concentration of phenolphthalein.

Table for Exercise D.1.6 Data for Ex. D.1.6.

Time (sec)	0	10.5	22.3	35.7	51.1	69.3	91.6	120.4	160.9
Hln (M)	0.005	0.0045	0.0040	0.0035	0.0030	0.0025	0.0020	0.0015	0.0010

Exercise D.1.7 Data from Michael J. Mombourquette, Queens University, Kingston, Ontario, Canada[3] for the decrease of CO in the reaction

$$CO(g) + NO_2(g) \rightarrow CO_2(g) + NO\ (g)$$

are shown in Table D.1.7A. and the decrease of N_2O_5 in the reaction

$$2N_2O_5(g) \rightarrow 4NO_2(g) + O_2$$

are shown in Table D.1.7B. Initially, 0.1 g/l of CO was mixed with 0.1 g/l of NO_2.

For each table,

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[1] http://wine1.sb.fsu.edu/chem1046/notes/Kinetics/Rxnrates.htm

[2] http://chemed.chem.purdue.edu/genchem/topicreview/bp/ch22/rate.html

[3] http://www.chem.queensu.ca/people/faculty/mombourquette/FirstYrChem/kinetics/index.htm

a. Graph the data.

b. Estimate the rate of change of the concentration of CO or of N_2O_5 for each of the times shown.

c. Draw a graph of the rate of reaction versus concentration of reactant.

Table for Exercise D.1.7 Data for Ex. D.1.7.

A.

Time (sec)	[CO] g/l
0	0.100
10	0.067
20	0.050
30	0.040

B.

Time (sec)	$[N_2O_5]$
0	0.0172
10	0.0113
20	0.0084
30	0.0062
40	0.0046
50	0.0035
60	0.0026

D.2 From Chapter 5: The derivative equation $P'(t) = k \times P(t)$

A crucial property of exponential functions established by the e^{kt} Rule is

Property D.2.1 Proportional Growth or Decay. If P is a function defined by

$$P(t) = Ce^{kt}$$

where C and k are numbers, then

$$P'(t) = k \times P(t)$$

Proof that $P(t) = Ce^{kt}$ implies that $P'(t) = kP(t)$:

$$\begin{aligned} P'(t) &= \left[Ce^{kt}\right]' \\ &= C\left[e^{kt}\right]' \\ &= Ce^{kt}k = k \times Ce^{kt} \\ &= k \times P(t) \end{aligned}$$

The reverse implication is also true, and is shown to be true in Chapter 18:

Property D.2.2 Exponential Growth or Decay
If P is a function and there is a number k for which

$$P'(t) = k \times P(t) \qquad \text{for all } t \geq 0$$

then there is a number C for which

$$P(t) = Ce^{kt}$$

Furthermore,

$$C = P(0) \qquad \text{so that} \qquad P(t) = P(0)e^{kt}$$

In the preceding equations, k can be either positive or negative. When k is negative, it is more common to emphasize this and write $-k$ and write $P(t) = e^{-kt}$, where in this context it is understood that k is a positive number.

In Chapter 1, we examined models of population growth, light decay, and penicillin clearance, all of which were of the form

$$P_{t+1} - P_t = R \times P_t$$

and found that

$$P_t = P_0 \times R^t$$

These are *discrete time* models in which the *average rate of change* of P_t is proportional to P_t. The exponential Growth or Decay Property D.2.2 is simply a *continuous time* model in which the *rate of change* of $P(t)$ is proportional to $P(t)$, and would be preferred in many instances. Bacterial populations may be visualized as growing continuously (and not in twenty minute bursts), the kidneys clear penicillin continuously (and not in five minute increments), and light decays continuously with depth (and not in one meter increments). Discrete time models are easy to comprehend and with short data intervals give good replications of data, but now that we know the definition of rate of change we can use continuous time or space models.

The equation

$$P' = kP \qquad \text{or} \qquad P'(t) = kP(t) \qquad \text{or} \qquad \frac{dP}{dt} = kP$$

derives from many models of biological and physical processes including population growth, drug clearance, chemical reaction, decay of radio activity – any system which can be described by:

Mathematical Model D.2.1 Proportional change. The rate of change of a quantity is proportional to the amount of the quantity.

For example in population studies, we commonly assume that

Mathematical Model D.2.2 Simple population growth. The growth rate of a population is proportional to the size of the population.

Let $P(t)$ be the size of a population at time, t. The component parts of the sentence in the Mathematical Model of simple population growth are symbolized by

a: The growth rate of a population : $P'(t)$

b: is proportional to : $= k\times$

c: the size of the population : $P(t)$

The sentence of the Mathematical Model is then written

$$\underbrace{P'(t)}_{a} \quad \underbrace{= k\times}_{b} \quad \underbrace{P(t)}_{c}$$

From the property of Exponential Growth and Decay

$$P(t) = C \times e^{kt} \qquad \text{and if } P(0) = P_0 \text{ is known} \qquad P(t) = P_0 e^{kt} \tag{D.4}$$

In the event that the rate of **decrease** of a quantity, $P(t)$, is proportional to the size of $P(t)$, then because $-P'(t)$ is the rate of decrease of $P(t)$,

$$-P' = k \times P(t), \qquad P' = -k \times P(t), \qquad \text{and} \qquad P(t) = P_0 e^{-kt},$$

where k is a positive number.

Example D.2.1 A distinction between discrete and continuous models. Suppose in year 2000 a population is at 5 million people and the population growth rate (excess of births over deaths) is 6 percent per year. One interpretation of this is to let $P(t)$ be the population size in millions of people at time t measured in years after 2000 and to write

$$P(0) = 5 \qquad P'(t) = 0.06 \times P(t)$$

Then, from the property of Exponential Growth or Decay D.2.2, we may write

$$P(t) = P(0)e^{0.06t} = 5e^{0.06t}$$

$P(t) = 5e^{0.06t}$ does not exactly match the hypothesis that 'population growth rate is 6 percent per year', however. By this equation, after one year,

$$P(1) = 5e^{0.06\times 1} = 5e^{0.06} \doteq 5 \times 1.0618$$

The consequence is that during the first year (and every year) there would be a 6.18 percent increase, a contradiction.

The discrepancy lies with the model equation $P'(t) = 0.6P(t)$. Instead, we may write

$$P(0) = 5 \qquad P'(t) = k \times P(t)$$

where k is to be determined. Then from Exponential Growth or Decay D.2.2 we may write

$$P(t) = 5e^{kt}$$

Now impose that $P(1) = 5 \times 1.06$, a 6 percent increase during the first year, and write

$$P(1) = 5e^{k\times 1} = 5 \times 1.06$$

This leads to

$$e^{k\times 1} = 1.06$$

We take the natural logarithm of both numbers and get

$$\begin{aligned} \ln\left(e^{k}\right) &= \ln 1.06 \\ k &= \ln 1.06 \\ &\doteq 0.05827 \end{aligned}$$

Then

$$P(t) = 5e^{0.05827t}$$

gives a description of the population t years after 2000. Each *annual* population is 6 percent greater than that of the preceding year. The continuous model of growth is actually

$$P(0) = 5 \qquad P'(t) = 0.05827P(t)$$

■

The growth of a bank savings account is similar to this simplified model of population growth. If you deposited \$5000 in 2000 at a true 6 percent annual interest rate, it may amount to

$$P(t) = 5000e^{0.05827t}$$

dollars t years after 2000. On the other hand, some banks advertise and compute interest on the basis of 6% interest with 'instantaneous' compounding, meaning that their model is

$$P'(t) = 0.06 \times P(t)$$

leading to

$$P(t) = 5000e^{0.06t}$$

They will say that their 'APR' (annual percentage rate) is $100 \times e^{0.06} = 6.18$ percent.

Example D.2.2 Geologists in the early nineteenth century worked out the sequential order of geological layers well before they knew the absolute dates of the layers. Their most extreme estimates of the age of the earth was in the order of 400 million years[4], about 1/10 of today's estimates based on decay of

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[4] Charles Darwin wrote in the Origin of Species that Earth was several hundred million years old, but he was opposed in 1863 by a dominant physical scientist, William Thompson (later to become Lord Kelvin) who estimated that Earth was between 24 and 400 million years old. His estimate was based on his calculation of the time it would take for Earth to cool from molten rock to today's temperatures in the upper layers of the Earth. See article by Philip England, Peter Molnar, and Frank Richter, GSA Today, 17, 1 (January 1, 2007).

radioactive material. Early applications of radiometric dating used the decomposition of uranium-238 first to thorium-234 and subsequently to lead-206. More recently Potassium-40 decomposition has been found to be useful (and zircon decay is currently the best available).

Potassium-40 decomposes to both argon-40 and calcium-40 according to

$$9\,(^{40}\mathrm{K}) \quad \longrightarrow \quad {}^{40}\mathrm{Ar} + 8\,(^{40}\mathrm{Ca})$$

When deposited, volcanic rock contains significant amounts of ^{40}K but is essentially free of ^{40}Ar because ^{40}Ar is a gas that escapes the rock under volcanic conditions. Once cooled, some volcanic rock will become essentially sealed capsules that contain ^{40}K and retain the ^{40}Ar that derives from decomposition of the ^{40}K.

Mathematical Model D.2.3 Potassium-40 decomposition. The rate of disintegration of ^{40}K is proportional to the amount of ^{40}K present.

If we let $K(t)$ be the amount of ^{40}K present t years after deposition of rock of volcanic origin and K_0 the initial amount of ^{40}K present, then

$$K(0) = K_0, \qquad K'(t) = -rK(t)$$

where r is a positive constant. The minus sign reflects the disintegration of ^{40}K. From the equation we may write

$$K(t) = K_0 e^{-rt}$$

The half-life of ^{40}K is 1.28×10^9 years, meaning that 1.28×10^9 years after deposition of the volcanic rock, the amount of ^{40}K in the rock will be $\frac{1}{2}K_0$. We use this information to evaluate r.

$$\begin{aligned}
\frac{1}{2}K_0 &= K_0 e^{-r\times 1280000000} \\
\frac{1}{2} &= e^{-r\times 1280000000} \\
\ln\frac{1}{2} &= -r \times 1280000000 \\
r &= \frac{\ln 2}{1280000000} \\
K(t) &= K_0 e^{-\frac{\ln 2}{1280000000}t}
\end{aligned}$$

Problem. Suppose a rock sample is found to have 5×10^{14} ^{40}K atoms and 2×10^{13} ^{40}Ar atoms. What is the age of the rock?

Solution. It is necessary to assume[5] that all of the ^{40}Ar derives from the ^{40}K, and that there has been no leakage of ^{40}K or ^{40}Ar into or out of the rock. Assuming so, then the number of ^{40}K atoms that have

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[5] Because ^{40}Ar is a gas at the temperatures that the rock was formed, no ^{40}Ar is originally in the rock.

decomposed (to either ^{40}Ca or ^{40}Ar) must be nine times the number of ^{40}Ar atoms, or $9 \times (2 \times 10^{13}) = 1.8 \times 10^{14}$ atoms. Therefore

$$K_0 = 5 \times 10^{14} + 1.8 \times 10^{14} = 6.8 \times 10^{14}$$

and

$$K(t) = 6.8 \times 10^{14} e^{-\frac{\ln 2}{1280000000}t}$$

We want the value of t for which $K(t) = 5 \times 10^{14}$. Therefore,

$$\begin{aligned} 5 \times 10^{14} &= 6.8 \times 10^{14} e^{-\frac{\ln 2}{1280000000}t} \\ \frac{5}{6.8} &= e^{-\frac{\ln 2}{1280000000}t} \\ \ln \frac{5}{6.8} &= -\frac{\ln 2}{1280000000}t \\ t &= 568,000,000 \end{aligned}$$

The rock is about 568 million years old. ▬

D.2.1 Two primitive modeling concepts.

Primitive Concept 1. Suppose you have a barrel (which could just as well be a blood cell, stomach, liver, or lake or ocean or auditorium) and $A(t)$ liters is the amount of water (glucose, plasma, people) in the barrel at time t minutes. If water is running into the barrel at a rate R_1 liters/minute and leaking out of the barrel at a rate R_2 liters/minute then

$$\begin{array}{ccccc} \text{Rate of change of water} & = & \text{Rate water enters} & - & \text{Rate water leaves} \\ \text{in the barrel} & & \text{the barrel} & & \text{the barrel} \\ & & & & \\ A'(t) & = & R_1 & - & R_2 \\ & & & & \\ \frac{\text{L}}{\text{min}} & & \frac{\text{L}}{\text{min}} & & \frac{\text{L}}{\text{min}} \end{array}$$

Primitive Concept 2. Similar to Primitive Concept 1 except that there is salt in the water. Suppose $S(t)$ is the amount in grams of salt in the barrel and C_1 is the concentration in grams/liter of salt in the stream entering the barrel and C_2 is the concentration of salt in grams/liter in the stream leaving the barrel. Then

$$\begin{array}{ccccc} \text{Rate of change of salt} & = & \text{Rate salt enters} & - & \text{Rate salt leaves} \\ \text{in the barrel} & & \text{the barrel} & & \text{the barrel} \\ & & & & \\ S'(t) & = & C_1 \times R_1 & - & C_2 \times R_2 \\ & & & & \\ \frac{\text{g}}{\text{min}} & & \frac{\text{g}}{\text{L}} \times \frac{\text{L}}{\text{min}} & & \frac{\text{g}}{\text{L}} \times \frac{\text{L}}{\text{min}} \end{array}$$

Observe that the units are g/m on both sides of the equation. Maintaining a balance in units often helps to find the correct equation.

Example D.2.3 Suppose a runner is exhaling at the rate of 2 liters per second. Then the amount of air in her lungs is decreasing at the rate of two liters per second. If, furthermore, the CO_2 partial pressure in the exhaled air is 50 mm Hg (approx 0.114 g CO_2/liter of air at body temperature of 310 K)[6], then she is exhaling CO_2 at the rate of 0.114 g/liter $\times 2$ liters/sec $= 0.228$ g/sec. ■

Example D.2.4 Classical Washout Curve. A barrel contains 100 liters of water and 300 grams of salt. You start a stream of pure water flowing into the barrel at 5 liters per minute, and a compensating stream of salt water flows from the barrel at 5 liters per minute. The solution in the barrel is 'well stirred' so that the salt concentration is uniform throughout the barrel at all times. Let $S(t)$ be the amount of salt (grams) in the barrel t minutes after you start the flow of pure water into the barrel.

Explore D.2.1 Draw a graph of what you think will be the graph of $S(t)$. In doing so consider

- What is $S(0)$?
- Does $S(t)$ increase or decrease?
- Will there be a time, t_*, for which $S(t_*) = 0$? If so, what is t_*?

■

Solution. First let us analyze S. We use Primitive Concept 2. The concentration of salt in the water flowing into the barrel is 0. The concentration of salt in the water flowing out of the barrel is the same as the concentration $C(t)$ of salt in the barrel which is

$$C(t) = \frac{S(t)}{100}\text{g/L}$$

Therefore

$$\begin{array}{ccccc}
\begin{array}{c}\text{Rate of change of salt}\\ \text{in the barrel}\end{array} & = & \begin{array}{c}\text{Rate salt enters}\\ \text{the barrel}\end{array} & - & \begin{array}{c}\text{Rate salt leaves}\\ \text{the barrel}\end{array}\\[2ex]
S'(t) & = & C_1 \times R_1 & - & C_2 \times R_2\\[2ex]
S'(t) & = & 0 \times 5 & - & \frac{S(t)}{100}\frac{\text{gr}}{\text{L}} \times 5\frac{\text{L}}{\text{min}}
\end{array}$$

Furthermore, $S(0) = 300$. Thus

$$\begin{aligned}
S(0) &= 300\\
S'(t) &= -0.05S(t).
\end{aligned}$$

---

[6]
$$\frac{\frac{50}{760}\text{A} \times 1\text{liter} \times 44\text{mol wt}}{0.08206\text{gas const} \times 310\text{K}} = 0.114\text{g}$$

From the Exponential Growth and Decay property D.2.2,

$$S(t) = 300e^{-0.05t}$$

Explore D.2.2 . Draw the graph of $S(t) = 300e^{-0.05t}$ on your calculator and compare it with the graph you drew in the previous Explore. ■ ■

Example D.2.5 Classical Saturation Curve.

Problem. Suppose a 100 liter barrel is full of pure water and at time $t = 0$ minutes a stream of water flowing at 5 liters per minute and carrying 3 g/liter of salt starts flowing into the barrel. Assume the salt is well mixed in the barrel and water overflows at the rate of 5 liters per minute. Let $S(t)$ be the amount of salt in the barrel at time t minutes after the salt water starts flowing in.

Explore D.2.3 Draw a graph of what you think will be the graph of $S(t)$. In doing so consider

- What is $S(0)$?
- Does $S(t)$ increase or decrease?
- Is there an upper bound on $S(t)$, the amount of salt in the barrel that will be in the barrel?

■

Solution: We analyze S; again we use Primitive Concept 2. The concentration of salt in the inflow is 3 g/liter. The concentration $C(t)$ of salt in the tank a time t minutes is

$$C(t) = \frac{S(t)}{100}$$

The salt concentration in the outflow will also be $C(t)$. Therefore

$$\begin{array}{ccccc}
\begin{array}{c}\text{Rate of change of salt}\\ \text{in the barrel}\end{array} & = & \begin{array}{c}\text{Rate salt enters}\\ \text{the barrel}\end{array} & - & \begin{array}{c}\text{Rate salt leaves}\\ \text{the barrel}\end{array} \\[2ex]
S'(t) & = & C_1 \times R_1 & - & C_2 \times R_2 \\[2ex]
S'(t) & = & 3 \times 5 & - & \frac{S(t)}{100} \times 5 \\[2ex]
\frac{\text{g}}{\text{min}} & & \frac{\text{g}}{\text{L}} \times \frac{\text{L}}{\text{min}} & & \frac{\text{g}}{\text{L}} \times \frac{\text{L}}{\text{min}}
\end{array}$$

Initially the barrel is full of pure water, so

$$S(0) = 0$$

We now have

$$\begin{aligned} S(0) &= 0 \\ S'(t) &= 15 - 0.05S(t) \end{aligned} \tag{D.5}$$

This equation is not in the form of $P'(t) = kP(t)$ because of the 15. Proceed as follows.

Equilibrium. Ask, 'At what value, E, of $S(t)$ would $S'(t) = 0$?' That would require

$$0 = 15 - 0.05E = 0, \text{or} E = 300\text{g}.$$

$E = 300$ g is the *equilibrium level* of salt in the barrel. We focus attention on the difference, $D(t)$, between the equilibrium level and the current level of salt. Thus

$$D(t) = 300 - S(t) \qquad \text{and} \qquad S(t) = 300 - D(t)$$

Now,

$$D(0) = 300 - S(0) = 300 - 0 = 300$$

Furthermore,

$$S'(t) \quad = \quad [300 - D(t)]' \quad = \quad -D'(t)$$

We substitute into Equations D.5

$$\begin{array}{rclcrcl} S(0) & = & 0 & \qquad & D(0) & = & 300 \\ S'(t) & = & 15 - 0.05S(t) & \qquad & -D'(t) & = & 15 - 0.05(300 - D(t)) \end{array}$$

The equations for D become

$$\begin{aligned} D(0) &= 300 \\ D'(t) &= -0.05D(t) \end{aligned}$$

This is in the form of the Exponential Growth and Decay Property D.2.2, and we write

$$D(t) = 300e^{-0.05t}$$

Returning to $S(t) = 300 - D(t)$ we write

$$S(t) = 300 - D(t) = 300 - 300e^{-0.05t}$$

The graph of $S(t) = 300 - 300e^{-0.05t}$ is shown in Figure D.4. Curiously, the graph of $S(t)$ is also called an *exponential decay curve.* $S(t)$ is not decaying at all; $S(t)$ is increasing. What is decaying exponentially is $D(t)$, the remaining salt capacity. ■

Explore D.2.4 Show that if $S(t) = 300 - 300e^{-0.05t}$, then

$$S(0) = 0, \qquad \text{and} \qquad S'(t) = 15 - 0.05S(t).$$

■

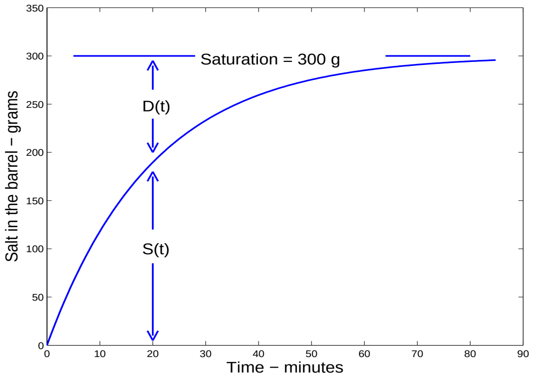

Figure D.4: The graph of $S(t) = 300 - 300e^{-0.05t}$ depicting the amount of salt in a barrel initially filled with 100 liters of pure water and receiving a flow of 5 L/m carrying 3 g/L. $D(t) = 300 - S(t)$.

D.2.2 Continuous-space analysis of light depletion.

As observed in Chapter 1, light intensity decreases as one descends from the surface of a lake or ocean. There we was divided the water into discrete layers and it was assumed that each layer absorbs a fixed fraction, F, of the light that enters it from above. This hypothesis led to the difference equation

$$I_{d+1} - I_d = -F \times I_d$$

Light is actually absorbed continuously as it passes down through a (homogeneous) water medium, not in discrete layers. We examine the light intensity, $I(x)$, at a distance, x meters, below the surface of a lake or ocean, assuming that the light intensity penetrating the surface is a known quantity, I_0.

We start by testing an hypothesis about light transmission in water that appears different from the hypothesis we arrived at in Chapter 1:

Mathematical Model D.2.4 Light Absorbance: The amount of light absorbed by a (horizontal) layer of water is proportional to the thickness of the layer and to the amount of light entering the layer (see Figure D.5).

The mathematical model of light absorbance implies, for example, that

1. The light absorbed by a water layer of thickness 2Δ is twice the light absorbed by a water layer of thickness Δ and
2. A layer that absorbs 10% of a dim light will absorb 10% of a bright light.

We know from experimental evidence that implication (1) is approximately true for thin layers and for low levels of turbidity. Implication (2) is valid for a wide range of light intensities.

Double Proportionality. The mathematical model of light absorbance the light absorbed by a layer is proportional to two things, the thickness of the layer and the intensity of the light entering the layer. We handle this double proportionality by assuming that the amount of light absorbed in a layer is proportional to the **product** of the thickness of the layer and the intensity of the light incident to the

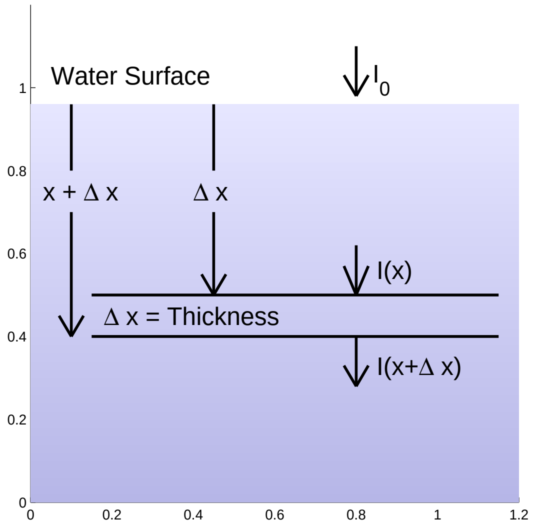

Figure D.5: Diagram of light depletion below the surface of a lake or ocean. $I(x)$ is light intensity at depth x due to light of intensity I_0 just below the surface of the water.

layer. That is, there is a number, K, such that if $I(x)$ is the light intensity at depth x and $I(x + \Delta x)$ is the light intensity at depth $x + \Delta x$, then

$$I(x + \Delta x) - I(x) \doteq -K \times \Delta x \times I(x) \qquad \text{(D.6)}$$

The product, $K \times \Delta x \times I(x)$, has the advantage that

1. For fixed incident light intensity, $I(x)$, the light absorbed, $I(x + \Delta x) - I(x)$, is proportional to the thickness, Δx, (proportionality constant $= -K \times I(x)$) and

2. For fixed thickness Δx, the light absorbed is proportional to the incident light, $I(x)$ (proportionality constant $= -K \times \Delta x$).

Equation D.6 can be rearranged to

$$\frac{I(x + \Delta x) - I(x)}{\Delta x} \quad \doteq \quad -KI(x)$$

The approximation ($\doteq$) improves as the layer thickness, Δx, approaches zero.

$$\text{As } \Delta x \to 0 \qquad \frac{I(x + \Delta x) - I(x)}{\Delta x} \to I'(x)$$

and we conclude that

$$I'(x) = -KI(x) \qquad \text{(D.7)}$$

The Exponential Growth and Decay Property D.2.2 implies that

$$\text{because} \qquad I'(x) = -KI(x), \qquad I(x) = I_0 e^{-Kx} \qquad \text{(D.8)}$$

Example D.2.6 Assume that 1000 w/m^2 of light is striking the surface of a lake and that 40% of that light is reflected back into the atmosphere. We first solve the initial value problem

$$\begin{aligned} I(0) &= 600 \\ I'(x) &= -KI(x) \end{aligned}$$

to get

$$I(x) = 600e^{-Kx}$$

If we have additional information that, say, the light intensity at a depth of 10 meters is 400 W/m^2 we can find the value of K. It must be that

$$I(10) = 600e^{-K\times 10} = 400$$

The only unknown in the last equation is K, and we solve

$$\begin{aligned} 600e^{-K\times 10} &= 400 \\ e^{-K\times 10} &= 400/600 = 5/6 \\ \ln\left(e^{-K\times 10}\right) &= \ln(2/3) \\ -K\times 10 &= \ln(2/3) \\ K &\doteq 0.040557 \end{aligned}$$

Thus we could say that

$$I(x) = 600\times e^{-0.040557x}$$

If we know, for example, that 30 W/m^2 of light are required for a certain species of coral to grow, we can ask for the maximum depth, $\overline{x}$, at which we might find that species. We would solve

$$\begin{aligned} I(\overline{x}) &= 30 \\ 600\times e^{-0.040557\overline{x}} &= 30 \\ \ln\left(e^{-0.040557\overline{x}}\right) &= \ln(30/600) \\ -0.040557\overline{x} &= \ln(1/20) \\ \overline{x} &= 73.9 \quad \text{meters} \end{aligned}$$

D.2.3 Doubling time and half-life

Suppose k and C are positive numbers. The *doubling time* of $F(t) = Ce^{kt}$ is a number t_{dbl} such that

$$\text{for any time } t \qquad F(t + t_{dbl}) = 2\times F(t).$$

That there is such a number follows from

$$F(t + t_{dbl}) \;=\; 2\times F(t)$$

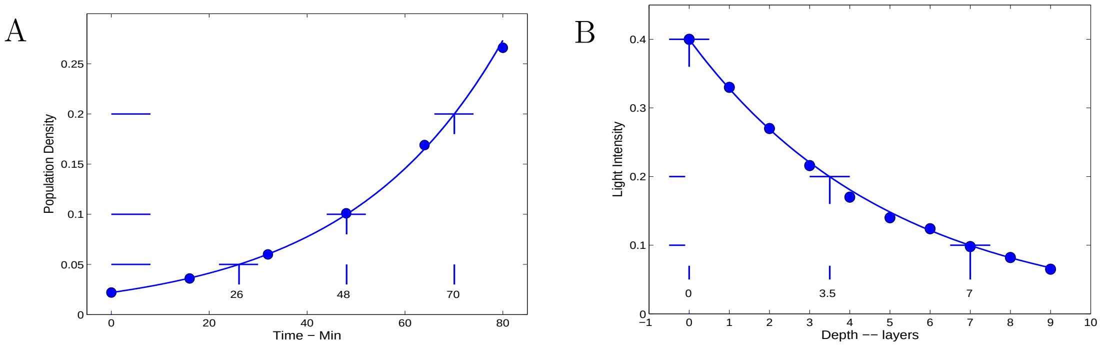

Figure D.6: A. Bacterial population density and ABS=$0.022e^{0.0315t}$, which has doubling time of 22 minutes. B. Light depletion and $I_d = 0.4e^{-0.196d}$ which has a 'half life' of 3.5 meters.

$$\begin{aligned}
Ce^{k\times(t+t_{dbl})} &= 2\times Ce^{kt} \\
Ce^{kt}\times e^{kt_{dbl}} &= 2\times Ce^{kt} \\
e^{kt_{dbl}} &= 2 \\
t_{dbl} &= \frac{\ln 2}{k}. \qquad \text{(D.9)}
\end{aligned}$$

The *half-life* of $F(t) = Ce^{-kt}$ is a number t_{half}, usually written as $t_{1/2}$, such that

$$\text{for any time } t \qquad F(t+t_{1/2}) = \frac{1}{2}\times F(t).$$

Using steps similar to those for the doubling time you can find that

$$t_{1/2} = \frac{\ln 2}{k}. \qquad \text{(D.10)}$$

Explore D.2.5 Write the steps similar to those for the doubling time to show that $t_{1/2} = (ln2)/k$. ■

In Figure D.6A is a graph of the bacterial density from Table **??** and of the equation

$$\text{Abs} = 0.022e^{0.0315}t, \qquad t_{dbl} = \frac{\ln 2}{0.0315} = 22.0 \quad \text{minutes.}$$

The bacterial density doubles every 22 minutes, as illustrated for the intervals [26,48] minutes and [48,70] minutes.

In Figure D.6B is a graph of light intensity decay from Figure **??** (repeated in Figure D.7) and of the Equation **??**

$$I_d = 0.400\times 0.82^d$$

Because $0.82 = e^{\ln 0.82} = e^{-0.198}$,

$$I_d = 0.400\times e^{-0.198d} \qquad \text{and} \qquad d_{1/2} = \frac{\ln 2}{0.198} = 3.5$$

Every 3.5 layers of muddy water the light intensity decays by one-half. $d_{1/2}$ is a distance and might be called 'half depth' rather than 'half life.' 'Half life' is the term used for all exponential decay, however, and you are well advised to use it.

Example D.2.7 *Problem.* Suppose a patient has taken 80 mg of Sotolol, a drug that regularizes heart beat, once per day for several days. Sotolol has a half-life in the body of 12 hrs. There will be an historical accumulation, H, of Sotolol in the body due to previous days' pills, and each day when the 80 mg pill is taken there will be $H + 80$ mg of Sotolol in the body that will decrease according to the 12 hr half-life. Find the historical accumulation, H, and the daily fluctuation.

Solution. Let A_t be the amount of Sotolol in the body at time t hours during the day, with $t = 0$ being the time the pill is taken each day. Then

$$A_0^- = H \qquad \text{and} \qquad A_0^+ = H + 80.$$

Furthermore, because the half-life is 12 hours, A_{24}^-, the amount left in the body just before the pill is taken the next day, is

$$A_{24}^- = \frac{1}{4}\,A_0^+ = \frac{1}{4}\,(H + 80),$$

and

$$A_{24}^- = H$$

Therefore,

$$\frac{1}{4}\,(H + 80) = H, \qquad H = \frac{4}{3}\,20 = 26.7\text{mg}$$

Furthermore, $A_0^+ = H + 80 = 106.7$mg so the amount of Sotolol in the body fluctuates from 26.7 mg to 106.7 mg, a four to one ratio. You are asked to compare this with taking 40 mg of Sotolol twice per day in Exercise D.2.5

D.2.4 Semilogarithm and LogLog graphs.

Functions $P(t)$ that satisfy an equation $P'(t) = k \times P(t)$ may be written $P(t) = P_0 e^{kt}$ and will satisfy the relation $\ln\, P(t) = (\ln\, P_0) + kt$. The graph of $\ln\, P(t)$ *vs* t is a straight line with intercept $\ln\, P_0$ and slope k. Similarly, if $P'(t) = -kP(t)$ the graph of $\ln\, P(t)$ *vs* t is a straight line with slope (in rectilinear coordinates) $-k$. A scientist with data $t, P(t)$ that she thinks is exponential may plot the graph of $\ln\, P(t)$ *vs* t. If the graph is linear, then a fit of a line to that data will lead to an exponential relation of the form $P(t) = Ae^{kt}$ or $P(t) = Ae^{-kt}$. She may then search for a biological process that would justify a model $P'(t) = \pm kP(t)$.

Example D.2.8 In Section **??** we showed the results of an experiment measuring the light decay as a function of depth. The data and a semilog graph of the data are shown in Figure D.7.

As shown in the figure,

$$\log_{10}\, I_d = -0.4 - 0.087 \times d$$

Depth Layer	I_d mW/cm^2
0	0.400
1	0.330
2	0.270
3	0.216
4	0.170
5	0.140
6	0.124
7	0.098
8	0.082
9	0.065

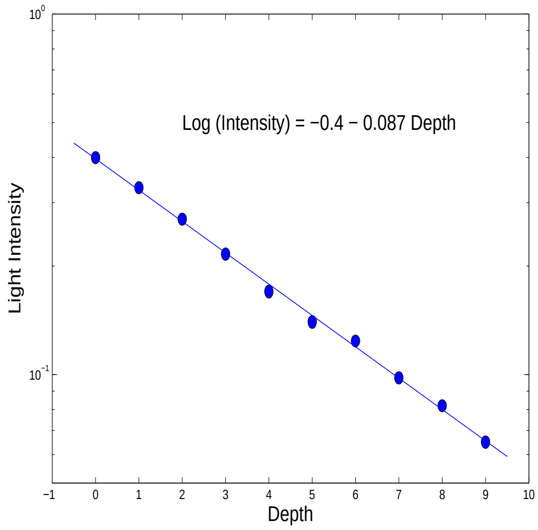

Figure D.7: Data and a semilog graph of the data showing experimental results of measuring light decrease with depth of water.

is a good approximation to the data. Therefore

$$I_d \doteq 10^{-0.4-0.087d}$$

$$= 0.4 \times 0.82^d$$

which is the same result obtained in Section **??**. As shown in the previous subsection, the relation

$$I'(d) = -k \times I(d)$$

corresponds to a process underlying light depletion in water.

D.2.5 Relative Growth Rates and Allometry.

If y is a positive function of time, the relative growth rate of y is

$$\frac{y'(t)}{y(t)} \qquad \text{Relative Growth Rate.} \tag{D.11}$$

The relative growth rate of y is sometimes called the fractional growth rate or the logarithmic growth rate.

Definition D.2.1 Allometry. . Two functions, x and y, of time are said to be allometrically related if there are numbers C and a such that

$$y(t) = C \times (\, x(t)\,)^a\,. \tag{D.12}$$

If x and y are allometric then

$$\log y = \log C\, x^a = \log C + a \log x, \tag{D.13}$$

for any base of log. Therefore if $\log y$ is plotted *vs* $\log x$ the graph should be a straight line.

Explore D.2.6 Show that if x and y satisfy Equation D.13, $\log y(t) = \log C + a \log x(t)$, then

$$\frac{y'(t)}{y(t)} = a \times \frac{x'(t)}{x(t)}.$$

Conclude that if x and y are allometric then the relative growth rate of y is proportional to the relative growth rate of x. ■

Shown in Figure D.8 is a graph of $\log_{10}$ of the weight of large mouth bass *vs* $\log_{10}$ of their length[7]. The data appear linear and we conclude that the weight is allometric to the length. An equation of a line close to the data is

$$\frac{\log y - 1.05}{\log x - 2.0} = \frac{2.6 - 1.05}{2.5 - 2.0}, \qquad \log y = -5.75 + 3.1 \log x$$

Then

$$y = 10^{-5.75}\, x^{3.1}$$

The weights of the bass are approximately proportional to the cube of the lengths. This is consistent with the fact that the volume of a cube is equal to the cube of the length of an edge. Many interesting allometric relations are not supported by underlying models, however (Exercises D.2.26, and D.2.25).

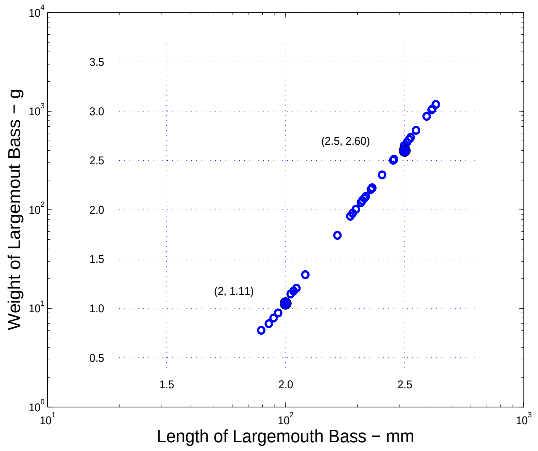

Figure D.8: Weight *vs* Length for large mouth bass plotted on a log-log graph.

Exercises for Section D.2, The derivative equation $P'(t) = k \times P(t)$

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[7] Data from Robert Summerfelt, Iowa State University.

Exercise D.2.1 Write a solution for each of the following derivative equations. Sketch the graph of the solution. For each, find the doubling time, t_{dbl}, or half life, $t_{1/2}$, which ever is applicable.

$$\begin{array}{llllllllllll}
\text{a.} & P(0) & = & 5 & P'(t) & = & 2P(t) & \text{b.} & P(0) & = & 5 & P'(t) = -2P(t) \\
\text{c.} & P(0) & = & 2 & P'(t) & = & 0.1P(t) & \text{d.} & P(0) & = & 2 & P'(t) = -0.1P(t) \\
\text{e.} & P(0) & = & 10 & P'(t) & = & P(t) & \text{f.} & P(0) & = & 10 & P'(t) = -P(t) \\
\text{g.} & P(0) & = & 0 & P'(t) & = & 0.01P(t) & \text{h.} & P(0) & = & 0 & P'(t) = -0.01P(t)
\end{array}$$

Exercise D.2.2 Write a solution for each of the following derivative equations. Sketch the graph of the solution. For each, find the half life, $t_{1/2}$, which is the time required to 'move half way toward equilibrium.'

Recall the solution in Example D.2.5 to solve $S'(t) = 15 - 0.05S(t)$.

$$\begin{array}{llllllllllll}
\text{a.} & S(0) & = & 0 & S'(t) & = & 10 - 2S(t) & \text{b.} & S(0) & = & 2 & S'(t) = 10 - 2S(t) \\
\text{c.} & S(0) & = & 5 & S'(t) & = & 10 - 2S(t) & \text{d.} & S(0) & = & 10 & S'(t) = 10 - 2S(t) \\
\text{e.} & S(0) & = & 0 & S'(t) & = & 20 - S(t) & \text{f.} & S(0) & = & 10 & S'(t) = 20 - S(t) \\
\text{g.} & S(0) & = & 20 & S'(t) & = & 20 - S(t) & \text{h.} & S(0) & = & 30 & S'(t) = 20 - S(t)
\end{array}$$

Exercise D.2.3 Find values of C and k so that $P(t) = Ce^{kt}$ matches the data.

$$\begin{array}{llllllllllll}
\text{a.} & P(0) & = & 5 & P(2) & = & 10 & \text{b.} & P(0) & = & 10 & P(2) = 5 \\
\text{c.} & P(0) & = & 2 & P(5) & = & 10 & \text{d.} & P(0) & = & 10 & P(5) = 10 \\
\text{e.} & P(0) & = & 5 & P(2) & = & 2 & \text{f.} & P(0) & = & 8 & P(10) = 6 \\
\text{g.} & P(1) & = & 5 & P(2) & = & 10 & \text{h.} & P(2) & = & 10 & P(10) = 20
\end{array}$$

Exercise D.2.4 Suppose a barrel has 100 liters of water and 400 grams of salt and at time $t = 0$ minutes a stream of water flowing at 5 liters per minute and carrying 3 g/liter of salt starts flowing into the barrel, the barrel is well mixed, and a stream of water and salt leaves the barrel at 5 liters per minute. What is the amount of salt in the barrel t minutes after the flow begins? Draw a candidate solution graph for this problem before computing the solution.

Exercise D.2.5 In Example D.2.7 it was shown that in a patient who takes 80 mg of Sotolol once per day, the daily fluctuation of Sotolol is from 26.7 mg to 106.7 mg. Sotolol has a half-life in the body of 12 hours. What is the fluctuation of Sotolol in the body if the patient takes two 40 mg of Sotolol at 12 hour intervals in the day? Would you recommend two 40 mg per day rather than one 80 mg pill per day?

Exercise D.2.6 A patient takes 10 mg of coumadin once per day to reduce the probability that he will experience blood clots. The half-life of coumadin in the body is 40 hours. What level, H, of coumadin will be accumulated from previous ingestion of pills and what will be the daily fluctuation of coumadin in the body.

Exercise D.2.7 Plot semilog graphs of the data sets in Table Ex. D.2.7 and decide which ones appear to be approximately exponential. For those that appear to be exponential, find numbers, C and k, so that

$$P(t) = Ce^{kt}$$

approximates the the data.

Table for Exercise D.2.7 Data sets for Exercise D.2.7

a.

t	$P(t)$
0	2.0
1	2.1
2	2.8
3	4.7
4	8.4
5	14.5

b.

t	$P(t)$
0	6.00
2	4.35
4	2.97
6	2.17
8	1.65
10	1.25

c.

t	$P(t)$
0	2.00
5	2.83
10	4.00
15	5.67
20	8.00
25	11.31

d.

t	$P(t)$
0	2.00
3	1.00
6	0.67
9	0.50
12	0.40
15	0.33

e.

t	$P(t)$
0	2.00
3	1.00
6	0.50
9	0.25
12	0.13
15	0.06

Exercise D.2.8 Shown in Table Ex. D.2.8 are data from *V. natrigens* growth reported in Chapter 1. Find numbers, C and k, so that

$$P(t) = Ce^{kt}$$

approximates the the data. Use your values of C and k and compute $P(0)$, $P(16)$, $P(32)$, $P(48)$, and $P(64)$ and compare them with the observed values in the Table Ex. D.2.8.

Table for Exercise D.2.8 Cell density of *V. natrigens* measured as light absorbance at 16-minute time increments.

pH 6.25	
Time (min)	Population Density
0	0.022
16	0.036
32	0.060
48	0.101
64	0.169

Exercise D.2.9 David Ho and colleagues[8] published the first study of HIV-1 dynamics within patients following treatment with an inhibitor of HIV-1 protease, ABT-538 which stops infected cells from producing new viral particles. Shown in Exercise Figure D.2.9A is a graph of plasma viral load before and after ABT-treatment was begun on day 1 for patient number 409 and in Exercise Figure D.2.9B is a semi-log graph of CD4 cell count following treatment.

a. By what percent is viral load diminished from day-1 to day-12?

b. The line in Figure D.2.9A has an equation, $y = 5.9 - 0.19x$. Remember that $y = \log_{10} V$ and x is days. Find V_0 and m so that the graph of $V(t) = V_0e^{-mt}$ in semilog coordinates is the line drawn in Exercise Figure D.2.9A.

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[8] David D. Ho, et al, *Nature* 373 (1995) 123:127.

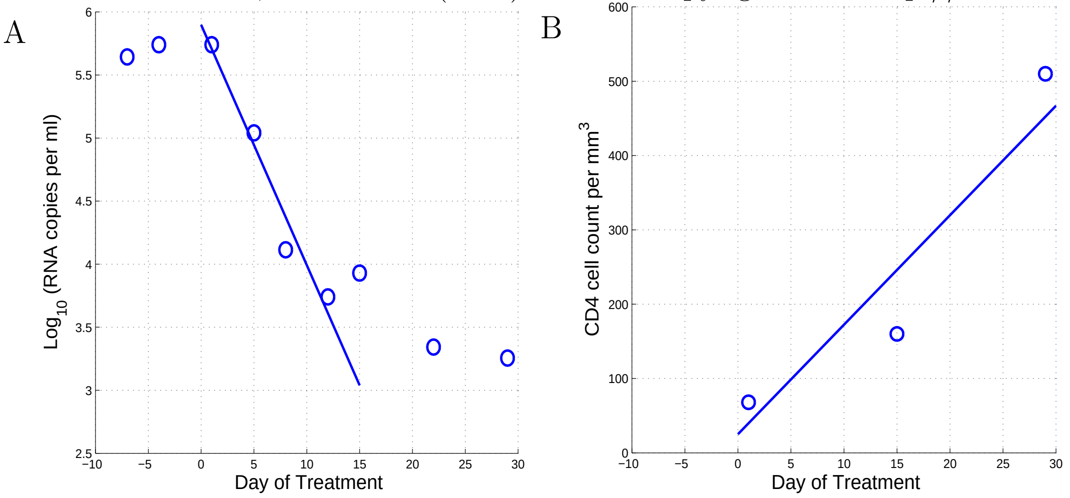
c. What is the half-life of the viral load?

d. From the previous step, $V'(t) = -0.43V(t)$. Suppose ABT-538 totally eliminates viral production during days 1 to 12. At what rate is the immune system of patient 409 eliminating virus before treatment.

e. Assume that CD4 cell counts increase linearly and the equation in Figure D.2.9B is y = 9.2 + 15.8 x. At what rate are CD4 cells being produced? Remember that y is CD4 count per mm^3 and there are about 6×10^6 mm^3 of blood in the human body.

After about 35 days, the HIV virus mutates into a form resistant to ABT-538 and pre-treatment viral loads soon return. Treatment with a protease inhibitor together with drugs that inhibit the translation of HIV RNA to DNA can decrease viral loads to levels below detection for the duration of treatment.

Figure for Exercise D.2.9 A. Count of HIV viral load during administration of ABT-538. B. Count of helper t-cell during the same period. Note that if $\log_{10}$ (RNA copies per ml) = 4, for example, then RNA copies per ml = 10^4. See Exercise D.2.9 Figures adapted by permission at no cost from Macmillan Publishing Group, Ltd. David D. Ho, Avidan U Neumann, Alan S. Perelson, Wen Chen, John M. Leonard & Martin Markowitz, *Nature* 373 (1995) 123:127 Copyright 1995 http://www.nature.com.



Exercise D.2.10 You inject two grams of penicillin into the 6 liter vascular pool of a patient. Plasma circulates through the kidney at the rate of 1.2 liters/minute and the kidneys remove 20 per cent of the penicillin that passes through.

1. Draw a schematic diagram showing the vascular pool and kidneys as separate entities, an artery leading from the vascular pool to the kidney and a vein leading from the kidney back to the vascular pool.

2. Let $P(t)$ be the amount of penicillin in the vascular pool t minutes after injection of penicillin. What is $P(0)$?

3. Use Primitive Concept 2 to write an equation for P'.

4. Write a solution to your equation.

Exercise D.2.11 Suppose a rock sample is found to have 8.02 μg of ^{40}K and 7.56 μg of ^{40}Ar. What is the age of the rock?

Exercise D.2.12 Suppose a rock sample is found to have 6.11 mg of ^{40}K and 0.05 mg of ^{40}Ar. What is the age of the rock?

Exercise D.2.13 Rubidium-87 decomposes to strontium-87 with a half-life of 50×10^9 years. Fortunately, rubidium and potassium occur in the same rock types and in the same minerals, usually in the ratio of 1 ^{87}Rb atom to approximately 600 ^{40}K atoms. Age determined by rubidium-87 to strontium-87 decomposition is an excellent check of ^{40}K to ^{40}Ar ages. However, ^{87}Sr may be lost from the rock or may be present but not derived from ^{87}Rb so the ^{87}Rb to ^{87}Sr age may not be as accurate as the ^{40}K to ^{40}Ar age.

a. Suppose a rock sample has 2.5×10^{11} atoms of ^{87}Rb and 1.5×10^{10} atoms of ^{87}Sr. What is the age of the rock?

b. Suppose a rock sample has 6.4μg of ^{87}Rb and 0.01μg of ^{87}Sr. What is the age of the rock?

Exercise D.2.14 A major advancement in archaeology was the development of carbon-14 dating in the 1950's by an American chemist Willard Libby, for which he received the 1960 Nobel Prize in Chemistry. Carbon-14 develops in the upper atmosphere as neutrons bombard nitrogen, and subsequently combines with oxygen to form carbon dioxide. About 1 in 10^{12} CO_2 atoms is formed with ^{14}C in today's atmosphere. Plants metabolize $^{14}CO_2$ (almost) as readily as $^{12}CO_2$, and resulting sugars are metabolized equally by animals that eat the plants. Consequently carbon from living material is 1 part in 10^{12} carbon-14. Upon death, no additional ^{14}C is absorbed into the material and ^{14}C gradually decomposes into nitrogen. Slightly confounding the use of radio carbon dating is the fact that the fraction of atmospheric $^{14}C0_2$ has not been historically constant at 1 molecule per 10^{12} molecules of $^{12}CO_2$.

Carbon-14 decomposes to nitrogen according to

$$^{14}_{6}\mathrm{C} \longrightarrow ^{14}_{7}\mathrm{N} + \beta^- + \overline{\nu} + \text{energy} \tag{D.14}$$

where β^- denotes an electron and $\overline{\nu}$ denotes an antineutrino. One of the neutrons of $^{14}_{6}$C looses an electron and becomes a proton.

Mathematical Model D.2.5 Carbon-14 decay. The rate of decomposition of $^{14}_{6}$C in a sample is proportional to the size of the sample. One-half of the atoms in a sample will decompose in 5730 years.

a. Write and solve a derivative equation that will show for a sample of ^{14}C initially of size C_0 what the size will be t years later.

b. In tissue living today, the amount of ^{14}C in one gram of carbon is approximately 10^{-12} grams. Assume for this problem that the same ratio in living material has persisted for the last 10,000 years. Also assume that upon death the only change in carbon of any form is the decrease in ^{14}C due to decomposition to nitrogen. Suppose a 100 gram sample of carbon from bone is found to have 3×10^{-11} grams of ^{14}C. What is the age of the sample?

c. Suppose that during the time 10,000 years ago until 2,000 years ago the amount of ^{14}C in one gram of carbon in living tissue was approximately 1.05×10^{-12} grams. Suppose a 100 gram sample of carbon from bone is found to have 3×10^{-11} grams of ^{14}C. What is the age of the sample?

Exercise D.2.15 Suppose solar radiation striking the ocean surface is 1250 W/m^2 and 20 percent of that energy is reflected by the surface of the ocean. Suppose also that 20 meters below the surface the light intensity is found to be 800 W/m^2.

a. Write an equation descriptive of the light intensity as a function of depth in the ocean.

b. Suppose a coral species requires 100 W/m^2 light intensity to grow. What is the maximum depth at which that species might be found?

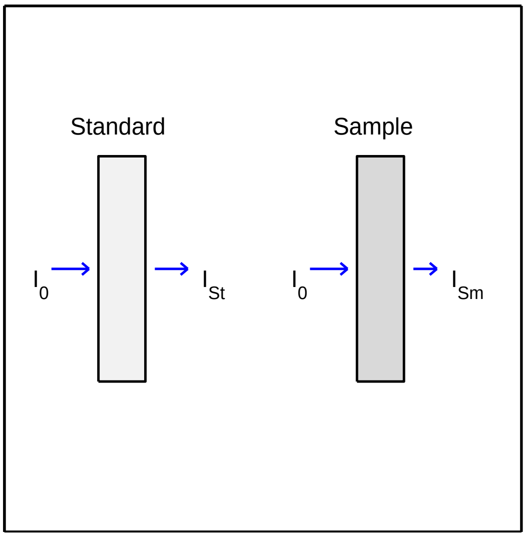
Exercise D.2.16 In two bodies of water, L_1 and L_2, the light intensities $I_1(x)$ and $I_2(x)$ as functions of depth x are measured simultaneously and found to be

$$I_1(x) = 800e^{-0.04x} \qquad \text{and} \qquad I_2(x) = 700e^{-0.05x}$$

Explain the differences in the two formulas in terms of the properties of water in the two bodies.

Exercise D.2.17 A spectrophotometer is used to measure bacterial cell density in a growth medium. Light is passed through a sample of the medium and the amount of light that is absorbed by the medium is an indicator of cell density. As cell density increases the amount of light absorbed increases. A standard is established by passing a light beam of intensity I_0 through a 0.5 cm layer of the growth medium without bacteria and the intensity I_{st} of the beam emerging from the medium is measured. See Figure Ex. D.2.17.

Figure for Exercise D.2.17 Diagram of spectrophotometer. A light beam of intensity I_0 enters the standard solution and the intensity I_{st} of the emerging beam is measured. A light beam of the same intensity I_0 enters the sample solution and the intensity I_{sm} of the emerging beam is measured. See Exercise D.2.17.



A light beam of the same intensity I_0 enters the sample solution and the intensity I_{sm} of the emerging beam is measured.

In the mathematical model of light absorbance (the amount of light absorbed by a layer of water is proportional to the thickness of the layer and to the amount of light entering the layer), the proportionality constant K is a measure of the opacity of the water. Recall that the solution Equation D.8 is $I(x) = I_0 e^{-Kx}$.

The bacteria in the sample placed in the spectrophotometer increase the opacity of the liquid. Explain why cell density is proportional to

$$\ln\left(\frac{I_{sm}}{I_{st}}\right)$$

The number $\ln(I_{sm}/I_{st})$ is called *absorbance.*

Exercise D.2.18 A patient comes into your emergency room and you start a penicillin infusion into the 6 liter vascular pool of 0.2 gms/min. Plasma circulates through the kidney at the rate of 1.2 liters/minute and the kidneys remove 20 per cent of the penicillin that passes through.

a. Draw a schematic diagram showing the vascular pool and kidneys as separate entities, an artery leading from the vascular pool to the kidney and a vein leading from the kidney back to the vascular pool.

b. Let $P(t)$ be the amount of penicillin in the vascular pool t minutes after injection of penicillin. What is $P(0)$?

c. Use Primitive Concept 2 to write an equation for P'.

d. Compute the solution to your equation and draw the graph of P.

e. The saturation level of penicillin in this problem is critically important to the correct treatment of your patient. Will it be high enough to control the infection you wish to control? If not, what should you do?

f. Suppose your patient has impaired kidney function and that plasma circulates through the kidney at the rate of 0.8 liters per minute and the kidneys remove 15 percent of the penicillin that passes through. What is the saturation level of penicillin in this patient, assuming you administer penicillin the same as before?

Exercise D.2.19 An egg is covered by a hen and is at 37° C. The hen leaves the nest and the egg is exposed to 17° C air.

a. Draw a graph representative of the temperature of the egg t minutes after the hen leaves the nest.

Mathematical Model D.2.6 Egg cooling. During any short time interval while the egg is uncovered, the change in egg temperature is proportional to the length of the time interval and proportional to the difference between the egg temperature and the air temperature.

b. Let $T(t)$ denote the egg temperature t minutes after the hen leaves the nest. Consider a short time interval, $[t, t + \Delta t]$, and write an equation for the change in temperature of the egg during the time interval $[t, t + \Delta t]$.

c. Argue that as Δt approaches zero, the terms of your previous equation get close to the terms of

$$T'(t) = -k(T(t) - 17) \tag{D.15}$$

d. Assume T(0) = 37 and find an equation for $T(t)$.

e. Suppose it is known that eight minutes after the hen leaves the nest the egg temperature is 35°C. What is k?

f. If the coldest temperature the embryo can tolerate is 32°C, when must the hen return to the nest?

Note: Equation D.15 is referred to as Newton's Law of Cooling.

Exercise D.2.20 Consider the following osmosis experiment in biology laboratory.

Material: A thistle tube, a 1 liter flask, some 'salt water', and some pure water, a membrane that is impermeable to the salt and is permeable to the water.

The bulb of the thistle tube is filled with salt water, the membrane is place across the open part of the bulb, and the bulb is inverted in a beaker of pure water so that the top of the pure water is at the juncture of the bulb with the stem. See the diagram.

Because of osmotic pressure the pure water will cross the membrane pushing water up the stem of the thistle tube until the increase in pressure inside the bulb due to the water in the stem matches the osmotic pressure across the membrane.

Our problem is to describe the height of the water in the stem as a function of time.

Mathematical Model D.2.7 Osmotic diffusion across a membrane. The rate at which pure water crosses the membrane is proportional to the osmotic pressure across the membrane minus the pressure due to the water in the stem.

Assume that the volume of the bulb is much larger than the volume of the stem so that the concentration of 'salt' in the thistle tube may be assumed to be constant.

Introduce notation and write a derivative equation with initial condition that will describe the height of the water in the stem as a function of time. Solve your derivative equation.

Exercise D.2.21 2 kilos of a fish poison, rotenone, are mixed into a lake which has a volume of $100 \times 20 \times 2 = 4000$ cubic meters. A stream of clean water flows into the lake at a rate of 1000 cubic meters per day. Assume that it mixes immediately throughout the whole lake. Another stream flows out of the lake at a rate of 1000 cubic meters per day.

What is the amount (p_t for discrete time or $P(t)$ for continuous time) of poison in the lake at time t days after the poison is applied?

a. Treat the problem as a discrete time problem with one-day time intervals. Solve the difference equation

$$p_0 = 2 \qquad p_{t+1} - p_t = -\frac{1000}{4000} p_t$$

b. Let t denote continuous time and $P(t)$ the amount of poison in the lake at time t. Let $[t, t+\Delta t]$ denote a short time interval (measured in units of days). An equation for the mathematical model is

$$P(t+\Delta t) - P(t) = -\frac{P(t)}{4000} \times \Delta t \times 1000$$

Show that the units on the terms of this equation balance.

c. Argue that

$$P(0) = 0, \qquad P'(t) = -0.25 P(t)$$

d. Compute the solution to this equation.

e. Compare the solution to the discrete time problem, p_t, with the solution to the continuous time problem, $P(t)$.

f. On what day, $\bar{t}$ will $P(\bar{t}) = 4\text{g}$?

Exercise D.2.22 Continuous version of Chapter Exercise **??**. Atmospheric pressure decreases with increasing altitude. Derive a dynamic equation from the following mathematical model, solve the dynamic equation, and use the data to evaluate the parameters of the solution equation.

Mathematical Model D.2.8 Mathematical Model of Atmospheric Pressure. Consider a vertical column of air based at sea level and assume that the temperature within the column is constant, equal to 20°C. The pressure at any height is the weight of air in the column above that height divided by the cross sectional area of the column. In a 'short' section of the column, by the ideal gas law the the mass of air within the section is proportional to the product of the volume of the section and the pressure within the section (which may be considered constant and equal to the pressure at the bottom of the section). The weight of the air above the lower height is the weight of air in the section plus the weight of air above the upper height.

Sea-level atmospheric pressure is 760 mm Hg and the pressure at 18,000 feet is one-half that at sea level.

Exercise D.2.23 When you open a bottle containing a carbonated soft drink, CO_2 dissolved in the liquid turns to gas and escapes from the liquid. If left open and undisturbed, the drink goes flat (looses its CO_2). Write a mathematical model descriptive of release of carbon dioxide in a carbonated soft drink. From your model, write a derivative equation descriptive of the carbon dioxide content in the liquid minutes after opening the drink.

Exercise D.2.24 *Decompression illness in deep water divers.* In the 1800's technology was developed to supply compressed air to under water divers engaged in construction of bridge supports and underwater tunnels. While at depth those divers worked without unusual physical discomfort. Shortly after ascent to the surface, however, they might experience aching joints, numbness in the limbs, fainting, and possible death. Affected divers tended to walk bent over and were said to have the "bends".

It was believed that nitrogen absorbed by the tissue at the high pressure below water was expanding during ascent to the surface and causing the difficulty, and that a diver who ascended slowly would be at less risk. The British Navy commissioned physician and mathematician J. S. Haldane[9] to devise a dive protocol to be followed by Navy divers to reduce the risk of decompression illness. Nitrogen flows quickly between the lungs and the plasma but nitrogen exchange between the plasma and other parts of the body (nerve, brain tissue, muscle, fat, joints, liver, bone marrow, for example) is slower and not uniform. Haldane used a simple model for nitrogen exchange between the plasma and the other parts of the body.

Mathematical Model D.2.9 Nitrogen absorption and release in tissue. The rate at which nitrogen is absorbed by a tissue is proportional to the difference in the partial pressure of nitrogen in the plasma and the partial pressure of nitrogen in the tissue.

Air is 79 percent nitrogen. Assume that the partial pressure of nitrogen in the lungs and the plasma are equal at any depth. At depth d,

$$\text{Plasma pp N}_2 = \text{Lung pp N}_2 \qquad = 0.79 \times \left(1 + \frac{d}{10}\right) \qquad \text{atmospheres.}$$

a. What is the partial pressure of nitrogen in a diver's lungs at the surface?

b. Suppose a diver has not dived for a week. What would you expect to be the partial pressure of nitrogen in her tissue?

---

[9] J. S. Haldane was the father of J. B. S. Haldane who, along with R. A. Fisher and Sewall Wright developed the field of population genetics.

c. A diver who has not dived for a week quickly descends to 30 meters. What is the nitrogen partial pressure in her lungs after descending to 30 meters?

d. Let $N(t)$ be the partial pressure of nitrogen in a tissue of volume, V, t minutes into the dive. Use the Mathematical Model D.2.9 Nitrogen absorption and release in tissue and Primitive Concept 2, to write an equation for N'.

e. Check to see whether (k is a proportionality constant)

$$N(t) = 0.79\left(1 + \frac{d}{10}\right) - 0.79\,\frac{d}{10}\,e^{-\frac{k}{V}t} \tag{D.16}$$

solves your equation from the previous step.

f. Assume $\frac{k}{V}$ in Equation D.16 is 0.0693 and $d = 30$. What is $N(30)$?

g. Haldane experimented on goats and concluded that on the ascent to the surface, $N(t)$ should never exceed two times Lung pp N_2. A diver who had been at depth 30 meters for 30 minutes could ascend to what level and not violate this condition if $\frac{k}{V} = 0.0693$?

Haldane supposed that there were five tissues in the body for which $\frac{k}{V} = 0.139$, 0.0693, 0.0347, 0.0173, 0.00924, respectively, and advised that on ascent to the surface, $N(t)$ should never exceed two times pp N_2 in any one of these tissues.

Exercise D.2.25 E. O. Wilson, a pioneer in study of area-species relations on islands, states in *Diversity of Life*, p 221, :

> "In more exact language, the number of species increases by the area-species equation, $S = C\,A^z$, where A is the area and S is the number of species. C is a constant and z is a second, biologically interesting constant that depends on the group of organisms (birds, reptiles, grasses). The value of z also depends on whether the archipelago is close to source ares, as in the case of the Indonesian islands, or very remote, as with Hawaii $\cdots$ It ranges among faunas and floras around the world from about 0.15 to 0.35."

Discuss this statement as a potential Mathematical Model.

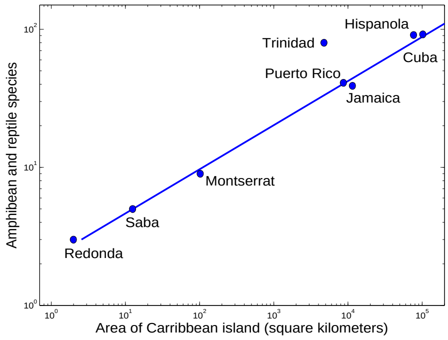
Exercise D.2.26 The graph of Figure D.2.26 showing the number of amphibian and reptile species on Caribbean Islands *vs* the areas of the islands is a classic example from P. J. Darlington, *Zoogeography: The Geographical Distribution of Animals*, Wiley, 1957, page 483, Tables 15 and 16.

a. Treat Trinidad as an unexplained outlier (meaning: ignore Trinidad) and find a power law, $S = C\,A^z$, relating number of species to area for this data.

b. Darlington observes that "$\cdots$ within the size range of these islands $\cdots$, division of the area by ten divides the amphibian and reptile fauna by two $\cdots$, but this ratio is a very rough approximation, and it might not hold in other situations." Is your power law consistent with Darlington's observation?

c. Why might Trinidad (4800 km^2) have nearly twice as many reptilian species (80) as Puerto Rico (8700 km^2) which has 41 species?

Figure for Exercise D.2.26 The number of amphibian and reptile species on islands in the Caribbean *vs* the areas of the islands. The data is from Tables 15 and 16 in P. J. Darlington, *Zoogeography: The Geographical Distribution of Animals*, Wiley, 1957, page 483.



Exercise D.2.27 The graph in Figure Ex. D.2.27 relates surface area to mass of a number of mammals. Assume mammal densities are constant (each two mammals are equally dense), so that the graph also relates surface area to volume.

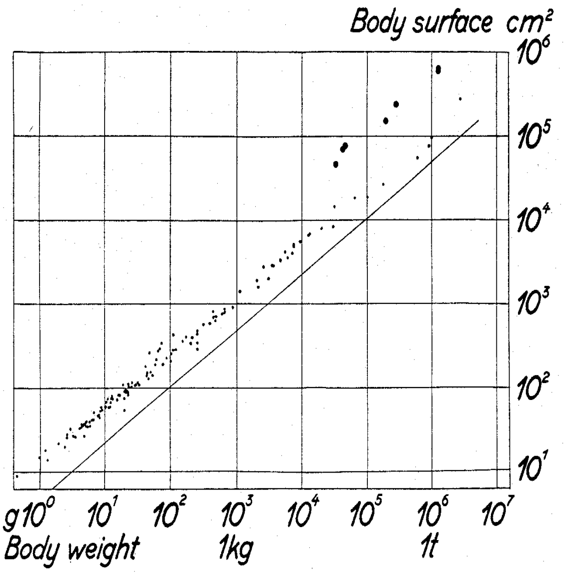
a. Find an equation relating the surface area, S, of a cube to the volume, V, of the cube.

b. Find an equation relating the surface area, $S = 4\pi r^2$, of a sphere of radius r to the volume, $V = \frac{4}{3}\pi r^3$, of the sphere.

c. Find an equation relating the surface area of a mammal to the mass of the mammal, using the graph in Figure Ex. D.2.27. Ignore the dark dots; they are for beech trees.

d. In what way are the results for the first three parts of this exercise similar?

Figure for Exercise D.2.27 Graph for Exercise D.2.27 relating surface area to mass of mammals. From A. M. Hemmingson, Energy metabolism as related to body size, and its evolution, Rep. Steno Mem. Hosp. (Copenhagen) 9:1-110. With kind permission from Dr. Peter R. Rossing, Director of Research, Steno Diabetes Center S/A. All rights reserved.



Exercise D.2.28 Body Mass Indices. The Body Mass Index (BMI) Mass/Height^2 was introduced by Adolphe Quetelet, a French mathematician and statistician in 1869. The Center for Disease Control and Prevention (CDC) notes that BMI is a helpful indicator of obesity and overweight in adults.

Mass is measured in kilograms and height is measured in meters.

From simple allometric considerations, BMI3 = Mass/Height^3 should be approximately a constant, C, and therefore a better comparative measure of obesity and overweight. If

$$BMI3 = \text{Mass/Height}^3 = C \quad \text{then} \quad BMI = \text{Mass/Height}^2 = C \times \text{Height}.$$

so that BMI should increase with height. CDC also states that "$\cdots$ women are more likely to have a higher percentage of body fat than men for the same BMI." If a male and a female both have BMI = 23 and are of average height for their sex (1.77 meters for males and 1.63 meters for females), then

$$\text{BMI3 for the male} = \frac{23}{1.77} = 13.0 \quad \text{and} \quad \text{BMI3 for the female} = \frac{23}{1.63} = 14.1$$

Thus BMI3 is larger for the female than for the male and may indicate a larger percentage of body fat for the female.

Shown are four Age, Weight, Height data points for boys. Compute BMI and BMI3 for the four points. Which of the two remains relatively constant with age?

Age (Boys)	8	12	16	20
Weight (kg) (50 percentile)	26	41	62	71
Height (m) (50 percentile)	1.28	1.49	1.74	1.77
BMI kg/m^2				
BMI3 kg/m^3				

Height and BMI are both given for boys in the 50th percentile.

We suggest that BMI3 might be more useful than BMI as an index of body fat. Other indices of body fat that have been suggested include M/H, $M^{1/3}/H$, $H/M^{1/3}$, and $cM^{1.2}/H^{3.3}$. The interested reader should visit the web site cdc.gov/nccdphp/dnpa/bmi and read the references there.

Exercise D.2.29 a. Complete the table of data for girls of age 8 to 20.

Age (Girls)	8	12	16	20
Weight (kg) (50 percentile)	26	42	54	58
Height (m) (50 percentile)	1.28	1.51	1.62	1.65
BMI kg/m^2				
BMI3 kg/m^3				

Height and BMI are both given for girls in the 50th percentile.

b. Which varies the least, BMI or BMI3?

D.3 From Chapter 7: The Equation $\mathbf{y}'' + \omega^2 y = 0$.

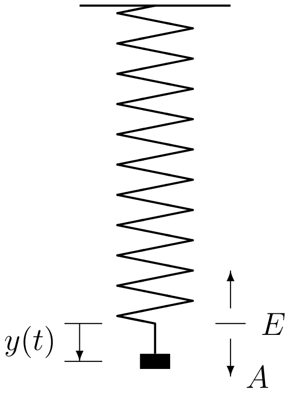
In this section:

A principal use of the sine and cosine functions are in the descriptions of *Harmonic Oscillations*. A mass suspended on a spring that is displaced from equilibrium, E, and released is an example of a simple harmonic oscillator. This spring-mass system is modeled by the equations

$$\begin{aligned} y(0) &= A \qquad \text{Initial displacement.} \\ y'(0) &= 0 \qquad \text{Initial velocity.} \\ y''(t) + \omega^2 y(t) &= 0 \end{aligned}$$

The equations are solved by

$$y(t) = A\cos(\omega t)$$



Harmonic oscillations are ubiquitous in the material world. The sine and cosine functions are called the harmonic functions and at least to first approximation are descriptive of sound waves, light waves, planetary motion, tidal motion, ear drum oscillations, swinging pendula, vibrations of an atom, alternating electrical current, earthquake waves, flutter of a leaf, (the list is quite long).

The most simple equation that applies to oscillating systems in which the resistance to motion is negligible is:

$$y''(t) + \omega^2 y(t) = 0 \tag{D.17}$$

Generally $y(t)$ is the displacement from equilibrium of some measure of the system. The constant ω measures the strength of the force that restores the system to equilibrium. Solutions to the equation are of the form

$$y(t) = A\sin(\omega t) + B\cos(\omega t) \tag{D.18}$$

where A and B are constants that are determined from information about the state of the system at time 0 (for example, $y(0) = 1$, $y'(0) = 0$ implies that $A = 0$ and $B = 1$). All such functions satisfy $y''(t) + \omega^2 y(t) = 0$. That there are no other solutions follows from the uniqueness of solutions to linear differential equations usually established in differential equation courses.

We first establish that if $y(t) = A\sin(\omega t)$ then $y''(t) + \omega^2 y(t) = 0$.

$$\begin{aligned}
y(t) &= A\sin(\omega t) \\[2em]
y'(t) &= A\cos(\omega t) \times [\omega t]' \\
&= A\cos(\omega t) \times \omega \\[2em]
y''(t) &= [A\omega\cos(\omega t)]' \\
&= A\omega(-\sin(\omega t)) \times [\omega t]' \\
&= -A\omega^2\sin(\omega t)
\end{aligned}$$

It is immediate then that $y''(t) + \omega^2 y(t) = 0$ for

$$y''(t) + \omega^2 y(t) = \left(-A\omega^2\sin(\omega t)\right) + \omega^2 \times A\sin(\omega t) = 0$$

You are asked to show in Exercises D.3.4 and D.3.5 that both $y(t) = B\cos(\omega t)$ and $y(t) = A\sin\omega t + B\cos(\omega t)$ solve $y''(t) + \omega^2 y(t) = 0$.

It is easy to visualize the motion of a mass suspended on a spring and we begin there. However, the mathematics involved is the same in many other systems; one of the powers of mathematics is that a single mathematical formulation may be descriptive of many systems.

It is shown in beginning physics courses that if $y(t)$ measures the displacement from the rest position of a body of mass m suspended on a spring (see Figure D.9) with spring constant k, then

$$my''(t) + ky(t) = 0$$

. The equation is derived by equating the two forces on the mass,

$$\begin{array}{rrcll}
\text{Newton's Second Law of Motion} & F_1 & = & \text{mass} \times \text{acceleration} & = m \times y'' \\[1em]
\text{Hooke's Spring Law} & F_2 & = & -k \times \text{spring elongation} & = -k \times y
\end{array}$$

$$F_1 = F_2 \qquad \text{implies that} \qquad my'' = -ky, \qquad \text{or} \qquad my'' + ky = 0. \tag{D.19}$$

Assume that the mass is held motionless a distance A below the equilibrium point and at time $t = 0$ the mass is released. Then

$$y(0) = A \qquad \text{and} \qquad y'(0) = 0$$

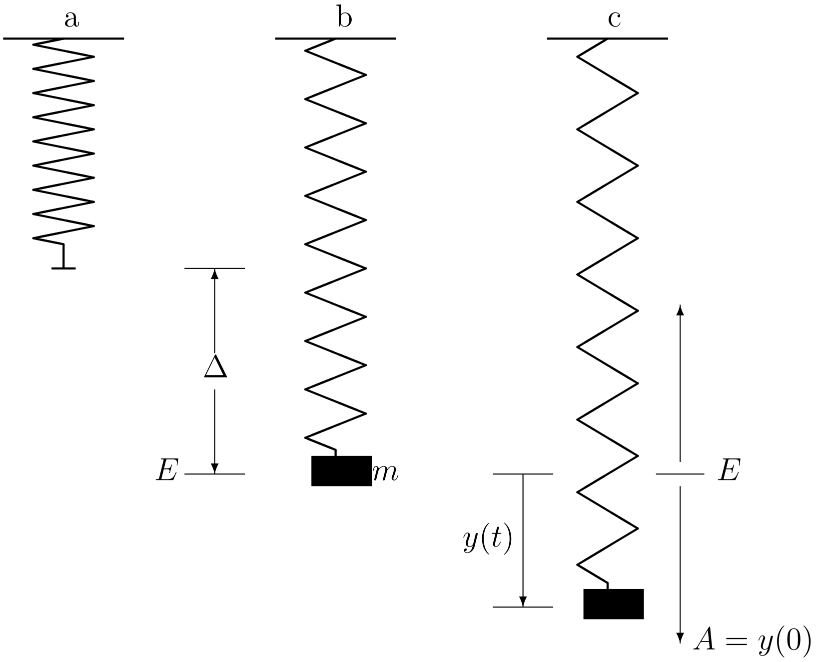


Figure D.9: Oscillation of a mass attached to a spring. a. Relaxed spring with no mass attached. b. A body of mass m is attached and stretches the spring a distance Δ to an equilibrium position E. c. The body is displaced a distance A below the equilibrium point E and released. $y(t)$ is the displacement of the body from E at time t and is positive when the body is below E.

If we let $\omega = \sqrt{\frac{k}{m}}$ so that $\omega^2 = \frac{k}{m}$ we have

$$y(0) = A \qquad y'(0) = 0 \qquad y''(t) + \omega^2 y(t) = 0 \tag{D.20}$$

You will show in Exercise D.3.4 that the function

$$y(t) = A\cos\omega t.$$

satisfies the three conditions of Equations D.20. It gives a good description of the motion of a body suspended on a spring.

The number ω is important. On the time interval $[0, 2\pi/\omega]$, $\cos\omega t$ completes one cycle and the position, $y(t)$, of the body progresses from A to $-A$ and back to A. Thus

$$\frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}} \qquad \text{is the } \textbf{period} \text{ of one oscillation.}$$

One time unit divided by the length of a period of oscillation gives the number of oscillations per unit time and

$$\frac{1}{\frac{2\pi}{\omega}} = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \qquad \text{is the } \textbf{frequency} \text{ of oscillations.}$$

If k/m is 'large' (stiff spring or small mass) the period will be 'short' and the body oscillates rapidly. If k/m is 'small' (weak spring or large mass) the period will be long, the frequency will be low and the body oscillates slowly.

Example D.3.1 Suppose a body of mass m is suspended from a spring with spring constant k.

1. If $m = 20$ gm $= 0.020$ Kg and $k = 0.125$ Newtons/meter and the initial displacement, $y_0 = 5$ cm $=$ 0.05 m, then

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{0.125\ \text{Kg}}{0.020\ \text{Kg-m/s}^2/\text{m}}} = 2.5/\text{s}$$

   and

$$y(t) = 0.05\cos(2.5t)$$

   The period of oscillation is

$$\frac{2\pi}{\omega} = \frac{2\pi}{2.51/\text{s}} = 2.51\ \text{s}.$$

   and the frequency of oscillation is approximately

$$60/2.51 = 23.9 \qquad \text{oscillations per minute.}$$

2. If $m = 5$ gm $= 0.005$ Kg (one-fourth the previous mass) and $k = 0.125$ Newtons/meter then the period and frequency of oscillation would be

$$\begin{aligned} \text{Period} = \quad & 2\pi\sqrt{\frac{m}{k}} = \quad 2\pi\sqrt{\frac{0.005}{0.125}} = 2\pi \times 0.2 \doteq 1.25 \quad \text{seconds per oscillation} \\ \text{Frequency} = \quad & \frac{1}{1.25} \quad \text{oscillations per second} \quad = \quad 48 \quad \text{oscillations per minute.} \end{aligned}$$

   Thus one-fourth the mass oscillates twice as fast.

3. If $m = 20gm$ and the spring extends 16 cm when the body is attached to it, then the spring constant, k is

$$k = \frac{\text{Force}}{\text{Extension}} = \frac{0.02 \times 9.8\ \text{Kg-Force}}{0.16\ \text{meter}} \times \frac{1\ \text{Newton}}{9.8\ \text{Kg-Force}} = 0.125 \qquad \text{Newton/meter}$$

It is a curious consequence of the previous analysis that the magnitude of the gravitational field, g, is not reflected in the model equation nor in the solution equation. The role of g is to determine equilibrium location, E. The period and frequency of the oscillations would be the same on the Moon as on Earth.

There are two important omissions in the previous analysis. We have ignored the mass of the spring (which will also be moving) and we have ignored resistance to movement (by the air and in the spring).

D.3.1 Resistance.

In most systems, the amplitudes of the oscillations decrease with time due to resistance to the movement or *friction* in the system. Resistance is a force directed opposite to the direction of motion and may be modeled by

$$\text{Resistance} = \quad -r \times y'(t)$$

Including the force of resistance with the force of the spring, Equation D.19 is modified to

$$my''(t) = -ky(t) - ry'(t)$$

or

$$my''(t) + ry'(t) + ky(t) = 0 \tag{D.21}$$

This is referred to as the equation of *damped* motion.

Example D.3.2 Suppose $m = 20$ gm $= 0.020$ kg, r $= 0.06$ Newtons/(meter/sec) and $k = 0.125$ Newtons/meter. Then the equation of damped motion is

$$0.02y''(t) + 0.06y'(t) + 0.125y(t) = 0 \tag{D.22}$$

We show that a solution to this equation is

$$y(t) = e^{-1.5t}\cos(2t)$$

$$\begin{aligned}
y'(t) &= \left[e^{-1.5t}\cos(2t)\right]' \\
&= \left[e^{-1.5t}\right]'\cos(2t) + e^{-1.5t}\left[\cos(2t)\right]' \\
&= e^{-1.5t}(-1.5)\cos 2t + e^{-1.5t}(-\sin 2t)\,2 \\
&= -1.5e^{-1.5t}\cos 2t - 2e^{-1.5t}(\sin 2t) \\
y''(t) &= \left[-1.5e^{-1.5t}\cos 2t - 2e^{-1.5t}(\sin 2t)\right]' \\
&= -1.5\left(\left[e^{-1.5t}\right]'\cos 2t + e^{-1.5t}\left[\cos 2t\right]'\right) - 2\left(\left[e^{-1.5t}\right]'\sin 2t + e^{-1.5t}\left[\sin 2t\right]'\right) \\
&= (-1.5)^2e^{-1.5t}\cos 2t + 2(-1.5)(-2)e^{-1.5t}\sin 2t - 2^2e^{-1.5}\cos 2t \\
&= -1.75e^{-1.5t}\cos 2t + 6e^{-1.5t}\sin 2t
\end{aligned}$$

Now we set up a table of coefficients and terms of Equation D.22.

$$\begin{array}{lrrcl} 0.02 & y''(t) & -1.75e^{-1.5t}\cos 2t & + & 6e^{-1.5t}\sin 2t \\ 0.06 & y'(t) & -1.5e^{-1.5t}\cos 2t & - & 2e^{-1.5t}\sin 2t \\ 0.125 & y(t) & e^{-1.5t}\cos 2t & & \end{array}$$

After substitution into Equation D.22 the coefficients of $e^{-1.5t}\cos 2t$ and $e^{-1.5t}\sin 2t$ are

$$0.02 \times (-1.75) + 0.06 \times (-1.5) + 0.125 \times 1 = 0.0 \qquad \text{and} \qquad 0.02 \times 6 + 0.06 \times (-2) = 0.0$$

so

$$y(t) = e^{-1.5t}\cos(2t) \qquad \text{solves} \qquad 0.02y''(t) + 0.06y'(t) + 0.125y(t) = 0.$$

In the previous problem, the resistance, $r = 0.06$ was selected to make the numbers in the solution (-1.5 and 2) reasonably tractable. The resistance, $r = 0.06$ is so great, however, that the oscillations are imperceptable after only two or three oscillations, as illustrated in Figure D.10A.

If $r = 0.002$ then the solution is

$$\begin{aligned} y(t) &= e^{-0.1t}\cos(\sqrt{6.2475}t) \\ &\doteq e^{-0.1t}\cos(2.4995t) \end{aligned}$$

A graph of $y(t) = e^{-0.1t}\cos(2.4995t)$ is shown in Figure D.10B.

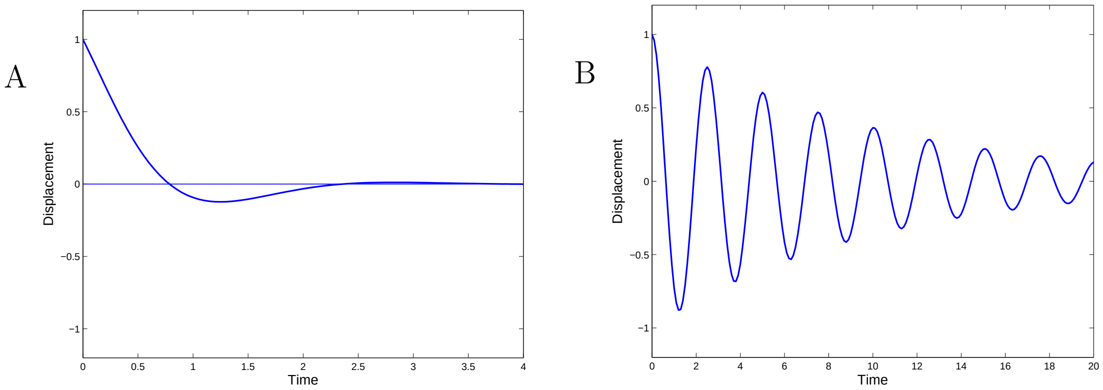


Figure D.10: A. Graph of $y(t) = e^{-1.5t}\cos(2t)$. B. Graph of $y(t) = e^{-0.1t}\cos(2.4995t)$.

If the resistance in a vibrating system is quite large (system is buried in molasses), the system may not vibrate at all but may just ooze back to equilibrium after a displacement. Equation D.21, $my''(t) + ry'(t) + ky(t) = 0$, may be written (divide by m)

$$y''(t) + 2by'(t) + cy(t) = 0 \tag{D.23}$$

The solutions to Equation D.23 are **Bolts of Lightning**

$$\left.\begin{array}{llll} 1. & y(t) & = & A\,e^{(-b+\sqrt{b^2-c})t} + B\,e^{(-b-\sqrt{b^2-c})t} & \text{if} \quad b^2-c>0 \\ \\ 2. & y(t) & = & e^{-bt}(A+Bt) & \text{if} \quad b^2-c=0 \\ \\ 3. & y(t) & = & A\,e^{-bt}\sin\sqrt{c-b^2}t + B\,e^{-bt}\cos\sqrt{c-b^2}t & \text{if} \quad b^2-c<0 \end{array}\right\} \qquad (D.24)$$

where A and B are constants determined by $y(0)$ and $y'(0)$. All three can be shown by substitution to solve Equation D.23.

A system with $b^2 - c > 0$ is 'over damped' and does not oscillate. For this condition, $b = \frac{r}{2m}$, $c = \frac{k}{m}$, and

$$b^2 - c = \left(\frac{r}{2m}\right)^2 - \frac{k}{m} = \frac{r^2 - 4km}{4m^2}.$$

The condition for overdamping, no oscillation in the system, is $r^2 > 4km$ – the square of the resistance is greater than 4 times the spring constant times the mass.

If the formulas above remind you of the roots to a quadratic polynomial, it is not an accident; the connection is shown in Exercise D.3.6.

The body suspended on a spring is easy to experiment with and typifies many oscillations that occur throughout nature. Other mechanical systems that have similar oscillations include the swinging pendulum and a rotating disc (as in the flywheel of a watch). Less apparent oscillating systems include diatomic molecules in which the distance between the two atoms oscillates very rapidly but can be approximated with the harmonic equations. In the next section we will give a simplified biological example of oscillations in predator-prey systems.

Exercises for Section D.3, The Equation $\mathbf{y}'' + \omega^2 y = 0$.

Exercise D.3.1 Show that the proposed solutions satisfy the equations and initial conditions.

	Solution	Derivative Equation	Initial conditions
a.	$y(t) = 2\sin t + \cos t$	$y'' + y = 0$	$y(0) = 1$ $y'(0) = 2$
b.	$y(t) = 4\cos 2t$	$y'' + 4y = 0$	$y(0) = 4$ $y'(0) = 0$
c.	$y(t) = \cos 3t - \sin 3t$	$y'' + 9y = 0$	$y(0) = 1$ $y'(0) = -3$
d.	$y(t) = -20\sin 5t + 15\cos 5t$	$y'' + 25y = 0$	$y(0) = 15$ $y'(0) = -100$
e.	$y(t) = 4\cos(3t + \pi/3)$	$y'' + 9y = 2$	$y(0) = 2$ $y'(0) = -6\sqrt{3}$
f.	$y(t) = \sin 2t - 2\cos 2t$	$y'' + 4y = 0$	$y(0) = -2$ $y'(0) = 2$
g.	$y(t) = 2\sin 3t + 3\cos 3t$	$y'' + 9y = 0$	$y(0) = 3$ $y'(0) = 6$
h.	$y(t) = 3\sin \pi t + 4\cos \pi t$	$y'' + \pi^2 y = 0$	$y(0) = 4$ $y'(0) = 3\pi$
i.	$y(t) = e^{-t}\sin t$	$y'' + 2y' + 2y = 0$	$y(0) = 0$ $y'(0) = 1$
j.	$y(t) = e^{-0.1t}\cos 2t$	$y'' + 0.2y' + 4.01y = 0$	$y(0) = 1$ $y'(0) = -0.1$
k.	$y(t) =$ $e^{-0.1t}\,(0.1\sin 2t + 2\cos 2t)$	$y'' + 0.2y' + 4.01y = 0$	$y(0) = 2$ $y'(0) = 0$

Exercise D.3.2 Find a number ω so that the proposed solution satisfies the derivative equation.

	Solution	Derivative equation
a.	$y(t) = 3\cos 5t$	$y\prime\prime + \omega^2 y = 0$
b.	$y(t) = 2\sin 3t + 5\cos 3t$	$y\prime\prime + \omega^2 y = 0$
c.	$y(t) = -4\cos \pi t$	$y\prime\prime + \omega^2 y = 0$
d.	$y(t) = 3e^{-t}\cos 5t$	$y\prime\prime + 2y' + \omega^2 y = 0$
e.	$y(t) = -4e^{-2t}\sin 3t$	$y\prime\prime + 4y' + \omega^2 y = 0$

Exercise D.3.3 Find a number k so that the proposed solution satisfies the derivative equation.

	Solution	Derivative equation
a.	$y(t) = e^{-2t}\cos 5t$	$y'' + ky' + 29y = 0$
b.	$y(t) = 5e^{-3t}\sin t$	$y'' + ky' + 10y = 0$
c.	$y(t) = 3e^{-2t}\cos 3t$	$y'' + ky' + 13y = 0$
d.	$y(t) = e^{-0.1t}\cos t$	$y'' + ky' + 1.01y = 0$
e.	$y(t) = \cos 5t$	$y'' + ky' + 25y = 0$

Exercise D.3.4 Show that if B and ω are constants and $y(t) = B\cos(\omega t)$, then

$$y(0) = B \qquad y'(0) = 0 \qquad \text{and} \qquad y''(t) + \omega^2 y(t) = 0.$$

Exercise D.3.5 Show that if A, B and ω are constants and $y(t) = A\sin(\omega t) + B\cos(\omega t)$, then

$$y(0) = B \qquad y'(0) = \omega A \qquad \text{and} \qquad y''(t) + \omega^2 y(t) = 0.$$

Exercise D.3.6 Recall Equation D.23

$$y''(t) + 2by'(t) + cy(t) = 0$$

and suppose that m is a number such that $y(t) = e^{mt}$ solves this equation. Compute $y'(t) = [\,e^{mt}\,]'$ and $y''(t)$. Substitute them into the equation, observe that e^{mt} is never zero, and conclude that

$$m^2 + 2bm + c = 0 \qquad \text{and} \qquad m = -b + \sqrt{b^2 - c} \qquad \text{or} \qquad m = -b - \sqrt{b^2 - c}$$

From this we conclude that

$$y(t) \qquad \text{is either} \qquad e^{(-b+\sqrt{b^2-c})t} \qquad \text{or} \qquad e^{(-b-\sqrt{b^2-c})t}$$

Both are solutions to Equation D.23. If $b^2 - c > 0$, these are the terms in solution 1 of Equations D.24. The exact condition $b^2 - c$ seldom occurs in nature. The issue for $b^2 - c < 0$ is how to interpret

$$e^{(-b+\sqrt{b^2-c})\,t} = e^{(-b+i\sqrt{c-b^2})t} = e^{-bt}e^{i\sqrt{c-b^2}\,t}$$

where $i = \sqrt{-1}$. **Warning: Incoming Bolt from the Blue.** The answer is that[10]

$$e^{i\sqrt{c-b^2}t} = \cos\sqrt{c-b^2}\,t + i\sin\sqrt{c-b^2}\,t.$$

This suggests (to some people at least) that

$$y(t) = e^{-bt}\cos\sqrt{c-b^2}\,t + ie^{-bt}\sin\sqrt{c-b^2}\,t$$

is a solution to Equation D.23 **BANG**. Because Equation D.23 has real number coefficients, some people think that the real and imaginary parts of $y(t)$ should each solve Equation D.23. They do, and with tenacity you can show that they do. See Exercise D.3.7.

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[10] We will try to convince you that this is reasonable in Chapter 9.

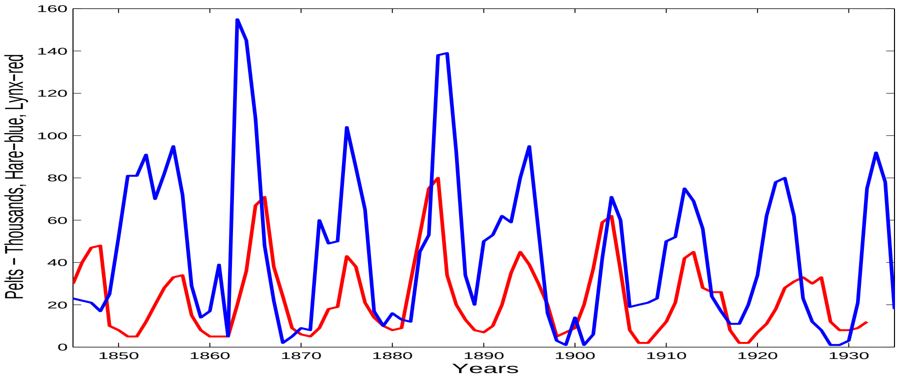

Figure D.11: Graph of snowshoe shoe hare and lynx pelts purchased by the Hudson Bay company for the years 1845 to 1945. Data read from Figures 3 (hare) and 16 (lynx) of D. A. MacLulich, Fluctuations in the number of varying hare, University of Toronto Studies, Biological Sciences, No. 43, 1937.

Exercise D.3.7 At least, show that

$$y(t) = e^{-bt}\cos(\sqrt{c-b^2}\;\; t)$$

solves

$$y''(t) + 2by'(t) + cy(t) = 0 \qquad \text{for} \qquad c - b^2 > 0$$

Exercise D.3.8 The complete solution to Exercise D.3.7 is given in the Solutions Section on the web. Try your hand with showing that

$$y(t) = e^{-bt}\cos(\sqrt{c-b^2}\;\; t)$$

solves

$$y''(t) + 2by'(t) + cy(t) = 0 \qquad \text{for} \qquad c - b^2 > 0$$

Get a big piece of paper.

D.4 From Chapter 7: Elementary predator-prey oscillation.

Predator-prey systems are commonly cited examples of periodic oscillation in biology. Data from trapping records of the snowshoe hare and lynx gathered by trappers and sold to the Hudson Bay Company are among the most popular first introduction. Shown in Figure D.11 is a graph showing the numbers of pelts purchased by the Hudson Bay Company for the years 1845 to 1935, and in Table D.1 are values read from the graph. [11]

Explore D.4.1 Do this. Complete the phase graph shown in Figure D.4.1 using data from Table D.1. Plot the points for the years 1915, 1917, and 1919 and draw the missing lines. ■

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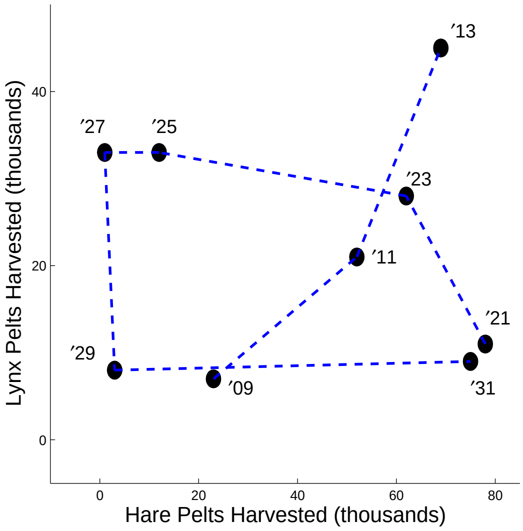
[11] Recent studies also demonstrate the fluctuations as shown on the web site http://lynx.uio.no/catfolk/sp-accts.htm. "Lynx density fluctuates dramatically with the hare cycle (Breitenmoser et al. Oikos 66 (1993), pp. 551-554). An ongoing long-term study of an unexploited population in good quality habitat in the Yukon found densities of 2.8 individuals (including kittens) per 100 km2 during the hare low, and 37.2 per 100 km2 during the peak (G. Mowat and B. Slough, unpubl. data)."

Table D.1: Part of the data read from D. A. MacLulich, *ibid.*

Hare Lynx Data								
Year	Hare	Lynx	Year	Hare	Lynx	Year	Hare	Lynx
1909	23	9	1919	20	10	1929	1	15
1911	52	20	1921	62	20	1931	21	15
1913	69	40	1923	80	25			
1915	24	50	1925	23	40			
1917	11	25	1927	8	40			

Explore Figure D.4.1 Phase graph axes for hare and lynx data.



The *phase graph* that you just drew is a good way to display the interaction between two populations. You should see a general counter clockwise direction to the graph. When you are in the right-most portion of the region with large hare population, the lynx population is increasing (the curve goes up). As you get to the upper right corner the lynx population has increased sufficiently that the hare population decreases (the curve goes to the left). And the pattern continues. We will return to the lynx-hare data in Exercise D.4.6 and find that there are exceptions to this pattern in the data.

Nils Chr. Stenseth, Wilhelm Falck, Ottar N. Bjrnstad, and Charles J. Krebs, argue in Population regulation in snowshoe hare and Canadian lynx: Asymmetric food web configurations between hare and lynx, *Proc. Nat. Acad. Sci. USA* **95** (1995), 5147-5152, that

> ” $\cdots$the classic view of a symmetric harelynx interaction is too simplistic. Specifically, we argue that the classic food chain structure is inappropriate: the hare is influenced by many predators other than the lynx, and the lynx is primarily influenced by the snowshoe hare.”

A Predator-Prey Model. Assume that there are two populations that interact as predator and prey in a reasonably isolated environment. Let $U(t)$ denote the number of prey and $V(t)$ denote the number of predators, and assume there are equilibrium values, U_e and V_e, so that U_e prey would provide enough food for V_e predators to just maintain their numbers (predator birth rate = predator death rate) and V_e predators would just balance the often excess birth rate of the prey (prey birth rate = prey death rate).

Shown in Figure D.12 is an axis system where the horizontal axis is $U(t)$ and the vertical axis is $V(t)$. An equilibrium point, (U_e, V_e), is plotted. If for some time, t, the populations are not at equilibrium, we let

$$\begin{aligned} u(t) &= U(t) - U_e \\ v(t) &= V(t) - V_e \end{aligned}$$

measure the departures from equilibrium.

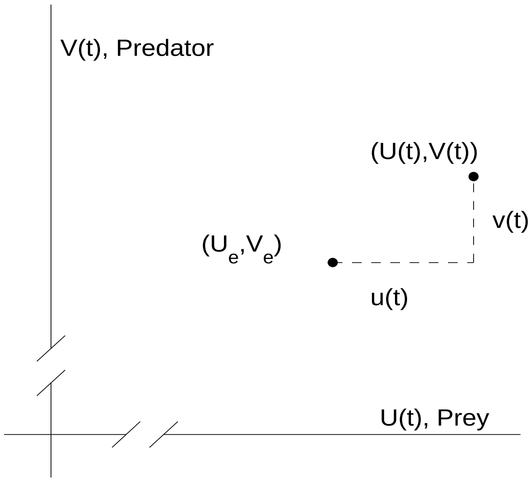


Figure D.12: Axes for a predator prey phase graph. The gaps in the axes allow $u(t)$ to be small compared to U_e and $v(t)$ to be small compared to V_e.

Suppose the populations $U(t)$ and $V(t)$ are in equilibrium and the predator population increases (perhaps some predators immigrate into the system). The excess predators would increase capture of prey, and we could expect the prey population to *decrease*. Alternatively, if the prey should become more numerous, the predators would have a greater food supply and their numbers may *increase*.

Mathematical Model D.4.1 For *small* deviations, $u(t)$ and $v(t)$, from equilibrium, we assume that

1. The rate of prey population *decrease*, $-U'(t)$, is proportional to the excess predator population, $v(t)$.
2. The rate of predator population *increase*, $V'(t)$, is proportional to the excess prey population, $u(t)$.

Thus from Part 1 we write

$$-U'(t) = a \times v(t) \qquad \text{or} \qquad U'(t) = -a \times v(t)$$

By the model, if the predator population exceeds equilibrium, $V(t) > V_e$ ($v(t) > 0$), then $U'(t) < 0$ and the prey population will decrease. However, if the predator population is less than normal, $V(t) < V_e$ ($v(t) < 0$), then $U'(t) > 0$ and the prey population increases. For this model, both populations must be assumed to be close to equilibrium. For example, a prey population greatly exceeding equilibrium, U_e might support a predator population slightly above equilibrium V_E and still grow.

Because $u(t) = U(t) - U_e$, $U(t) = U_e + u(t)$ and

$$U'(t) \;=\; u'(t).$$

We write

$$u'(t) = -a \times v(t)$$

Similarly,

$$v'(t) = b \times u(t)$$

Explore D.4.2 Show that Part 2 of the Mathematical Model D.4.1 leads to the equation

$$v'(t) = bu(t)$$

where b is a proportionality constant. ■

The two equations

$$\begin{aligned} u'(t) &= -av(t) \\ v'(t) &= bu(t) \end{aligned}$$

describe the dynamics of the predator – prey populations. There are two unknown functions, u and v, and the equations are linked, because u' is related to v and v' is related to u. There is a general procedure to obtain a single equation involving only u, as follows:

$$\begin{aligned} u'(t) &= -av(t) && \text{First Original Equation.} \\ [u'(t)]' &= [-av(t)]' && \text{Differentiate First Eq.} \\ u''(t) &= -av'(t) && \\ u''(t) &= -a(bu(t)) && \text{Substitute Second Eq.} \\ u''(t) + (ab)u(t) &= 0 && \end{aligned}$$

Now we let $\omega = \sqrt{ab}$ so that $\omega^2 = ab$ and write

$$u''(t) + \omega^2 u(t) = 0 \tag{D.25}$$

and see that it is equivalent to the dynamic equation in Equations D.20. To complete the analogy, we need $u(0)$ and $u'(0)$.

Suppose we have a predator-prey pair of populations and because of some disturbance to the environment (rain, cold, or fire, for example) at a time, $t = 0$, the populations are at (U_0, V_0), close to but different from the equilibrium values (U_e, V_e). Let the departures from equilibrium be

$$u_0 = U_0 - U_e \qquad \text{and} \qquad v_0 = V_0 - V_e$$

Then clearly we will use $u(0) = u_0$. Also, from $u'(t) = -av(t)$ we will get $u'(0) = -av(0) = -av_0$. Thus we have the complete system

$$u(0) = u_0 \qquad u'(0) = -av_0 \qquad u''(t) + \omega^2 u(t) = 0 \tag{D.26}$$

From Equation D.17 $y''(t) + \omega^2 y(t) = 0$, and its solution, Equation D.18, $y(t) = A\sin(\omega t) + B\cos(\omega t)$, we conclude that $u(t)$ will be of the form

$$u(t) = A\sin(\omega t) + B\cos(\omega t)$$

where A and B are to be determined. Observe that

$$u'(t) = A\omega\cos(\omega t) - B\omega\sin(\omega t)$$

Now,

$$\begin{aligned} u(0) &= A\sin(\omega 0) + B\cos(\omega 0) = A \times 0 + B \times 1 = B \\ \\ u'(0) &= A\omega\cos(\omega 0) - B\omega\sin(\omega 0) = A\omega \times 1 - B\omega \times 0 = A\omega \end{aligned}$$

It follows that

$$B = u_0 \qquad \text{and} \qquad A\omega = -av_0, \qquad \text{so that} \qquad A = -\frac{a}{\omega}v_0$$

and the solution is

$$u(t) = -\frac{a}{\omega}v_0\sin(\omega t) + u_0\cos(\omega t)$$

Remembering that $\omega = \sqrt{ab}$ we may write

$$\begin{aligned} u(t) &= -\frac{a}{\sqrt{ab}}v_0\sin(\sqrt{ab}\,t) + u_0\cos(\sqrt{ab}\,t) \\ &= -v_0\sqrt{\frac{a}{b}}\sin(\sqrt{ab}\,t) + u_0\cos(\sqrt{ab}\,t) \end{aligned} \tag{D.27}$$

Explore D.4.3 Also remember the first equation, $u'(t) = -av(t)$ and show that

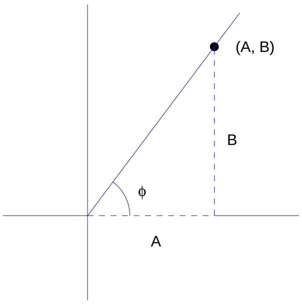
$$v(t) = v_0\cos(\sqrt{ab}\,t) + u_0\sqrt{\frac{b}{a}}\sin(\sqrt{ab}\,t) \tag{D.28}$$

■

A little trigonometry It often happens that a periodic oscillation is the sum of two oscillations of the same frequency as in Equations D.27 and D.28. When that happens, the two can be combined into a single sine function. Suppose $F(t) = A\sin(\omega t) + B\cos(\omega t)$ where A, B, and ω are numbers.

Let ϕ be an angle such that

$$\cos\phi = \frac{A}{\sqrt{A^2+B^2}}$$
$$\sin\phi = \frac{B}{\sqrt{A^2+B^2}}$$



Then

$$\begin{aligned} F(t) &= A\sin(\omega t) + B\cos(\omega t) \\ &= \sqrt{A^2+B^2}\left(\frac{A}{\sqrt{A^2+B^2}}\sin(\omega t) + \frac{B}{\sqrt{A^2+B^2}}\cos(\omega t)\right) \\ &= \sqrt{A^2+B^2}\,(\cos\phi\sin(\omega t) + \sin\phi\cos(\omega t)) \\ &= \sqrt{A^2+B^2}\sin(\omega t + \phi) \end{aligned}$$

The last step uses $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$.

Exercises for Section D.4, Elementary predator-prey oscillation.

Exercise D.4.1 Suppose that $a = b = 1$ and $u_0 = 3$ and $v_0 = 4$ in the prey equation D.27 so that

$$u(t) = -4\sin(t) + 3\cos(t).$$

(u_0 and v_0 are 'small' disturbances. We might suppose, for example, that the equilibrium populations are $U_e = 300$, $V_e = 200$, with 3 and 4 'small' with respect to 300 and 200).

a. Sketch the graph of $u(t) = -4\sin(t) + 3\cos(t)$.

b. Let ϕ (the Greek letter *phi*) denote the angle between 0 and 2π whose sine is $\frac{3}{5}$ and whose cosine is $\frac{-4}{5}$. Show that

$$\begin{aligned} u(t) &= 5(\cos\phi\sin t + \sin\phi\cos t) \\ &= 5\sin(t+\phi) \end{aligned}$$

c. Plot the graph of $5\sin(t+\phi)$ and compare it with the graph of $-4\sin t + 3\cos t$.

Exercise D.4.2 Compare Equation D.27, $u(t) = -v_0\sqrt{\frac{a}{b}}\sin(\sqrt{ab}\,t) + u_0\cos(\sqrt{ab}\,t)$ for the two cases:
Case 1: $a = b = 1$ and $u_0 = 3$ and $v_0 = 4$ (in the previous problem).
Case 2: $a = 4$, $b = 1$ and $u_0 = 3$ and $v_0 = 4$

What is the biological interpretation of the change from $a = 1$ to $a = 4$?

Exercise D.4.3 a. Find the formula for the *predator* population Equation D.28 using the parameters, $a = b = 1$, $u_0 = 3$ and $v_0 = 4$.

b. Let ψ (Greek letter *psi*) be the angle between 0 and 2π for which

$$\cos\psi = \frac{3}{5} \qquad \text{and} \qquad \sin\psi = \frac{4}{5}$$

and show that

$$v(t) = 5\sin(t + \psi)$$

Exercise D.4.4 Develop the *predator* harmonic equation.

a. Examine the steps leading to Equations D.26 and show that

$$v(0) = v_0 \qquad v'(0) = bu_0 \qquad v''(t) + \omega^2 v(t) = 0 \qquad \text{with} \quad \omega^2 = ab \tag{D.29}$$

b. Rewrite this system for $a = b = 1$, $u_0 = 3$ and $v_0 = 4$ and conclude that the solution is

$$v(t) = 3\sin t + 4\cos t$$

c. Find a formula for $v(t)$ (predator) using the formula from Exercise D.4.1 $u(t) = -4\sin t + 3\cos t$ (prey) and the equation $u'(t) = -av(t)$ $(a = 1)$.

Exercise D.4.5 The previous three exercises show that for $a = b = 1$, $u_0 = 3$ and $v_0 = 4$

$$\begin{aligned} u(t) &= -4\sin(t) + 3\cos(t) &&= 5\sin(t + 2.498) \\ v(t) &= 3\sin t + 4\cos t &&= 5\sin(t + 0.927) \end{aligned}$$

Graphs of u and v are displayed in two ways in Figure D.13. In Figure D.13A are the conventional graphs $u(t)$ *vs* t and $v(t)$ *vs* t. In Figure D.13B is a $V(t) = V_e + v(t)$ *vs* $U(t) = U_e + u(t)$ (U_e and V_e not specified).

At time $t = 0$ the excess predator and prey populations are both positive ($u_0 = 3$ and $v_0 = 4$).

a. Replicate Figure D.13 on your paper.

b. What are $u'(0)$ and $v'(0)$?

c. On your replica of Figure D.13A, draw tangents to the graphs of u and v at $u(0)$ and $v(0)$.

d. On your replica of Figure D.13B draw a line from (U_0, V_0) to $(U_0 + u'(0), V_0 + v'(0))$. Note that the distance from (U_e, V_e) to (U_0, V_0) is 5.

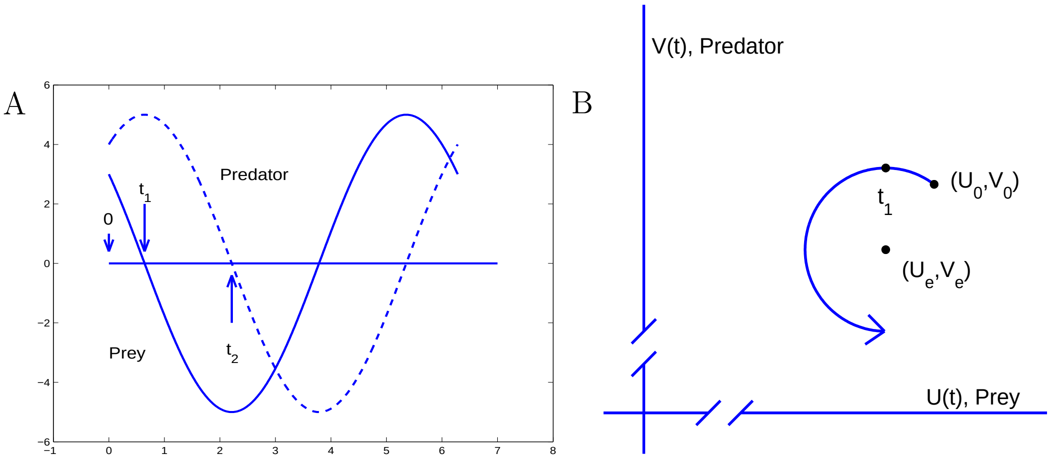


Figure D.13: A. Conventional and B. phase graphs of predator prey functions. For scale, the distance from (U_e, V_e) to (U_0, V_0) is 5.

e. In Figure D.13A, at the time $t = 0$ the prey curve has negative slope and the predator curve has positive slope. The graph in Figure D.13B moves from (U_0, V_0) to the left (prey is decreasing) and upward (predator is increasing).

1. Discuss the dynamics of the predator and prey populations at time t_1.
2. Discuss the dynamics of the predator and prey populations at time t_2 marked on Figure D.13A and mark the corresponding point on Figure D.13B.

f. Show that

$$(\,u(t)\,)^2 + (\,v(t)\,)^2 = u_0^2 + v_0^2$$

What is the significance of this equation?

Exercise D.4.6 J. D. Murray *Mathematical Biology*, Springer, New York, 1993, p 66 observes an exception to the phase graph for the Lynx-Hare during years 1874-1904. The data are shown in Figure D.14 along with a table of part of the data.

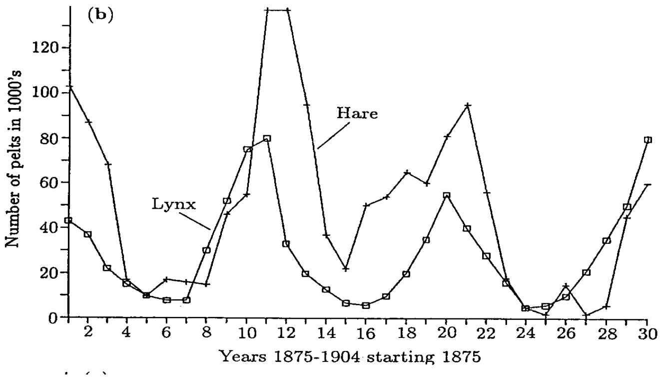


Figure D.14: Snowshoe Hare and Lynx data for the years 1874-1904.

Table D.2: Lynx-Hare data read Figures 3 (hare) and 16 (lynx) of D. A. MacLulich, Fluctuations in the number of varying hare, University of Toronto Studies, Biological Sciences, No. 43, 1937.

Hare Lynx Data Read From the Graph								
Year	Hare	Lynx	Year	Hare	Lynx	Year	Hare	Lynx
1875	104	43	1879	10	10	1883	45	53
1876	85	38	1880	16	8	1884	53	75
1877	65	21	1881	13	9	1885	138	80
1878	17	14	1882	12	31	1886	139	34

a. Read data points in Figure D.14 for the years 1887 and 1888.

b. Use the data in Table D.2 and your two data points to make a phase plot for the years 1875 - 1888.

c. Discuss the peculiarity of this predator-prey phase plot.

Murray notes that the 1875 - 1887 data seems to show that the 'hares are eating the lynx', and cites some explanations that have been offered, including a possible hare disease that could kill the lynx (no such disease is known) and variation in trapping practice in years of low population density.

Exercise D.4.7 For the algebraically robust. For Equations D.27 and D.28,

$$u(t) = -v_0\sqrt{\frac{a}{b}}\sin(\sqrt{ab}\,t) + u_0\cos(\sqrt{ab}\,t) \qquad v(t) = v_0\cos(\sqrt{ab}\,t) + u_0\sqrt{\frac{b}{a}}\sin(\sqrt{ab}\,t)$$

show that

$$\sqrt{\frac{b}{a}}\,(\,u(t)\,)^2 + \sqrt{\frac{a}{b}}\,(\,v(t)\,)^2 = \sqrt{\frac{b}{a}}\,(\,u_0\,)^2 + \sqrt{\frac{a}{b}}\,(\,v_0\,)^2\,.$$

What is the significance of this equation?

Exercise D.4.8 Suppose the prey population may be affected by over crowding even with low predator presence.

Mathematical Model D.4.2 For *small* deviations, $u(t)$ and $v(t)$, from equilibrium, we assume that

1. The prey population *decreases*, $-U'(t)$, when there is excess prey population $u(t)$ and when there is excess predator population, $v(t)$.
2. The predator population *increase*, $V'(t)$, is proportional to the excess prey population, $u(t)$.

Step 1 can be interpreted at least two ways:
A. $-U'(t)$ could be proportional to the *product* of $u(t)$ and $v(t)$, or
B. $-U'(t)$ could be proportional to the *sum* of $av(t) + cu(t)$ where a and c are numbers.

Both interpretations are relevant. Here we choose interpretation B. Step 2 is the same as for Model D.30 and write

$$\begin{aligned} -U'(t) &= a\,v(t) + c\,u(t) \\ \\ V'(t) &= b\,u(t) \end{aligned}$$

Again $U'(t) = u'(t)$ and $V'(t) = v'(t)$, so that

$$\begin{aligned} u'(t) &= -a\,v(t) - c\,u(t) \\ v'(t) &= b\,u(t) \end{aligned} \tag{D.30}$$

a. Use Equations D.30 to show that

$$v''(t) + c\,v'(t) + ab\,v(t) = 0 \tag{D.31}$$

Hint: Compute

$$[\,v'(t)\,]' = [\,b\,u(t)\,]', \qquad \text{use} \qquad u'(t) = -a\,v(t) - c\,u(t) \qquad \text{and} \qquad b\,u(t) = v'(t)$$

b. Equation D.31 may be compared with Equation D.21 for damped harmonic motion (harmonic motion with resistance). For $a = 2.02$, $b = 0.5$, and $c = 0.2$ the equation becomes

$$v''(t) + 0.2\,v'(t) + 1.01\,v(t) = 0 \tag{D.32}$$

Show that

$$v(t) = e^{-t/10}\sin t$$

is a solution to this equation.

c. Show that if $v(t) = e^{-t/10}\sin t$ then

$$u(t) = 2e^{-t/10}\cos t - 0.2e^{-t/10}\sin t$$

d. Show that

$$u^2 + 0.4\,uv + 4.04\,v^2 = 4e^{-t/5}$$

e. Plot a graph of $u(t)$ *vs* $v(t)$. It is of interest that $u^2 + 0.4\,uv + 4.04\,v^2 = 4$ is an ellipse in the u-v plane.

D.5 From Chapter 7: Periodic systems.

Many biological and physical systems exhibit periodic variation governed by feed back of information from the state of the system to the driving forces of the system. An excess of predators (state of the system) drives down (driving force) the prey population. An elongation of a spring (state of the system) causes (driving force) the suspended mass to move up toward the equilibrium position.

Examples of periodically varying feed back systems are presented. Exercises are distributed through the three subsections.

Explore D.5.1 Chamelons are a group of lizards that change their color to match the color of their environment. What is the color of a chamelon placed on a mirror? ■

D.5.1 Control switches.

Some street lights and household night lights have photosensitive switches that turn the lights on at sunset and turn them off at sunrise. In Figure D.5.1 is a household night light with a photosensitive switch and a mirror. The mirror can be adjusted so that the light from the bulb is reflected back to the photosensitive switch. What will be the behavior of the switch at night? With sunlight shining on it?

Explore D.5.2 You will find it interesting to perform an experiment. In a dark room, hold a mirror about 3 inches from a night light so that it reflects light from a night light back to the photosensitive switch. Move the mirror about 8 inches from the night light and note the change in the activity of the night light. ■

We propose the following mathematical model for the system.

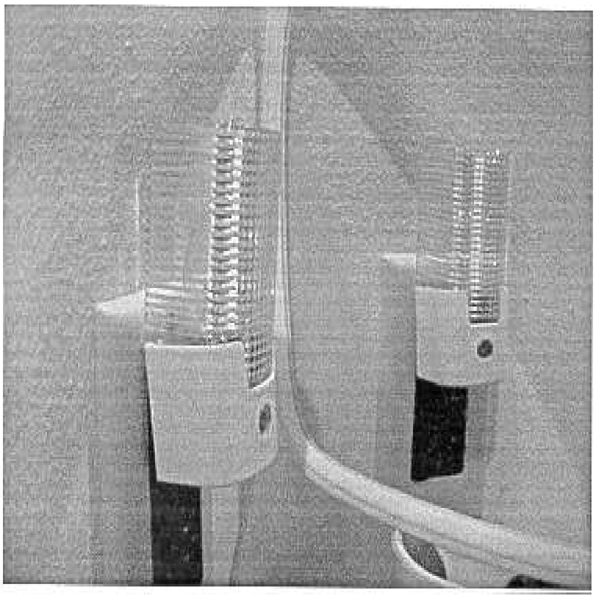
Mathematical Model D.5.1 Night lights. I There is a voltage in the photosensitive switch that increases at a rate proportional to the intensity of light striking the switch and the intensity of the light leaving the bulb decreases at a rate proportional to the voltage.

The intensity of the reflected light that strikes the switch is proportional to the intensity of the light leaving the bulb and inversely proportional to the square of the distance from the bulb to the mirror.

Exercise D.5.1 a. The first paragraph should remind you of a predator prey system. Assuming so, is the voltage the predator or the prey?

b. Write equations for the mathematical Model D.5.1.

Figure for Exercise D.5.1 A night light with a mirror that can be positioned to reflect light back to the photosensitive switch.



Exercise D.5.2 We propose a second mathematical model for night lights.

Mathematical Model D.5.2 Night lights. II There is a voltage in the photosensitive switch that increases at a rate proportional to the intensity of light striking the switch and dissipates at a constant rate when no light strikes the switch. The light is either on or off; it turns on when the voltage falls below a certain threshold and turns off when the voltage exceeds another threshold.

The intensity of the reflected light that strikes the switch is proportional to the intensity of the light leaving the bulb and inversely proportional to the square of twice the distance from the bulb to the mirror.

Let $v(t)$ be the voltage in the photosensitive switch at time t and $i(t)$ be the illumination striking the photosensitive switch at time t.

a. Write an equation descriptive of

> There is a voltage in the photosensitive switch that increases at a rate proportional to the intensity of light striking the switch and dissipates at a constant rate when no light strikes the switch.

b. Because the light is either on or off, it is easiest to treat the mathematical model D.5.2 Night Light II as a discrete system. Choose an increment time $\delta > 0$ and for $n = 0, 1, 2, \cdots N$, let

$$v_k = v(k \times \delta) \qquad \text{and} \qquad i_k = i(k \times \delta),$$

and assume that

$$v'(k \times \delta) \doteq \frac{v_{k+1} - v_k}{\delta}.$$

Write a discrete analog of your previous equation

c. Let $v_{on} < v_{off}$ be threshold values and $sw(v)$ be a 'switch' function defined by

$$sw(v(t)) = \begin{array}{lll} 1 & \text{for} & v(t) \leq v_{on} \\ (1 + sign(v'(t)))/2 & \text{for} & v_{on} < v(t) < v_{off} \\ 0 & \text{for} & v_{off} \leq v(t) \end{array}$$

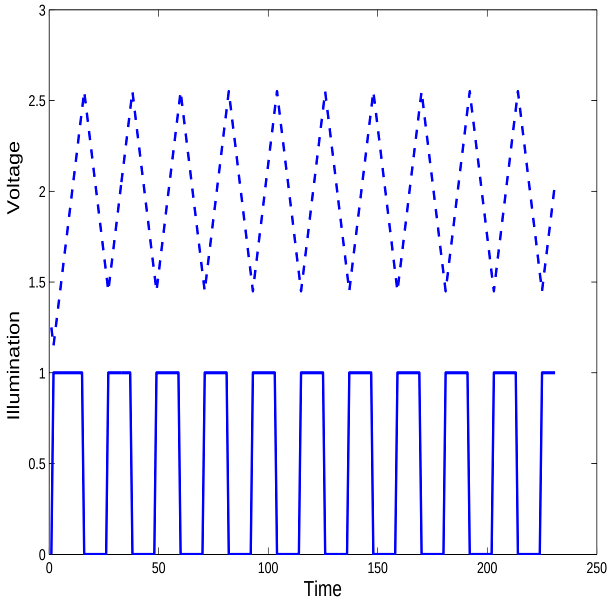
Figure D.5.2 illustrates the solutions to the equation

$$\begin{aligned} v_0 &= 1.25 \\ i_0 &= 0 \\ v_{k+1}+ &= v + k + (2 \times i_k - 1) \times \delta \\ \\ i_{k+1} &= sw(v_k), \end{aligned} \tag{D.33}$$

where $v_{on} = 1.5$ and $v_{off} = 2.5$, and $N = 230$.

If you have adequate computing, replicate Figure D.5.2. Else, compute (v_1, i_1), (v_2, i_2), (v_3, i_3), (v_4, i_4), and (v_5, i_5).

Figure for Exercise D.5.2 Solutions to Equations D.33. Light intensity is the solid curve and voltage is the dashed curve.



Exercise D.5.3 Thermostats control the furnaces on houses. They turn the furnace on when the temperature falls below a temperature set by the home owner and turn the furnace off when the temperature exceeds a temperature set by the home owner.

The control system of a thermostat is shown at http://www.howstuffworks.com/home-thermostat.htmin. There is a bi-metalic coil that expands or contracts according to the temperature of the coil. There is a small glass tube containing a dollop of mercury at the top of the coil that tilts as the coil expands or contracts. Two wires are at one end of the tube. When the temperature is "low" the tube tilts so that the mercury completes the connection between the two wires. When the temperature is "high" the tube tilts so that the mercury is at the opposite end of the tube from the wires and the connection is broken. As the temperature moves between the "low" and "high" temperatures the mercury slowly moves towards the center of the tube until a threshold angle is reached and it flows to the opposite end of the tube and either opens or closes the connection between the two wires.

a. Write a mathematical model descriptive of temperature inside a house when the furnace is not running and the outside temperature is below the temperature inside the house. You may wish to review Exercises **??** and D.2.19 and D.5.2.

b. Write equations that describe your mathematical model.

c. Draw a graph descriptive of the temperature inside a house in northern Minnesota for one day in January.

d. Draw a graph descriptive of the temperature inside a house in Virginia for one day in January.

D.5.2 Earthquakes.

The San Andreas fault in California is an 800 mile zone of contact between two tectonic plates, with the continental crust on the east and the oceanic crust on the west. As the oceanic crust moves north and rubs against the continental crust, at some points along the fault faces of the crust lock together and the earth bends — until a threshold distortion is surpassed. Then the faces of the crusts abruptly slide past one another sending shock waves out across the earth, and the crusts returns to a more relaxed condition. The maximum slippage recorded between two crusts is a 21 foot displacement of a road during the 1906 earthquake in the San Francisco region.

A simple model of this system was described by Steven Gao of Kansas State University. Consider a body of mass m on a horizontal platform, a spring with one end attached to the body and the other end moving along the platform at a rate v. There are two frictions associated with the body, the starting friction, F_{start}, and the sliding friction, F_{slide},

$$F_{slide} < F_{start}.$$

If the body is not moving relative to the platform, F_{start} is the force required to initiate movement. If the body is in motion along the platform, F_{slide} is the force required to continue motion.

The spring has a spring constant k; an elongation of length E in the spring causes a force of magnitude $k \times E$ on the body. Let L be the length of the spring when there is no tension on the spring.

In this model, the horizontal platform is the continental crust and the body and spring are the oceanic crust.

Mark a point on the platform as the zero point, let $x(t)$ be the distance from zero to the forward face of the body, and let $y(t)$ be the distance from zero to the forward end of the spring. Let $E(t)$ be the extension of the spring.

Assume the initial conditions:

$$x(0) = 0, \qquad y(0) = L, \qquad \text{so that} \qquad E(0) = 0$$

Figure D.15: San Andreas Fault. Photograph from Wallace, Robert E., ed., 1990, The San Andreas fault system, California: U.S. Geological Survey Professional Paper 1515, 283 p. [http://pubs.usgs.gov/pp/1988/1434/].

Exercise D.5.4 1. What is $y'(t)$ for all t?

2. Write a formula for $y(t)$.
3. The force of the spring on the body will be $k \times E(t) = k \times (y(t) - L - x(t))$. What is the force at time $t = 0$?
4. At what time, t_1, will the body first move (will the force on the body $= F_{start}$)?

During the first motion of the body, the net force, F, on the body will be

$$\begin{aligned} F &= k \times (y(t) - L - x(t)) - F_{slide} \\ &= k \times (L + vt - L - x(t)) - F_{slide} \\ &= k \times (vt - x(t)) - F_{slide} \end{aligned}$$

Newton's second law of motion gives

$$F = ma = mx'',$$

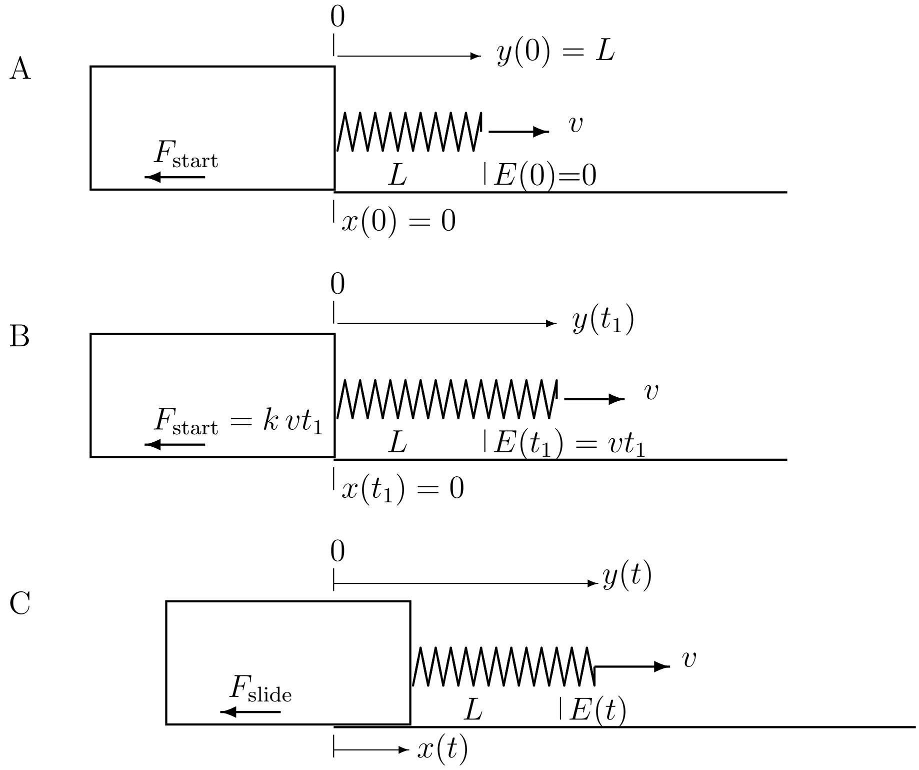


Figure D.16: A. Initial condition of a block with spring pull and starting friction. B. The block will first move at time t_1 when the force of the spring elongation, $k\,vt_1$, exactly matches the starting friction, F_{Start}. C. Motion of a block with spring pull and sliding friction.

so that

$$mx'' = k \times (vt - x(t)) - F_{slide}$$

or

$$x'' + \omega^2 x = \frac{k}{m} vt - \frac{F_{slide}}{m} \qquad\qquad \omega^2 = \frac{k}{m} \tag{D.34}$$

Let t_1 be the time at which the first motion starts. Then

$$x(t_1) \;=\; 0 \tag{D.35}$$

$$x'(t_1) \;=\; 0 \tag{D.36}$$

Reader Beware: Incoming Lightning Bolt! In Chapter 18, Second order and systems of two first order differential equations, you will learn how to find the function, $x(t)$, that satisfies equations D.34, D.35, and D.36:

$$\begin{aligned} x(t) \;&=\; \frac{-F_{start} + F_{slide}}{k} \cos(\omega\,(t - t_1)) \\ &\qquad -\frac{v}{\omega} \sin(\omega(t - t_1)) + vt - \frac{F_{slide}}{k} \end{aligned} \tag{D.37}$$

Exercise D.5.5 Show that the function, $x(t)$ defined in Equation D.37 satisfies equation D.35, D.36, and D.34.

Exercise D.5.6 Equation D.37 is valid until $x'(t)$ next equals to zero. Use the parameters

$$F_{start} = 5, \qquad F_{slide} = 4, \qquad k = 1, \qquad v = 0.1, \qquad \text{and} \qquad m = 1$$

and draw the graph of x. An elongation of $E = F_{start}/k = 5$ initiates motion at time $t_1 = F_{\text{start}}/kv = 50$, and $\omega^2 = k/m = 1$ with these parameters. Find the time and value of x at which $x' = 0$.

Using the parameters of the previous exercise, the body moved 2.33 units in a time of 3.33 units; the velocity of the forward end of the spring is 0.1 so the forward end moved 0.33 units during the motion. At the time the body stopped moving, the elongation of the spring was 5 - 2.33 + 0.33 = 3. After 20 more time units, the elongation will again reach 5 and the motion of the body will be repeated. A graph of the motion of the body is shown in Figure D.17

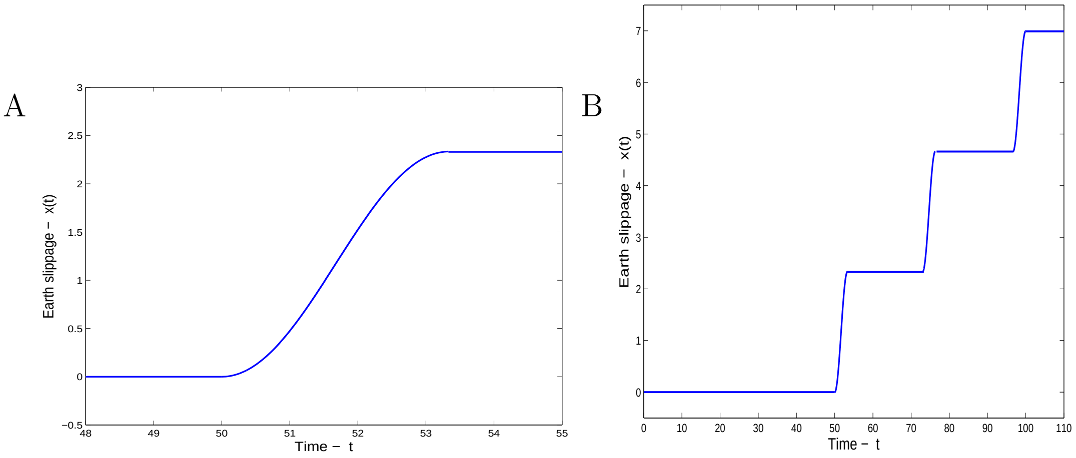


Figure D.17: A. Graph of the first slippage, $x(t)$, for Equation D.37 and the parameter values of Exercise D.5.6. B. Graph of three slippage events with quiescence between events.

The model we describe exhibits periodic 'relief' of the tension in the spring. Do earth quakes exhibit periodicity in their recurrence? If so it would greatly simplify the prediction of earth quakes! The dates of significant earth quakes and their magnitudes (4.8 - 6.7) that have occurred in the Los Angeles area fault since 1920 are shown in Table D.3. You may search the table for periodicity. As emphasized by Dr. Gao, the fault is a complex system; many small earthquakes occur every month; relief in on section may increase strain at other points. His model does, however, suggest the nature of the mechanics of an earth quake. The U.S. Geological Survey web site, http://pubs.usgs.gov/gip/earthq3/, contains very interesting discussion of earthquakes.

D.5.3 The circadian clock.

Exciting current studies in molecular biology and genetics are illuminating the molecular and neural systems that control the daily rhythm of our lives, wakes us up in the morning, puts us to sleep at night, and causes 'jet lag' when we travel to different time zones or even without travel when daylight savings time is initiated in the Spring. The molecular clocks that control circadian rhythms have been identified

Table D.3: Significant earthquakes in the Los Angeles, CA area 1920 - 1994. Mgtd = magnitude. http://pasadena.wr.usgs.gov/info/index.html

Year	Mgtd.	Year	Mgtd.	Year	Mgtd.	Year	Mgtd.
1920	4.9	1941	4.8	1973	5.2	1989	5.0
1930	5.2	1952	5.2	1979	5.2	1990	5.3
1933	6.4	1969	5.1	1981	6.0	1991	5.8
1938	5.2	1970	5.2	1987	5.9	1994	6.7
1941	4.8	1971	6.7	1988	5.0		

in fruit flies, mammals, *Neurospora* (fungus), *Arabidopsis* (plant), and cyanobacteria. They are markedly similar and It appears that there has been multiple evolution of the same basic mechanism. The *Neurospora* mechanism is described here from a review, J. C. Dunlap, *Ann. Rev. Genet.* **30** (1996), 579.

In *Neurospora*, there is a gene, *frq* (frequency), that is transcribed into a mRNA also called *frq* that codes for a protein, denoted FRQ. FRQ stimulates metabolic activities associated with daylight. A high level of FRQ also acts to shut off transcription of the gene *frq* so that the concentration of mRNA *frq* decreases. There is a potentially oscillating system: *frq* increases and causes an increase in FRQ that causes a decrease in *frq* so that FRQ decreases (is no longer transcribed from *frq* and is naturally degraded as are most proteins). However, there is resistance in the system and periodic stimulus from daylight is necessary to keep the system active and to entrain it to the daily 24 hour rhythm.

Circadian time (CT) begins with 0 at dawn, 6 is noon, 12 is dusk, 18 is midnight, and 24 = 0 is dawn. The *Neurospora* circadian cycle begins at midnight, CT 18, at which time both *frq* and FRQ are at low levels, but transcription of *frq* begins, say at a fixed rate. After a 3 hour time lag translation of *frq* to create FRQ begins. At dawn, CT 0, there is a marked increase in transcription of *frq* and an almost immediate increase of translation to FRQ. A high level of FRQ inhibits the transcription of *frq* and *frq* levels peak during CT 2 - 6 and declines steadily until CT 18. FRQ levels peak during CT 6 - 10 and also decline until CT 18 (the protein FRQ is constantly degraded and with decreasing levels of *frq* the turnover is more rapid than production).

Exercise D.5.7 a. Draw a graph representative of the concentration of the mRNA *frq* as a function of time (use CT).

b. Draw a graph representative of the concentration of the protein FRQ as a function of time (use CT).

c. Draw a phase diagram with concentration of *frq* on the horizontal axis and concentration of FRQ on the vertical axis covering one 24 hour period.

D.6 From Chapter 8, Applications of Derivatives

The next three exercises could form the basis for a semester project.

Exercise D.6.1 Sickle cell anemia is an inherited blood disease in which the body makes sickle-shaped red blood cells. It is caused by a single mutation from glutamic acid to valine at position 6 in the protein Hemoglobin B. The gene for hemoglobin B is on human chromosome 11; a single nucleotide change in the codon for the glutamic acid, GAG, to GTG causes the change from glutamic acid to valine. The location of a genetic variation is called a *locus* and the different genetic values (GAG and GTG) at the location are called *alleles*.

People who have GAG on one copy of chromosome 11 and GTG on the other copy are said to be heterozygous and do not have sickle cell anemia and have elevated resistance to malaria over those who have GAG on both copies of chromosome 11. Those who have GTG on both copies of chromosome 11 are said to be homozygous and have sickle cell anemia – the hydrophobic valine allows aggregation of hemoglobin molecules within the blood cell causing a sickle-like deformation that does not move easily through blood vessels.

Let A denote presence of GAG and a denote presence of GTG on chromosome 11, and let AA, Aa and aa denote the various presences of those codons on the two chromosomes of a person (note: $Aa = aA$); AA, Aa and aa label are the genotypes of the person with respect to this locus. It is necessary to assume *non-overlapping generations*, meaning that all members of the population are simultaneously born, grow to sexual maturity, mate, leave offspring and die. Let P, Q and R denote the frequencies of AA, Aa and aa genotypes in a breeding population and let p and q denote the frequencies of the alleles A and a among the chromosomes in the same population. The frequencies P, Q, and R are referred to as *genotype frequencies* and p and q are referred to as *allele frequencies*. In a population of size, N, there will be $2N$ chromosomes and $P \times 2N + Q \times N$ of the chromosomes will be A.

In a mating of AA with Aa adults, the chromosome in the fertilized egg (zygote) obtained from AA must be A and the with probability 1/2 the chromosome obtained from Aa will be A and with probability 1/2 will be a. Therefore, the zygote will be AA with probability 1/2 and will be Aa with probability 1/2.

a. Show that $p = P + \frac{1}{2}Q$ and $q = \frac{1}{2}Q + R$.

b. Assume a closed population (no migration) with random mating and no selection. Complete the table showing probabilities of zygote type for the various mating possibilities, the frequencies of the mating possibilities, and the zygote genotype frequencies. Include zeros with the zygote type probabilities but omit the zeros in the zygote genotype frequencies. Random mating assumes that the selection of mating partners is independent of the genotypes of the partners.

Adult Male	×	Female	Zygote types and probabilities			Random mating frequency	Zygote Genotype frequencies		
			AA	Aa	aa		AA	Aa	aa
AA	×	AA	1	0	0	P^2	P^2		
AA	×	Aa	1/2	1/2	0	PQ	$\frac{1}{2}PQ$	$\frac{1}{2}PQ$	
AA	×	aa	0	1	0	PR		PR	
cAa	×	AA							
Aa	×	Aa				QQ	$\frac{1}{4}QQ$	$\frac{1}{2}QQ$	$\frac{1}{4}QQ$
Aa	×	aa				QR		$\frac{1}{2}QR$	$\frac{1}{2}QR$
aa	×	AA				PR		PR	
aa	×	aA							
aa	×	aa							
Sum						1	Σ_{AA}	Σ_{Aa}	Σ_{aa}

c. When the table is complete, you should see that

$$\begin{aligned}
\Sigma_{Aa} &= \frac{1}{2}PQ + PR + \frac{1}{2}QP + \frac{1}{2}Q^2 + \frac{1}{2}QR + RP + \frac{1}{2}QR \\
&= PQ + 2PR + \frac{1}{2}Q^2 + QR \\
&= 2P\left(\frac{1}{2}Q + R\right) + Q\left(\frac{1}{2}Q + R\right) \\
&= (2P + Q)\left(\frac{1}{2}Q + R\right) \\
&= 2\left(P + \frac{1}{2}Q\right)\left(\frac{1}{2}Q + R\right) \\
&= 2pq
\end{aligned}$$

Show that

$$\Sigma_{AA} = p^2 \qquad \text{and} \qquad \Sigma_{aa} = q^2$$

This means that under the random mating hypothesis, the *zygote genotype* frequencies of the offspring population are determined by the *allele* frequencies of the adults. This is referred to as the Hardy-Weinberg theorem. If the probability of an egg growing to adult and contributing to the next generation of eggs is the same for all eggs, independent of genotype, then the allele frequencies, p and q, are constant after the first generation.

Random mating does not imply the promiscuity that might be imagined. It means that the selection of mating partner is independent of the genotype of the partner. In the United States, blood type would be a random mating locus; seldom does a United States young person inquire about the blood type of an attractive partner. In Japan, however, this seems to be a big deal, to the point that dating services arranging matches also match blood type. The major histocompatibility complex (MHC) of a young person would seem to be fairly neutral; few people even know their MHC type. It has been demonstrated, however, that young women are repulsed by the smell of men of the same MHC type as their own[12].

---

[12] Claus Wedekind, et al, MHC-Dependent mate preferences in humans, *Proceedings: Biological Sciences*, 260 (1995) 245-249.

d. Show that if the frequency of A in an egg generation is $\hat{p}$, then the frequency of A in the next egg generation will also be $\hat{p}$.

e. Suppose that because of malaria, an AA type egg, either male or female, has probability 0.8 of reaching maturity and mating and because of sickle cell anemia an aa type has only 0.2 probability of mating, but that an Aa type has 1.0 probability of mating. This condition is called *selection*. Then the distribution of genotypes in the egg and the mating populations will be

Genotype	AA	Aa	aa	
Egg	p^2	$2pq$	q^2	
Adult	$0.8p^2/F$	$2pq/F$	$0.2q^2/F$	where $F = 0.8p^2 + 2pq + 0.2q^2$

Find the frequency of A in the adult population. Note: This will also be the frequency of A in the next egg population.

f. We call $F(p)$ the *balance* of the population, and because $p + q = 1$

$$F = F(p) = 0.8p^2 + 2p(1-p) + 0.2(1-p)^2$$

You will be asked in Exercise D.6.4 to show that when the probability of reproduction depends on the genotype (*selection* is present), during succeeding generations, allele frequency, p, moves toward the value of local maximum of F.

1. Show that $F(p) = 1 - 0.2p^2 - 0.8(1-p)^2$.
2. Find the value $\hat{p}$ of p that maximizes $F(p)$.

Exercise D.6.2 Consider two alleles A and a at a locus of a random mating population and the fractions of AA, Aa and aa zygotes that reach maturity and mate are in the ratio $1 + s_1 : 1 : 1 + s_2$ where s_1 and s_2 can be positive, negative, or zero, but $s_1 \geq -1$ and $s_2 \geq -1$. The balance function is

$$F(p) = (1+s_1)p^2 + 2pq + (1+s_2)q^2 = (1+s_1)p^2 + 2p(1-p) + (1+s_2)(1-p)^2 = 1 + s_1\,p^2 + s_2\,(1-p)^2$$

where p and q are the frequencies of A and a among the zygotes.

a. Sketch the graphs of F and find the values $\hat{p}$ of p in [0,1] for which $F(p)$ is a maximum for

1. $s_1 = 0.2$ and $s_2 = -0.3$.
2. $s_1 = 0$ and $s_2 = -0.2$.
3. $s_1 = -0.2$ and $s_2 = -0.3$.
4. $s_1 = 0.2$ and $s_2 = 0.3$.

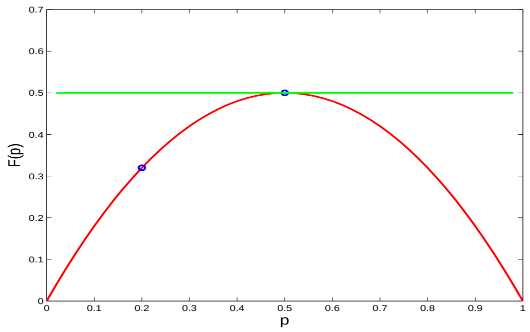
b. Suppose that $s_1 + s_2 \neq 0$ and $0 \leq s_2/(s_1 + s_2) \leq 1$. Is it true that $\hat{p}s_2/(s_1 + s_2)$ is the value of p in [0,1] for which $F(p)$ is a maximum?

Exercise D.6.3 Use the notation of the previous exercise, Exercise D.6.2.

a. Run the following MATLAB program.

```
close all;clc;clear
p=0.2;q=1-p;s1=-0.3;s2=-0.5;

for k = 1:12
    pp(k) = p;
    AA = p^2; Aa = 2*p*q; aa = q^2;
    F(k) = (1+s1)*AA + Aa + (1+s2)*aa;
    AA_n = (1+s1)*AA/F(k); Aa_n = Aa/F(k);
    aa_n = (1+s2)*aa/F(k);
    p = AA_n+0.5*Aa_n; q = 0.5*Aa_n+aa_n;
end
[pp.’ F.’]
x = [0:0.02:1];
y = 1+s1*x.^2 +s2*(1-x).^2;
plot(x,y,’r’,’linewidth’,2)
xlabel(’p’,’fontsize’,16); ylabel(’F(p)’,’fontsize’,16)
hold
plot(pp,F,’o’,’linewidth’,2)
```



b. The program computes the allele frequency, p, of allele A through 12 generations of selection and produces the graph shown to the right of the program. Mark the initial and second, third and fourth values of p. What is the limiting value of p?

c. Change the second line of the program to `p=0.9; q=1-p; s1=-0.3; s2=-0.5;` , and run the new program.

d. Change the second line of the program to `p=0.2; q=1-p; s1=0.3; s2=0.5;` , and run the new program.

e. Use some other values of `p` , `s1` , and `s2` , and run the program.

Exercise D.6.4 The program shown in Exercise D.6.3 assumes a population with an A-allele frequency of p=0.2 and computes future populations assuming that there is selection pattern AA: 0.7; Aa: 1.0; aa: 0.5. The heterozygote, Aa, is favored over either of the two homozygotes, AA and aa; the condition is referred to as *over dominance* and occurs in sickle cell anemia. The A-allele frequency of p=0.2 is out of balance with the selection forces acting on the population, and in subsequent generations p moves toward the value $\hat{p}$ that maximizes the balance function,

$$F(p) = (1+s_1)\,p^2 + 2\,p\,q + (1+s_2)\,q^2 = (1+s_1)\,p^2 + 2\,p\,(1-p) + (1+s_2)\,(1-p)^2 = 1 + s_1p^2 + s_2(1-p)^2.$$

This exercise proves that it always happens in the case of over dominance.

a. Consider two alleles A and a at a locus of a random mating population with non-overlapping generations and the fractions of AA, Aa and aa zygotes that reach maturity and mate are in the ratio $1+s_1 : 1 : 1+s_2$ where $-1 \le s_1 < 0$ and $-1 \le s_2 < 0$. Show that the maximum value of

$$F(p) = 1 + s_1\, p^2 + s_2\, (1-p)^2 \quad \text{occurs at} \quad \hat{p} = \frac{s_2}{s_1+s_2} \quad \text{and} \quad F(\hat{p}) = 1 + \frac{s_1 s_2}{s_1+s_2} < 1.$$

b. Assume the egg allele frequencies in the first generation are $A \quad p_0$ and $a \quad q_0 = 1 - p_0$ and that the egg genotype frequencies are $AA \quad p_0^2$, $Aa \quad 2p_0q_0$ and $aa \quad q_0^2$. After selection the adult genotype frequencies are

$$\begin{array}{ccc} AA & Aa & aa \\ \dfrac{(1+s_1)p_0^2}{F(p_0)} & \dfrac{2p_0q_0}{F(p_0)} & \dfrac{(1+s_2)q_0^2}{F(p_0)} \end{array} \quad \text{where} \quad F(p_0) = 1 + s_1\, p_0^2 + s_2\, (1-p_0)^2$$

Show that the frequency, p_1 of A in the adult population (and therefore of the resulting egg population) is

$$p_1 = \frac{(1+s_1)p_0^2 + p_0(1-p_0)}{F(p_0)}$$

c. For the n^{th} generation

$$p_{n+1} = \frac{(1+s_1)p_n^2 + p_n(1-p_n)}{F(p_n)} \quad \text{where} \quad F(p_n) = 1 + s_1 p_n^2 + s_2(1-p_n)^2 \tag{D.38}$$

It is the sequence $\{p_0, p_1, p_2, \cdots\}$ that we wish to show converges to $\hat{p}$.

We will consider only the case $0 < p_0 < \hat{p}$.

In Grungy Algebra I, shown below, we find that

$$p_{n+1} - p_n = -\frac{p_n\,(1-p_n)\,(s_1+s_2)}{F(p_n)}\,(\hat{p} - p_n) \tag{D.39}$$

d. Show that

$$-\frac{p_n\,(1-p_n)\,(s_1+s_2)}{F(p_n)} > 0$$

e. Show that if $\hat{p} - p_n$ is positive then $p_{n+1} - p_n$ is positive, so that $p_{n+1} > p_n$.

f. Show that

$$\begin{aligned} \frac{p_n\,(1-p_n)}{F(p_n)} &= \frac{p_n\,(1-p_n)}{1 + s_1 p_n^2 + s_2(1-p_n)^2} \\ &\le \frac{p_n\,(1-p_n)}{1 - p_n^2 - (1-p_n)^2} \\ &= \frac{1}{2} \end{aligned}$$

$$\frac{p_n\,(1-p_n)}{F(p_n)} \le \frac{1}{2}. \tag{D.40}$$

g. Show that

$$-\frac{p_n\,(1-p_n)\,(s_1+s_2)}{F(p_n)} \le 1$$

h. Conclude that

$$p_{n+1}-p_n \le (\hat{p}-p_n), \quad \text{so that} \quad p_{n+1} \le \hat{p}$$

i. We now know that $\{p_0, p_1, p_2, \cdots\}$ is an increasing sequence bounded above by $\hat{p}$. Does $\{p_0, p_1, p_2, \cdots\}$ converge to $\hat{p}$?

j. Show that in the extreme case where $s_1 = s_2 = -1$,

$$\hat{p} = \frac{1}{2}, \quad \text{and for any } p_0, \quad p_1 = \frac{1}{2}, \quad \text{and} \quad p_2 = \frac{1}{2}, \quad p_3 = \frac{1}{2} \cdots.$$

k. Now assume $s_1 > -1$ or $s_2 > -1$. In Not So Grungy Algebra II (shown below) we show that

$$\hat{p} - p_{n+1} = \left[(s_1+s_2)\frac{p_n(1-p_n)}{F(p_n)} + 1 \right] (\hat{p}-p_n) \tag{D.41}$$

and in Mystical Algebra III we show that

$$0 \le (s_1+s_2)\frac{p_n(1-p_n)}{F(p_n)} + 1 < 1. \tag{D.42}$$

Use Equations D.41 and D.42 to show that $\{p_0, p_1, p_2, \cdots\}$ converges to $\hat{p}$.

l. **Grungy Algebra I.** Proof of Equation D.39. There are at least four mistakes in the following equations that you should correct. Begin with Equation D.38,

$$\begin{aligned}
p_{n+1} &= \frac{(1+s_1)p_n^2 + p_n(1-p_n)}{F(p_n)} \\
p_{n+1} - p_n &= \frac{(1+s_1)p_n^2 + p_n(1-p_n)}{F(p_n)} - p_n \\
&= p_n \left[\frac{(1+s_1)p_n + (1-p_n)}{1 + s_1 p_n^2 + s_2(1-p_n)^2} - 1 \right] \\
&= p_n \, \frac{(1+s_1)p_n + (1-p_n) - 1 + s_1 p_n^2 + s_2(1-p_n)^2}{F(p_n)} \\
&= p_n \, \frac{p_n + s_1 p_n + 1 - p_n - 1 - s_1 p_n^2 - s_2(1-p_n)^2}{F(p_n)} \\
&= p_n \, \frac{s_1 p_n - s_1 p_n^2 - s_2(1-p_n)^2}{F(p_n)} \\
&= p_n\,(1-p_n)\,\frac{s_1 p_n - s_2(1-p_n)}{F(p_n)}
\end{aligned}$$

$$\begin{aligned} &= p_n\,(1-p_n)\,(s_1+s_2)\,\frac{\dfrac{s_2}{s_1+s_2}-p_n}{F(p_n)} \\ p_{n+1}-p_n &= -p_n\,(1-p_n)\,(s_1+s_2)\,\frac{\hat{p}-p_n}{F(p_n)} \end{aligned}$$

m. **Not So Grungy Algebra II.** Proof of Equation D.41. Check the last step. Begin with Equation D.39:

$$\begin{aligned} p_{n+1}-p_n &= -\frac{p_n\,(1-p_n)\,(s_1+s_2)}{F(p_n)}\,(\hat{p}-p_n) \\ p_n-p_{n+1} &= \frac{p_n\,(1-p_n)\,(s_1+s_2)}{F(p_n)}\,(\hat{p}-p_n) \\ p_n-\hat{p}+\hat{p}-p_{n+1} &= \frac{p_n\,(1-p_n)\,(s_1+s_2)}{F(p_n)}\,(\hat{p}-p_n) \\ \hat{p}-p_{n+1} &= \left[\frac{p_n\,(1-p_n)\,(s_1+s_2)}{F(p_n)}\right](\hat{p}-p_n) \end{aligned}$$

n. **Mystical Algebra III.** Proof of Inequalities D.42. Remember that $0 > s_1 \geq -1$ and $0 > s_2 \geq -1$ and either $s_1 > -1$ or $s_2 > -1$. Then

$$\begin{array}{ccccccc} 0 & > & s_1+s_2 & > & -2 & & \\ 0 & > & (s_1+s_2)\,\dfrac{p_n(1-p_n)}{F(p_n)} & > & -2\,\dfrac{p_n(1-p_n)}{F(p_n)} & \overset{??}{\geq} & -1 \\ 1 & > & (s_1+s_2)\,\dfrac{p_n(1-p_n)}{F(p_n)}+1 & \geq & 0 & & \end{array}$$

D.7 Chapter D Suggested solutions.

Exercise D.1.3. $I'(x) = -KI(x)$ where $I(x)$ is light intensity at depth x K is a constant.

Exercise D.1.4. $P'(t) = -K\dfrac{P(t)}{V}$ where $P(t)$ is the amount of penicillin in the body, V is the volume of the vascular pool, and K is a constant.

Exercise D.1.5.

Exercise D.1.6.

Exercise D.1.7.

Exercise D.2.1.

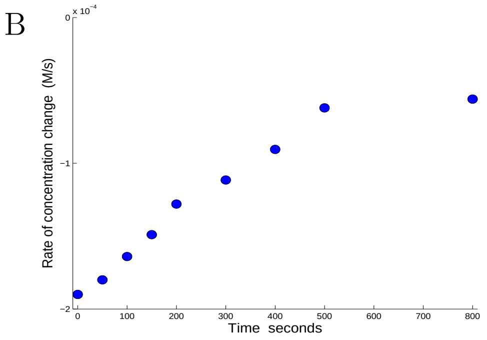


Figure D.18: **Exercise D.1.5.** B. Rate of change of butyl chloride concentration.

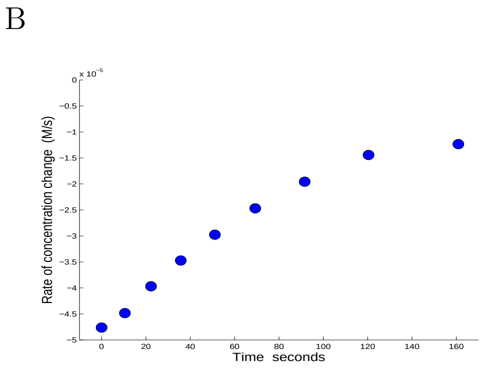


Figure D.19: **Exercise D.1.6.** B. Rate of change of phenolphthalein concentration.

a. $P(t) = 5e^{2\,t} \quad t_{Dbl} = 0.35$ c. $P(t) = 2e^{0.1\,t} \quad t_{Dbl} = 3.47$

e. $P(t) = 10e^{t} \quad t_{Dbl} = 0.69$ g. $P(t) = 0 \quad t_{Dbl} =$ Undefined or 0

Exercise D.2.2.

a. $S(t) = 5 - 5e^{-2\,t} \quad t_{1/2} = 0.35$ c. $S(t) = 10 \quad t_{Dbl} =$ Undefined or 0

e. $S(t) = 20 - 20e^{-t} \quad t_{1/2} = 0.69$ g. $S(t) = 20 \quad t_{Dbl} =$ Undefined or 0

Exercise D.2.3.

a. $S(t) = 5e^{0.35\,t}$ c. $S(t) = 2e^{0.32\,t}$

e. $S(t) = 5e^{-0.458\,t}$ g. $S(t) = 2.5e^{0.693\,t}$

Exercise D.2.4. $S(t) = 300 + 100e^{-0.05\,t}$.

Exercise D.2.5. Let A_t be the amount of Sotolol in the body at time t with $t = 0$ hours the time of the first pill. Then

$$A_0^- \;=\; H$$

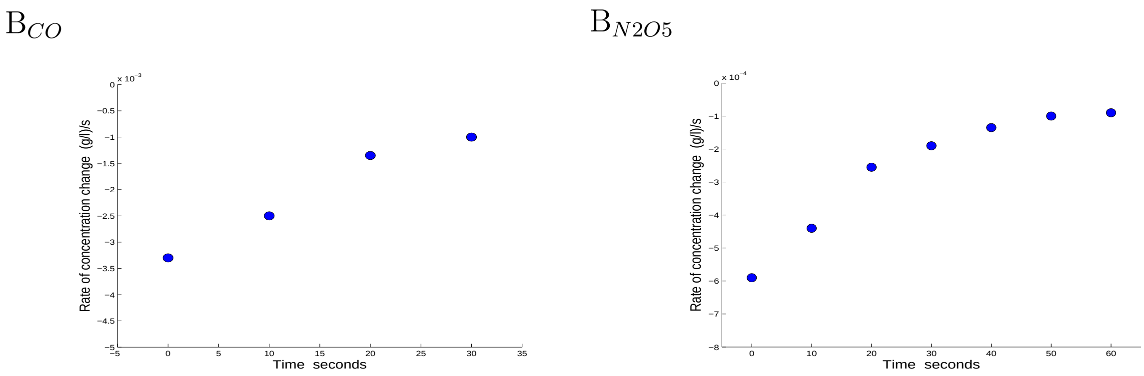

B_{CO} B_{N2O5}

Figure D.20: **Exercise D.1.7.** B_{CO}. Rate of change of carbon dioxide concentration. B_{N2O5}. Rate of change of N_2O_5 concentration.

$$\begin{aligned}
A_0^+ &= H + 40 \\
A_{12}^- &= \frac{1}{2}A_0^+ = \frac{1}{2}(H+40) \\
A_{12}^+ = A_{12}^- + 40 &= \frac{1}{2}H + 60 \\
A_{24}^- &= \frac{1}{2}A_{12}^+ = \frac{1}{4}H + 30 \\
H &= A_{24}^- = \frac{1}{4}H + 30 = 40.
\end{aligned}$$

The amount of Sotolol in the body fluctuates between 40 and 80 mg, a two to one ratio, substantially less than the four to one ratio from 26.7 to 106.7 fluctuation when taking 80 mg once per day.

Exercise D.2.7. The semilog graph of data set a appears to be nonlinear. The semilog graph of data c. appears to be linear and $P(t) = 2e^{0.173\,t}$ fits the data.

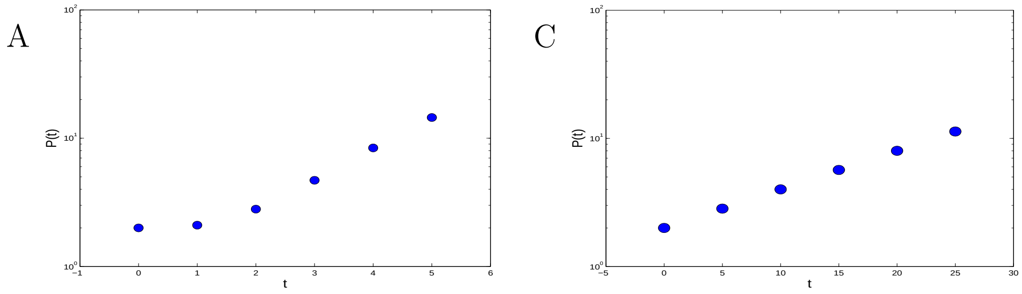


Figure D.21: **Exercise D.2.7.** Semilog graphs of data sets A and C.

Exercise D.2.8. $P(t) = 0.022e^{0.0319\,t}$

Exercise D.2.9.

a. $\dfrac{10^{5.75} - 10^{3.75}}{10^{5.75}} = 0.99 = 99$ percent.

b.

$$\begin{aligned}
\log_{10} V(t) &= 5.9 - 0.19t \\
V(t) &= 10^{5.9-0.19t} = 10^{5.9} \times 10^{-0.19t} \\
V(t) &= 790000 \times \left(e^{\ln 10}\right)^{-0.19t} = 790000e^{-0.43t}
\end{aligned}$$

c. $t_{1/2} = (\ln 2)/0.43 = 1.6$ days.

d. The immune system is eliminating virus at the rate of 43 percent per day.

e. $15.8 \times 6 \times 10^6 = 95 \times 10^6$ CD4 cells per day.

Exercise D.2.10. c. $P(t) = 2e^{-0.04\,t}$.

Exercise D.2.11. $[(\ln(8.02/76.06))/(-\ln 2)] \times 1.28 \times 10^9 = 4.15 \times 10^9$ years old.

Exercise D.2.13. a. $[(\ln(2.5/2.65))/(-\ln 2)] \times 50 \times 10^9 = 4.20 \times 10^9$ years old.

Exercise D.2.14. a. $[(\ln(3/10.0))/(-\ln 2)] \times 5730 = 9953$ years old.

Exercise D.2.15. b. 206 meters.

Exercise D.2.16. In L_1, the intensity of the light that penetrates the surface is higher (800 *vs* 700) than for L_2 and the water is clearer (smaller decay constant, -0.04 *vs* -0.05) in L_1 than in L_2.

Exercise D.2.17. Increased opacity of the sample over that of the standard is proportional to cell density. Let k_{sm} and k_{st} be the opacities of the sample and standard, respectively and δ be the thickness of each.

$$\begin{aligned}
I_{st} &= I_0 e^{-k_{st}\delta} \\
I_{sm} &= I_0 e^{-k_{sm}\delta} \\
k_{sm} - k_{st} &= \frac{1}{\delta}\ln\left(\frac{I_{sm}}{I_{st}}\right)
\end{aligned}$$

Exercise D.2.18. f. For the patient with the kidney that removes only 15% of the penicillin that passes through it, $P(t) = 10 - 10e^{-0.02t}$, saturation level = 10 g.

Exercise D.2.19. f. The hen should return within 21.8 minutes.

Exercise D.2.20. Let $h(t)$ be the height of water in the stem of the thistle tube t minutes after the beginning of the experiment, P_{Oz} the osmotic pressure across the membrane (assumed to be constant), δ the density of the salt water, and g the acceleration of gravity. The pressure, $P(t)$, due to water in the stem is $g \times \delta \times h(t)$. The rate, $R(t)$ at which water crosses the membrane is proportional to $P_{Oz} - P(t)$ and $h'(t)$ is proportional to $R(t)$.

Conclude that

$$h(t) \quad = \quad \frac{P_{Oz}}{g\delta}\left(1 - e^{-g\delta K\,t}\right)$$

where K is a proportionality constant.

Exercise D.2.21. See Figure D.22.

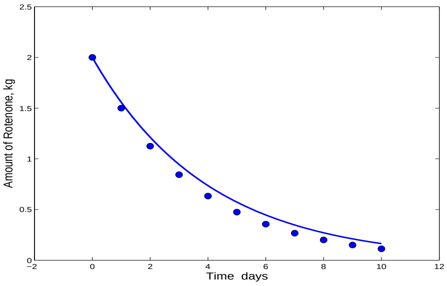


Figure D.22: **Exercise D.2.21.** Graphs of $p_t = 2 \times 0.75^t$, $t = 0, 1, \cdots, 10$ and $P(t) = 2e^{-0.25t}$, $0 \leq t \leq 10$.

Exercise D.2.22. $P(h) = e^{-0.00012634\,h}$.

Exercise D.2.24. g. She could ascend to $d = 8.1$ meters.

Exercise D.2.25. Wilson's statement is a very interesting statement describing a relation between the number of species to be found on an island and the area of the island and has been confirmed in several locations. It does not, however, describe the competition or other interactions between species that might lead to the relationship and therefore does not meet our definition of a Mathematical Model. The relation has been derived from very elementary first principles by J. Harte, T. Zillio, E. Conlisk, and A. B. Smith, Maximum entropy and the state-variable approach to macroecology, *Ecology* 89 (2008), 2700-2711, that applies when the number of individuals per species is small.

Explore D.4.3.

$$\begin{aligned}
-av(t) = u'(t) \quad &= \quad v_0\sqrt{\frac{a}{b}}\sqrt{ab}\cos(\sqrt{ab}\,t) - u_0\sqrt{ab}\sin(\sqrt{ab}\,t) \\
v(t) \quad &=
\end{aligned}$$

Explore D.5.2. The light switches off and on rapidly. When farther from the mirror, the light continues to switch off and on, but at a less rapid pace, because the light intensity striking the photoreceptor on the light is lower.

Exercise D.5.1.

a. The voltage is analogous to the predator.

b. Let $v(t)$ be the voltage and $i(t)$ be the light intensity. Then

$$i'(t) = -a \times v(t), \qquad \text{and} \qquad v'(t) = b \times i(t)$$

Exercise D.5.2.

a. There are numbers a and b such that

$$v'(t) = a \times i(t) - b$$

b.

$$\frac{v_{k+1} - v_k}{\delta} = a \times i_k - b, \qquad \text{or} \qquad v_{k+1} = v_k + (a \times i_k - b) \times \delta$$

c. Matlab code to generate Figure D.5.2 follows.

```
close all;clc;clear
a=2;  b=1;  Von=1.5;  Voff=2.5;  delta=0.1;

volt(1)=1.25; illum(1)=0;
for k=1:230
    volt(k+1) = volt(k) +(a*illum(k)-b)*delta;
    if volt(k+1) <= Von
        illum(k+1)=1;
    elseif Voff <= volt(k+1)
            illum(k+1)=0;
    else
        illum(k+1)=(1+sign(volt(k+1)-volt(k)))/2;
    end
end
plot(volt,'--','linewidth',2)
hold
plot(illum,'linewidth',2)
xlabel('Time','fontsize',14)
ylabel('Illumination                Voltage','fontsize',14)
```

Exercise D.5.4.

a. $y'(t) = v$.

b. $y(t) = L + vt$.

c. At time zero, the force on the block is zero.

d. At time $t_1 = F_{start}/(kv)$ the block will first move.

Exercise D.5.5.

$$x'' + \omega^2 x = \frac{k}{m}vt - \frac{f_{slide}}{m} \qquad \omega^2 = \frac{k}{m} \qquad x(t_1) = 0, \qquad x'(t_1) = 0$$

$$\begin{aligned}
x(t) &= \frac{-F_{start} + F_{slide}}{k}\cos\left(\omega\left(t - t_1\right)\right) - \frac{v}{\omega}\sin\left(\omega(t - t_1)\right) + vt - \frac{F_{slide}}{k}\\
x'(t) &= -\omega\frac{-F_{start} + F_{slide}}{k}\sin\left(\omega\left(t - t_1\right)\right) - \omega\frac{v}{\omega}\cos\left(\omega(t - t_1)\right) + v\\
x''(t) &= -\omega^2\frac{-F_{start} + F_{slide}}{k}\cos\left(\omega\left(t - t_1\right)\right) + \omega^2\frac{v}{\omega}\sin\left(\omega(t - t_1)\right)
\end{aligned}$$

$$\begin{aligned}
x(t_1) &= \frac{-F_{start} + F_{slide}}{k} + vt_1 - \frac{F_{slide}}{k}\\
&= \frac{-F_{start} + F_{slide}}{k} + vF_{start}/(kv) - \frac{F_{slide}}{k} = 0\\
x'(t_1) &= -\omega\frac{v}{\omega} + v = 0\\
x'' + \omega^2 x &= -\omega^2\frac{-F_{start} + F_{slide}}{k}\cos\left(\omega\left(t - t_1\right)\right) + \omega^2\frac{v}{\omega}\sin\left(\omega(t - t_1)\right)\\
&\quad +\omega^2\left(\frac{-F_{start} + F_{slide}}{k}\cos\left(\omega\left(t - t_1\right)\right) - \frac{v}{\omega}\sin\left(\omega(t - t_1)\right) + vt - \frac{F_{slide}}{k}\right)\\
&= \omega^2 vt - \omega^2\frac{F_{slide}}{k} = \frac{k}{m}vt - \frac{F_{slide}}{m}
\end{aligned}$$

Exercise D.5.6. Graphs of

$$x(t)\frac{-5+4}{1}\cos\left(\sqrt{1/1}\,(t-50)\right) - \frac{0.1}{\sqrt{1/1}}\sin\left(\sqrt{1/1}(t-50)\right) + 0.1t - \frac{4}{1} \tag{D.43}$$

and its translations in time are shown in Figure D.23. At time $t = 53.33$, $x' \doteq 0$ and $x \doteq 2.33$. The slippage is complete as the movement was 2.33. The next slippage begins at $t \doteq 73.33$.

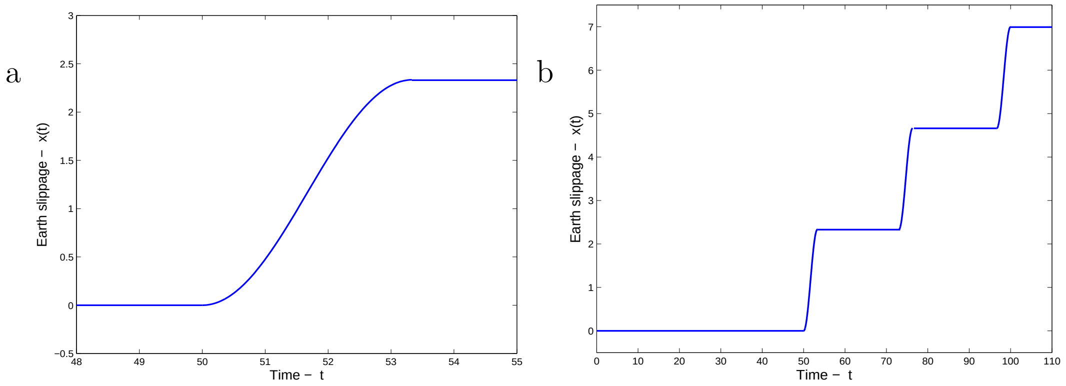


Figure D.23: a. Graph of earthquake Equation D.43 and b. repeated instance of slippage.

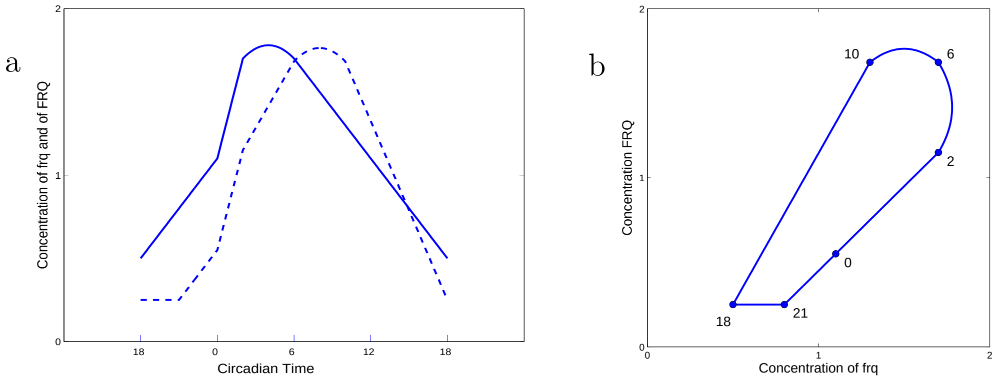


Figure D.24: a. Hypothetical graphs of the levels of messenger RNA, frq, (solid curve) and its protein product FRQ (dashed curve). b. Phase plane of frq and FRQ.

Exercise D.5.7. See Figure D.24

Exercise D.3.1.

a.
$$\begin{aligned} y(t) &= 2\sin t + \cos t & y(0) &= 2\times 0 + 1 = 1 \\ y' &= 2\cos t - \sin t & y'(0) &= 2\times 1 - 0 = 2 \\ y''(t) &= -2\sin t - \cos t \end{aligned}$$

$$y''(t) + y(t) = (-2\sin t - \cos t) + (2\sin t + \cos t) = 0$$

c.
$$\begin{aligned} y(t) &= \cos 3t - \sin 3t & y(0) &= 1 - 0 = 1 \\ y' &= -3\sin 3t - 3\cos 3t & y'(0) &= -3\times 0 - 3\times 1 = -3 \\ y''(t) &= -9\cos 3t + 9\sin 3t \end{aligned}$$

$$y''(t) + 9y(t) = (-9\cos 3t + 9\sin 3t) + 9(\cos 3t - \sin 3t) = 0$$

e.
$$\begin{aligned} y(t) &= 4\cos(3t+\pi/3) & y(0) &= 4\times 1/2 = 2 \\ y' &= -12\sin(3t+\pi/3) & y'(0) &= -12\times\sqrt{3}/2 = -6\sqrt{3} \\ y''(t) &= -36\cos(3t+\pi/3) \end{aligned}$$

$$y''(t) + 9y(t) = (-36\cos(3t+\pi/3)) + 9(4\cos(3t+\pi/3)) = 0$$

Exercise D.3.1, Continued

g.
$$\begin{aligned} y(t) &= 2\sin 3t + 3\cos 3t & y(0) &= 2\times 0 + 3\times 1 = 3 \\ y' &= 6\cos 3t - 9\sin 3t & y'(0) &= 6\times 1 - 9\times 0 = 6 \\ y''(t) &= -18\sin 3t - 27\cos 3t \end{aligned}$$

$$y''(t) + 9y(t) = (-18\sin 3t - 27\cos 3t) + 9(2\sin 3t + 3\cos 3t) = 0$$

i.
$$\begin{aligned} y(t) &= e^{-t}\sin t & y(0) &= 1\times 0 = 0 \\ y' &= e^{-t}\cos t - e^{-t}\sin t & y'(0) &= 1\times 1 - 1\times 0 = 1 \\ y''(t) &= -e^{-t}\sin t - 2e^{-t}\cos t + e^{-t}\sin t \\ &\quad = -2e^{-t}\cos t \end{aligned}$$

$$y''(t) + 2y'(t) + 2y(t) = (-2e^{-t}\cos t) + 2(e^{-t}\cos t - e^{-t}\sin t) + 2(e^{-t}\sin t) = 0$$

k.
$$\begin{aligned}
4.01\times \quad y(t) &= e^{-0.1t}(0.1\sin 2t + 2\cos 2t) && y(0) = 1\times(0+2) = 2\\
+0.2\times \quad y' &= -4.01e^{-0.1t}\sin 2t && y'(0) = -4.01\times 1\times 0 = 0\\
+1\times \quad y''(t) &= -4.01e^{-0.1t}(2\cos 2t - 0.1\sin 2t)\\
= \quad & e^{-0.1t}\left((8.02-8.02)\cos 2t + (0.401-0.802+0.401)\sin 2t\right) = 0
\end{aligned}$$

Exercise D.3.2. a. $\omega = 5$. c. $\omega = \pi$.

Exercise D.3.3. a. $k = 4$. c. $k = 4$.

Exercise D.3.5.

$$\begin{aligned}
y(t) &= A\sin(\omega t) + B\cos(\omega t) && y(0) = A\sin(\omega 0) + B\cos(\omega 0) = B\\
y'(t) &= A\omega\cos(\omega t) - B\omega\sin(\omega t) && y'(0) = A\omega\cos(\omega 0) - B\omega\sin(\omega 0) = A\omega\\
y''(t) &= -A\omega^2\sin(\omega t) - B\omega^2\cos(\omega t)\\
y''(t) + \omega^2 y(t) &=
\end{aligned}$$

$$-A\omega^2\sin(\omega t) - B\omega^2\cos(\omega t) + \omega^2(A\sin(\omega t) + B\cos(\omega t)) = 0$$

Exercise D.3.7.

$$\begin{aligned}
y(t) &= e^{-bt}\cos\sqrt{c-b^2}\,t\\
y'(t) &= -\sqrt{c-b^2}\,e^{-bt}\sin\sqrt{c-b^2}\,t - be^{-bt}\cos\sqrt{c-b^2}\,t\\
y''(t) &= -(c-b^2)e^{-bt}\cos\sqrt{c-b^2}\,t + 2b\sqrt{c-b^2}\,e^{-bt}\sin\sqrt{c-b^2}\,t + b^2e^{-bt}\cos\sqrt{c-b^2}\,t\\
&= (-c+2b^2)e^{-bt}\cos\sqrt{c-b^2}\,t + 2b\sqrt{c-b^2}\,e^{-bt}\sin\sqrt{c-b^2}\,t
\end{aligned}$$

$$y''(t) + 2by'(t) + cy(t) =$$

$$\begin{aligned}
+1\times \quad & \left\{(-c+2b^2)e^{-bt}\cos\sqrt{c-b^2}\,t \quad + \quad 2b\sqrt{c-b^2}\,e^{-bt}\sin\sqrt{c-b^2}\,t\right\}\\
+2b\times \quad & \left\{-\sqrt{c-b^2}\,e^{-bt}\sin\sqrt{c-b^2}\,t \quad - \quad be^{-bt}\cos\sqrt{c-b^2}\,t\right\}\\
+c\times \quad & e^{-bt}\cos\sqrt{c-b^2}\,t\\
= \quad & (-c+2b^2-2b^2+c)e^{-bt}\cos\sqrt{c-b^2}\,t\\
& \qquad +(-2b\sqrt{c-b^2} + 2b\sqrt{c-b^2})e^{-bt}\sin\sqrt{c-b^2}\,t\\
= \quad & 0
\end{aligned}$$

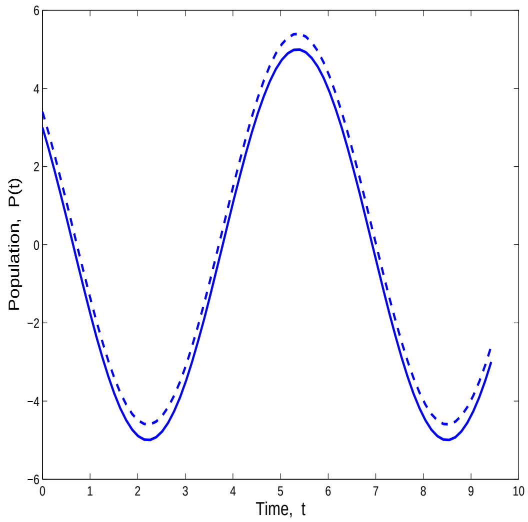


Figure D.25: The solid curve is the graph of $u(t) = -4\sin t + 3\cos t$. The dashed curve is the graph of $u(t) = 5\sin(t+\phi)$ translated 0.4 units upward; without translation, the two graphs are identical.

Exercise D.4.1.

a. See the solid curve in Figure D.25.

b.

$$\begin{aligned} u(t) &= -4\sin t + 3\cos t \qquad \cos\phi = -4/5, \qquad \sin\phi = 3/5 \\ &= 5\left(\frac{-4}{5}\sin t + \frac{3}{5}\cos t\right) \\ &= 5\left(\cos\phi\sin t + \sin\phi\cos t\right) \\ &= 5\sin(t+\phi) \qquad \text{Use formula} \quad \sin(A+B) = \sin A\cos B + \cos A\sin B. \end{aligned}$$

c. The dashed curve in Figure D.25 is the graph of $y = 5\sin(t+\phi)$ translated vertically 0.4 units. Without translation, the graphs are identical.

Exercise D.4.3. For $a = b = 1$, $u_0 = 3$ and $v_0 = 4$, $v(t) = v_0\cos(\sqrt{ab}\,t) + u_0\sqrt{\frac{b}{a}}\sin(\sqrt{ab}\,t)$ becomes $v(t) = 4\cos(t) + 3\sin(t)$. With $\psi = \arccos(3/5)$, $v(t) = 5(\sin\psi\cos t + \cos\psi\sin t) = 5\sin(t+\psi)$.

Exercise D.4.5. 1. At time t_1 the predator is at its maximum and the prey is decreasing at its maximum rate of decrease. 2. At time t_2 the prey is at its minimum and the predator is decreasing at its maximum rate of decrease.

Exercise D.4.6. a. 1887: Lynx 29, Hare 57; 1888: Lynx 17, Hare 18.

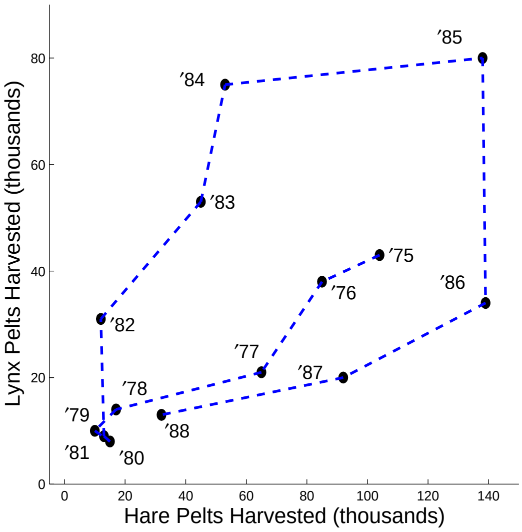


Figure D.26: Phase graph of pelts harvested by the Hudson Bay Company, 1875 to 1888. There is clearly a clockwise rotation.

b. See Figure D.26.

c. There is clearly a clockwise rotation to the path. This means that low hare population leads to increased lynx population and high hare population leads to decreased lynx population, for example, which is contrary to real population forces.

Exercise D.4.8. See Figure D.27

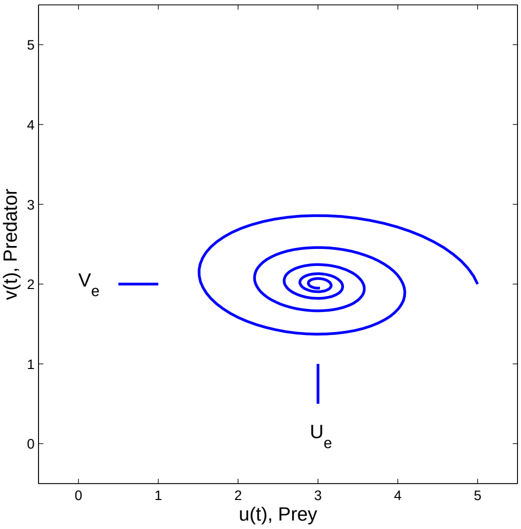


Figure D.27: **Exercise D.4.8.** Phase graph of $u(t) = 2e^{-t/10}\cos t - 0.2e^{-t/10}\sin t$ *vs* $v(t) = e^{-t/10}\sin t$. The graph is drawn around an hypothesized equilibrium point $(U_e, V_e) = (3, 2)$.

Chapter 13

Partial derivatives and Diffusion.

Where are we going?

Most measurable biological quantities are dependent on more than one variable; they are functions of two or more variables. The concept of derivative of a function of one variable is extended to functions of two variables in this chapter. Conditions for local maxima and minima of functions of two variables are presented.

Functions of two variables may describe diffusion of disease, invasive species, heat, or chemicals in space and time dimensions. An equation that relates partial derivatives with respect to to a space variable and with respect to time is introduced and used to quantify diffusion processes.

13.1 Partial derivatives of functions of two variables.

Most measurable biological quantities are dependent on more than one variable. Corn yield is measurably dependent on rainfall, number of degree-days and available nitrogen, potassium, and phosphorus. Brain development in children is dependent on several nutritional factors as well as environmental factors such as rest and sociological experiences.

A function of two variables is a rule that assigns to each ordered number pair in a set called its domain a number in a set called its range. Examples include

$$F(x,y) = x^2 + y^2 \qquad F(x,y) = xe^{-y^2} \qquad F(x,y) = \sin(x+y) \qquad F(x,y) = \sqrt{x}\ln y$$

The domains of the first three functions implicitly are all number pairs (x,y). The first function assigns to the number pair (x,y) the number x^2+y^2; it assigns to (-2,3) the number 13, for example. The domain of $F(x,y) = \sqrt{x}\ln y$ implicitly is the set of number pairs (x,y) for which $x \geq 0$ and $y > 0$. In each example, the first variable of F is x and the second variable of F is y.

Graphs of functions of two variables can be visualized in three-dimensional space (3-space) with the domain D lying in a horizontal x,y plane and the vertical axis being $z = F(x,y)$. Shown in Figure 13.1A is the graph of $F(x,y) = 2$ which is a horizontal plane a distance 2 above the x,y plane. Shown in Figure 13.1B is a graph of the function $F(x,y) = x(1-x) + y(1-y)^2$. Additional graphs of functions of

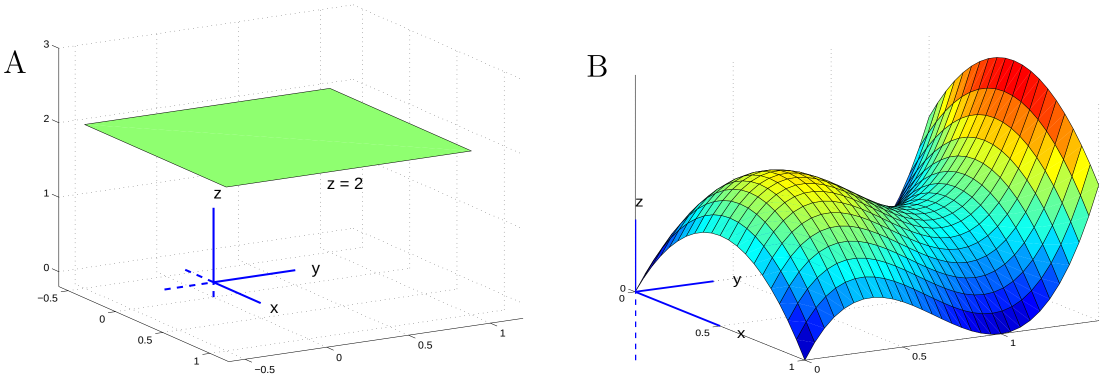


Figure 13.1: Graph in 3-dimensional space of A. $F(x,y) = 2$ and B. $F(x,y) = x(1-x) + y(1-y)^2$.

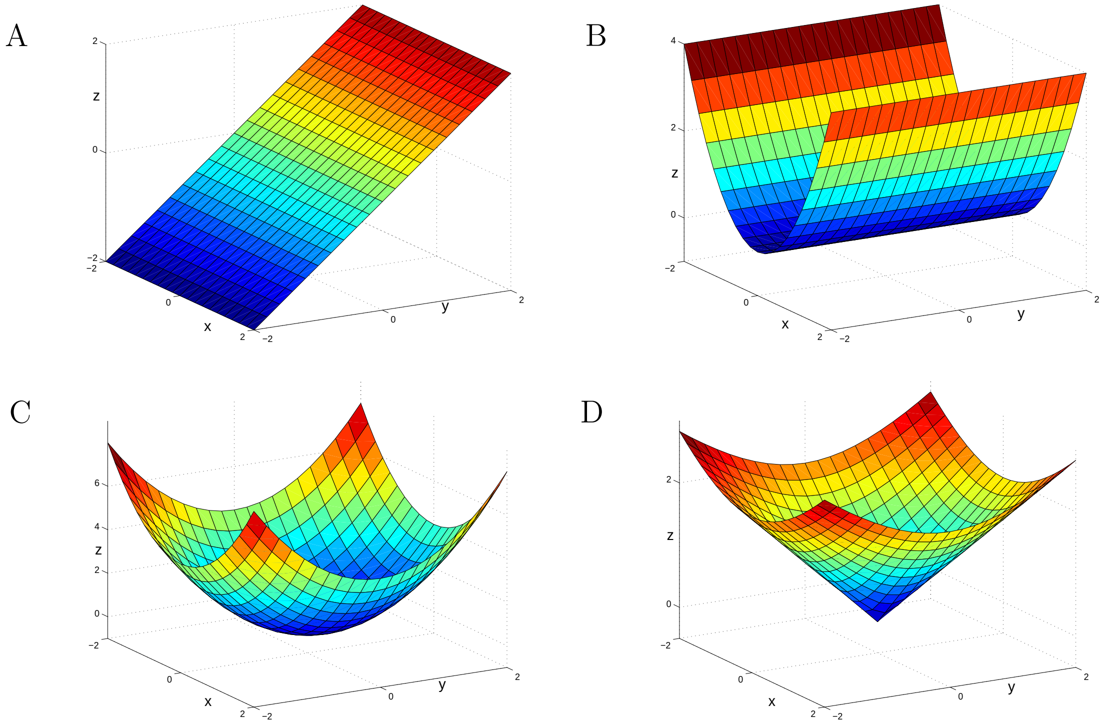


Figure 13.2: Graphs in 3-dimensional space of A. $F(x,y) = y$, B. $F(x,y) = x^2$, C. $F(x,y) = x^2 + y^2$, D. $F(x,y) = \sqrt{x^2 + y^2}$.

two variables are shown in Figure 13.2. In drawing such graphs, it is customary to use a **"right-handed axis system."** Visualize your right hand holding the z-axis and aligned so that your thumb lies on and points in the direction of the positive z-axis. Then in a right-handed system, your fingers will point from the positive x-axis to the positive y-axis.

A linear function of two variables is a function of the form $F(x,y) = ax + by + c$ where a, b, and c are numbers. For example, $F(x,y) = 2x + 3y - 6$ is a linear function and the graph of $z = 2x + 3y - 6$ in Figure 13.3 is a plane in three-dimensional space. The portion of the graph of $z = 2x + 3y - 6$ that lies in the plane $y = 0$ (marked A) is the line $z = 2x - 6$ in the x, z plane. The x, z plane is the set of all points $(x, 0, z)$ in 3-space. The portion of the graph of $z = 2x + 3y - 6$ that lies in the plane $x = 0$ (marked B) is the line $z = 3y - 6$ in the y, z plane. The portion of the graph of $z = 2x + 3y - 6$ that lies in the plane $z = 0$ (marked C) is the line $0 = 2x + 3y - 6$ in the $(x, y, 0)$ plane.

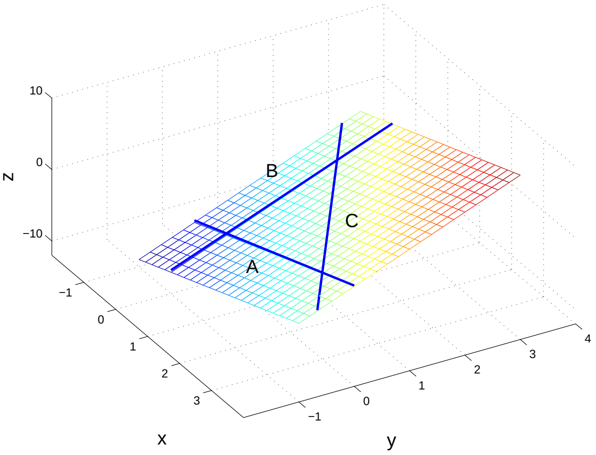


Figure 13.3: Graph of the plane $z = 2x + 3y - 6$ in three dimensional space. The line marked 'A' is the slice of that plane with $y = 0$. The line marked 'B' is the slice with $x = 0$, and the line marked 'C' is the 'level curve' with $z = 0$.

Definition 13.1.1 Partial Derivative. Suppose F is a function of two variables and (a, b) is in the domain of F. The partial derivative of F with respect to its first variable, denoted by F_1, is the ordinary derivative of F with respect to its first variable *with the second variable held constant.* Similarly, the partial derivative of F with respect to its second variable, denoted by F_2, is the ordinary derivative of F with respect to its second variable *with the first variable held constant.*

Second order derivatives are denoted by $F_{1,1}$, $F_{1,2}$, $F_{2,1}$, and $F_{2,2}$ where $F_{i,j}$ is the derivative of F_i with respect to the $j\underline{th}$ variable.

The limit definitions of partial derivatives are

$$F_1(a,b) = \lim_{h \to 0} \frac{F(a+h,b) - F(a,b)}{h} \qquad F_2(a,b) = \lim_{h \to 0} \frac{F(a,b+h) - F(a,b)}{h}. \tag{13.1}$$

The Leibnitz notation can be particularly helpful in writing partial derivatives of $F(x, y)$.

$$F_1(a,b) = \frac{\partial F}{\partial x}(a,b), \qquad F_2(a,b) = \frac{\partial F}{\partial y}(a,b),$$

$$F_{1,1}(a,b) = \frac{\partial^2 F}{\partial x^2}(a,b), \qquad F_{1,2}(a,b) = \frac{\partial^2 F}{\partial y \partial x}(a,b), \qquad F_{2,2}(a,b) = \frac{\partial^2 F}{\partial y^2}(a,b).$$

When notations for the domain variables of F are clear, as in $F = F(x,y)$ it is helpful to write, for examples,

$$\begin{array}{rclcrcl} F_1(x,y) &=& F_x(x,y), & & F_{1,1}(x,y) &=& F_{xx}(x,y), \\ & & & \text{and} & & & \\ F_2(x,y) &=& F_y(x,y), & & F_{2,1}(x,y) &=& F_{yx}(x,y). \end{array}$$

Some examples of partial derivatives are:

$$F(x,y) = x^2 + y^2$$

$$\begin{array}{rclrclrcl} F_1(x,y) &=& 2x & F_{1,2}(x,y) &=& 0 & F_2(x,y) &=& 2y \\ F_{1,1}(x,y) &=& 2 & F_{2,1}(x,y) &=& 0 & F_{2,2}(x,y) &=& 2 \end{array}$$

$$F(x,y) = xe^{-y^2}$$

$$\begin{array}{rclrclrcl} F_1(x,y) &=& e^{-y^2} & F_{1,2}(x,y) &=& e^{-y^2} \times (-2y) & F_2(x,y) &=& xe^{-y^2} \times (-2y) \\ F_{1,1}(x,y) &=& 0 & F_{2,1}(x,y) &=& e^{-y^2} \times (-2y) & F_{2,2}(x,y) &=& -2xe^{-y^2} + 4xy^2e^{-y^2} \end{array}$$

$$F(x,y) = \sin(x+y)$$

$$\begin{array}{rclrclrcl} F_x(x,y) &=& \cos(x+y) & F_{xy}(x,y) &=& -\sin(x+y) & F_y(x,y) &=& \cos(x+y) \\ F_{xx}(x,y) &=& -\sin(x+y) & F_{yx}(x,y) &=& -\sin(x+y) & F_{yy}(x,y) &=& -\sin(x+y) \end{array}$$

$$F(x,y) = x^{1/2}\ln y$$

$$\begin{array}{rclrclrcl} \dfrac{\partial F(x,y)}{\partial x} &=& (1/2)x^{-1/2}\ln y & \dfrac{\partial^2 F(x,y)}{\partial x \partial y} &=& (1/2)x^{-1/2}y^{-1} & \dfrac{\partial F(x,y)}{\partial y} &=& x^{1/2}y^{-1} \\ \dfrac{\partial^2 F(x,y)}{\partial^2 x} &=& -(1/4)x^{-3/2}\ln y & \dfrac{\partial^2 F(x,y)}{\partial y \partial x} &=& (1/2)x^{-1/2}y^{-1} & \dfrac{\partial^2 F(x,y)}{\partial y^2} &=& -x^{1/2}y^{-2} \end{array}$$

In all of these cases, and usually, $F_{1,2} = F_{2,1}$. *Always when* $F_{1,2}$ *and* $F_{2,1}$ *are continuous they are equal.*

Definition 13.1.2 Limit and continuity of a function of two variables. Suppose F is a function of two variables with domain D and (a,b) is a point of D and for every positive number δ there is a point (x,y) of D such that $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$, and L is a number. Then

$$\lim_{(x,y)\to(a,b)} F(x,y) = L$$

means that if ϵ is a positive number, there is a positive number δ such that if (x,y) is in D and $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$ then $|F(x,y) - L| < \epsilon$. The statement that F is continuous at (a,b) means that

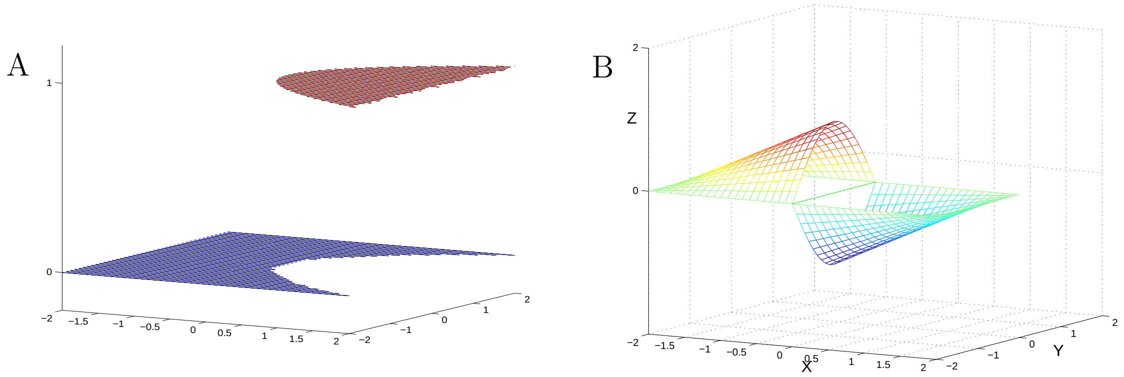
$$\lim_{(x,y)\to(a,b)} F(x,y) = F(a,b)$$

or that there is a number $\delta_0 > 0$ such that for every point (x,y) of D distinct from (a,b), $\sqrt{(x-a)^2 + (y-b)^2} > \delta_0$.

Explore 13.1.1 Identify the points of the graphs in Explore Figure 13.1.1 A and B at which the graphs are discontinuous.

Explore Figure 13.1.1 A. $F(x,y) = 1$ if $x^2 \leq y$; else $F(x,y) = 0$.

B. $F(x,y) = -(1+x/2)\sin(\pi y/2)$ for $x < 0$ and $-2 < y < 0$, $F(x,y) = (1-x/2)\sin(\pi y/2)$ for $x > 0$ and $-2 < y < 0$.



In Figure 13.4, slices of the graph of

$$F(x,y) = x(1-x)^2 + y^2(1-y)$$

corresponding to $x = 0.2$ and $y = 0.4$ are shown. The line tangent to the slice $x = 0.2$ at the point $(0.2, 0.6, 0.0.432)$ is drawn. Its slope is

$$F_2(0.2, 0.6) = 0 + 2y(1-y) - y^2\Big|_{(x,y)=(0.2,0.6)} = 0.12.$$

The line tangent to the slice $y = 0.4$ at the point $(0.4, 0.4, 0.24)$ is also drawn. Its slope is

$$F_1(0.3, 0.4) = (1-x)^2 - 2x(1-x)\Big|_{(x,y)=(0.4,0.4)} = -0.12$$

Definition 13.1.3 Local linear approximation, tangent plane. Suppose F is a function of two variables, (a,b) is a number pair in the domain of F, and F_1 and F_2 exist and are continuous on the interior of a circle with center (a,b). Then the local linear approximation to F at (a,b) is the linear function

$$L(x,y) = F(a,b) + F_1(a,b) \times (x-a) + F_2(a,b) \times (y-b) \tag{13.2}$$

The graph of L is the *tangent plane* to the graph of F at $(a, b, F(a,b))$, and F is said to be differentiable at (a,b).

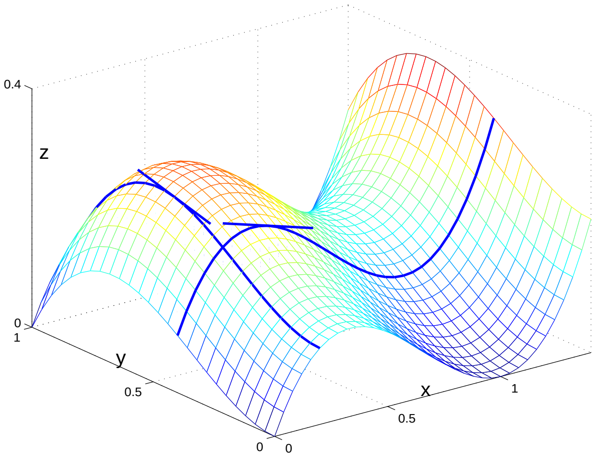


Figure 13.4: Graph in 3-dimensional space of $F(x,y) = x(1-x)^2 + y^2(1-y)$ and slices through the graph at $x = 0.2$ and $y = 0.4$. Observe that this is still a righthanded system although the x- and y-axes are not the same as in previous graphs.

For $F(x,y) = x^2 + y^2$, $F_1(x,y) = 2x$, and $F_2(x,y) = 2y$. At the point (1,2), F_1 and F_2 are continuous on the circle of radius 1 and center (1,2) (actually continuous everywhere), and

$$F(1,2) = 5, \qquad F_1(1,2) = 2x|_{x=1,y=2} = 2, \qquad F_2(1,2) = 4,$$

The local linear approximation to F at (1,2) is

$$L(x,y) = 5 + 2(x-1) + 4(y-2).$$

A graph of F and L appear in Figure 13.5. The graph of L is below the graph of F except at the point of tangency, (1,2,5).

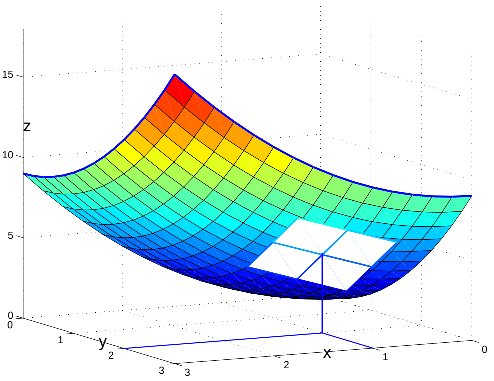


Figure 13.5: Graph in 3-dimensional space of $z = F(x,y) = x^2 + y^2$ and a square lying in the tangent plane at (1,2,5), $L(x,y) = 5 + 2(x-1) + 4(y-2)$, which is below the graph of F except at the point of tangency, (1,2,5). View is from below the graph. Is this a righthanded axis system?

For a function, $F(x,y)$, of two variables to have a tangent it is necessary that $F_1(x,y)$ and $F_2(x,y)$ be continuous. For functions f of one variable, the tangent line to the graph of f at $(a, f(a))$ is simply the line $y = f(a) + f'(a)(x-a)$; $f'(a)$ must exist, but there is no requirement that $f'(x)$ be continuous or even exist for $x \neq a$. For functions of two variables to have a linear approximation, more is required. The example,

$$F(x,y) = \sqrt{|xy|}$$

illustrates the need for conditions beyond the existence of $F_1(a,b)$ and $F_2(a,b)$ in order that there be a tangent plane at $(a, b, F(a,b))$. See Figure 13.6.

$$\begin{array}{rrcl@{\qquad}rcl} \text{For} & F(x,y) & = & \sqrt{|xy|}, & F(0,0) & = & 0 \\ & F(x,0) & = & 0, & F_1(0,0) & = & 0, \\ & F(0,y) & = & 0, & F_2(0,0) & = & 0. \end{array}$$

so the local linear approximation at (0,0) might be

$$L(x,y) = 0 + 0 \times (x-0) + 0 \times (y-0) = 0,$$

the graph of which is the horizontal plane $z = 0$. However, the slice of the graph through $y = x$ for which

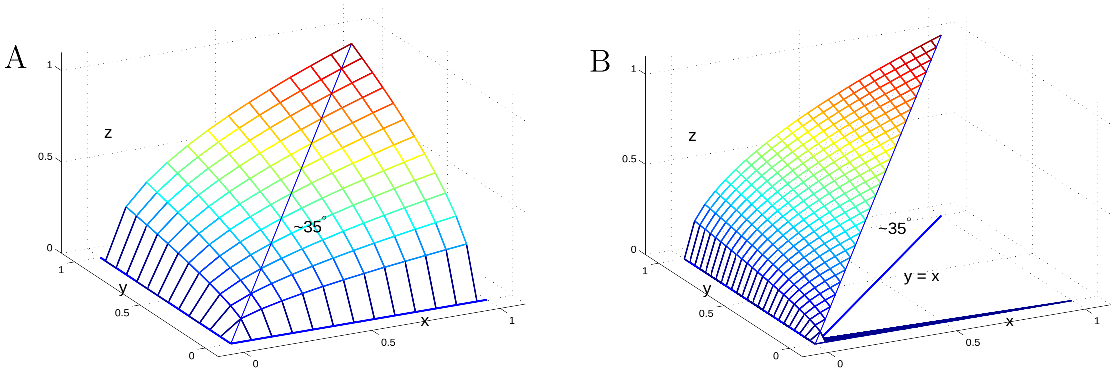


Figure 13.6: A. Graph of $F(x,y) = \sqrt{|xy|}$ on $x \geq 0$, $y \geq 0$. The rest of the graph is obtained by rotation of the portion shown about the z-axis 90, 180 and 270 degrees. There is no local linear approximation to F at (0,0) and no tangent plane to the graph of F at (0,0,0). B. Cut away of the graph of A showing the angle between the surface and the horizontal plane above the line $y = x$.

$z = |x|$ shown in Figure 13.6B is a line that pierces the purported tangent plane, $L(x,y) = 0$, at an angle of about 35 degrees. Thus over the line $y = x$, $L(x,y) = 0$ is not tangent to the graph of F. We do not accept the graph of $L(x,y)$ as tangent to the graph of F. Also, F is not differentiable at (0,0). In this case

$$F_1(x,y) = \frac{1}{2}\sqrt{\frac{|y|}{|x|}} \quad \text{for} \quad x > 0, \quad \text{and} \quad -\frac{1}{2}\sqrt{\frac{|y|}{|x|}} \quad \text{for} \quad x < 0,$$

$$F_1(0,0) = 0, \quad \text{and} \quad F_1(0,y) \quad \text{does not exist for} \quad y \neq 0$$

F_1 is neither continuous nor even defined throughout the interior of any circle with center (0,0). The conditions of Definition 13.1.3 for a local linear approximation are not met.

Explore 13.1.2 Compute $F_2(x,y)$ for $F(x) = \sqrt{|xy|}$. Is F_2 continuous on the interior of a circle with center (0,0)?

Property 13.1.1 A property of tangents to functions of one variable. Suppose f is a function of one variable and at a number a in its domain, $f'(a)$ exists. The graph of $L(x) = f(a) + f'(a)(x - a)$ is the tangent to the graph of f at $(a, f(a))$. Then

$$\begin{aligned} \lim_{x\to a} \frac{|f(x) - L(x)|}{|x-a|} &= \lim_{x\to a} \left| \frac{f(x) - f(a) - f'(a)(x-a)}{x-a} \right| \\ &= \left| \lim_{x\to a} \frac{f(x) - f(a)}{x-a} - f'(a) \right| \qquad (13.3) \\ &= 0. \end{aligned}$$

It is sometimes said that the numerator, $f(x) - L(x)$, in $\dfrac{|f(x) - L(x)|}{|x-a|}$ 'goes to zero faster' than does the denominator, $|x - a|$. *It is this property that is the defining characteristic of 'tangent'.* Only the existence of $f'(a)$ is required. It is not required that $f'(x)$ be continuous.

As was apparent in Figure 13.5B for the function $F(x, y) = \sqrt{|xy|}$, something more than existence of $F_1(a, b)$ and $F_2(a, b)$ is required in order to have a tangent plane for $F(x, y)$ at a point (a, b). A sufficient condition is that F_1 and F_2 exist and be continuous on a circle with center (a, b).

Property 13.1.2 A property of local linear approximations to functions of two variables. Suppose F is a function of two variables, (a, b) is a number pair in the domain of F, and F_1 and F_2 exist and are continuous on the interior of a circle with center (a, b). Then $L(x, y) = F(a, b) + F_1(a, b)(x - a) + F_2(a, b)(y - b)$ is the local linear approximation to F at (a, b), and

$$\lim_{(x,y)\to(a,b)} \frac{|F(x, y) - L(x, y)|}{\sqrt{(x-a)^2 + (y-b)^2}} = 0 \qquad (13.4)$$

The graph of L is the tangent plane to F at the point $(a, b, F(a, b))$.

The proof of Property 13.1.2 involves some interesting analysis that you can understand, but is long enough that we have not included it. We use this property in proving asymptotic stability of systems of difference equations in Chapter 16.

Exercises for Section 13.1, Partial derivatives of functions of two variables.

Exercise 13.1.1 Draw three dimensional graphs of

a. $F(x,y) = 2$ b. $F(x,y) = x$

c. $F(x,y) = x^2$ d. $F(x,y) = (x+y)/2$

e. $F(x,y) = 0.2x + 0.3y$ f. $F(x,y) = (x^2+y^2)/4$

g. $F(x,y) = 0.5\,xe^{-y}$ h. $F(x,y) = \sin y$

i. $F(x,y) = 0.5\,x + \sin y$ j. $F(x,y) = x \sin y$

k. $F(x,y) = \sqrt{x^2+y^2}$ l. $F(x,y) = xy$

m. $F(x,y) = \dfrac{1}{0.4 + x^2 + y^2}$ n. $F(x,y) = e^{(} - x^2 - y^2)$

o. $F(x,y) = |xy|$ p. $F(x,y) = \sin(x^2+y^2)$

Exercise 13.1.2 Find the partial derivatives, F_1, F_2, $F_{1,1}$, $F_{1,2}$, $F_{2,1}$ and $F_{2,2}$ of the following functions.

a. $F(x,y) = 3x - 5y + 7$ b. $F(x,y) = x^2 + 4xy + 3y^2$

c. $F(x,y) = x^3y^5$ d. $F(x,y) = \sqrt{xy}$

e. $F(x,y) = \ln(x \times y)$ f. $F(x,y) = \frac{x}{y}$

g. $F(x,y) = e^{x+y}$ h. $F(x,y) = x^2e^{-y}$

i. $F(x,y) = \sin(2x+3y)$ j. $F(x,y) = e^{-x}\cos y$

Exercise 13.1.3 Is the plane $z = 0$ a tangent plane to the graph of $F(x,y) = \sqrt{x^2+y^2}$ shown in Figure 13.2D.

Exercise 13.1.4 Find $F_1(a,b)$ and $F_2(a,b)$ for

a. $F(x,y) = \dfrac{x}{1+y^2}$ $(a,b) = (1,0)$

b. $F(x,y) = \sqrt{x^2+y^2}$ $(a,b) = (1,2)$

c. $F(x,y) = e^{-xy}$ $(a,b) = (0,0)$

d. $F(x,y) = \sin x \cos y$ $(a,b) = (\pi/2, \pi)$

Exercise 13.1.5 Find the local linear approximation, $L(x,y)$, to $F(x,y)$ at the point (a,b). For each case, compute

$$\frac{|F(x,y) - L(x,y)|}{\sqrt{(x-a)^2 + (y-b)^2}} \quad \text{for} \quad (x,y) = (a+0.1, b), \quad \text{and} \quad (x,y) = (a+0.01, b+0.01).$$

$$\begin{array}{lrcll} \text{a.} & F(x,y) = 4x + 7y - 16 & (a,b) & = & (3,2) \\ \text{b.} & F(x,y) = xy & (a,b) & = & (2,1) \\ \text{c.} & F(x,y) = \dfrac{x}{y+1} & (a,b) & = & (1,0) \\ \text{d.} & F(x,y) = xe^{-y} & (a,b) & = & (1,0) \\ \text{e.} & F(x,y) = \sin \pi(x+y) & (a,b) & = & (1/2, 1/4) \end{array}$$

Exercise 13.1.6 For $P = nRT/V$, find $\frac{\partial}{\partial V}P$ and $\frac{\partial}{\partial T}P$. For fixed T, how does P change as V increases? For fixed V, how does P change as T increases?

Exercise 13.1.7 Draw the graph of $F(x,y)$ and the graph of the plane tangent to the graph of F at the point (a,b). A MATLAB program to solve c. is shown in the answer to this exercise, Exercise 13.1.7.

$$\begin{array}{lrcl} \text{a.} & F(x,y) = 0.5x + y + 1 & (a,b) & = (0.5,1) \\ \text{b.} & F(x,y) = (x^2 - y^2)/2 & (a,b) & = (1,1) \\ \text{c.} & F(x,y) = (x^2 + y^2)/2 & (a,b) & = (1,0.5) \\ \text{d.} & F(x,y) = e^{x\,y/2} & (a,b) & = (1,0.2) \\ \text{e.} & F(x,y) = \sqrt{9 - x^2 - y^2} & (a,b) & = (1,0.2) \\ \text{f.} & F(x,y) = 4 - x^2 - y^2 & (a,b) & = (1,0.2) \\ \text{g.} & F(x,y) = 2/(1 + x^2 + y^2) & (a,b) & = (1,1) \\ \text{h.} & F(x,y) = e^{-x^2-y^2} = e^{-x^2}\, e^{-y^2} & (a,b) & = (1,0.5) \end{array}$$

Exercise 13.1.8 Let F be defined by

$$\begin{aligned} F(x,y) &= x^2 \quad \text{for} \quad y > 0 \\ &= 0 \quad \text{for} \quad y \le 0 \end{aligned}$$

1. Sketch a graph of F in three dimensional space.
2. Is $F_1(x,y)$ continuous on the interior of a circle with center $(0,0)$?
3. Let $L(x,y) = 0$ for all (x,y). Is it true that

$$\lim_{(x,y)\to(0,0)} \frac{F(x,y) - L(x,y)}{\sqrt{x^2+y^2}} = 0 \quad ?$$

4. Are you willing to call the plane $z = 0$ a tangent plane to the graph of F?

13.2 Maxima and minima of functions of two variables.

Figure 13.7B shows a slice of the graph of $F(x,y) = x(1-x)^2 + y^2(1-y)$ through $x = 1/3$ and a slice through $y = 2/3$. The point $(1/3, 2/3, 8/27)$ at which these slices intersect appears to be and is a local maximum for F. The two tangents to these slices are horizontal as would be expected at an interior local maximum for a function of a single variable. These two slices were selected by solving two equations for x and y:

$$\begin{aligned} F_1(x,y) &= 1 - 4x + 3x^2 = 0 \\ F_2(x,y) &= 2y - 3y^2 = 0. \end{aligned}$$

There are four solutions, called *critical points* of F,

$$(x,y) = (1/3, 2/3), \quad (x,y) = (1/3, 0), \quad (x,y) = (1, 2/3) \quad \text{and} \quad (x,y) = (1,0).$$

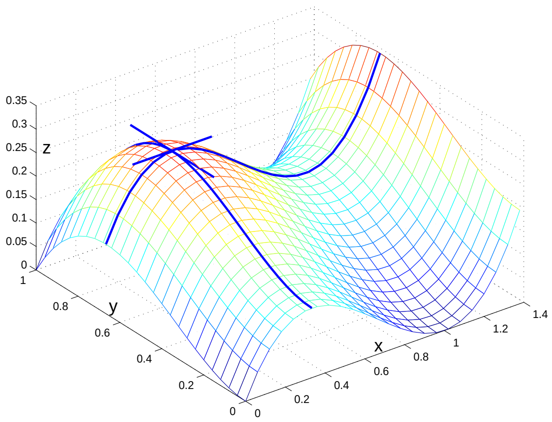


Figure 13.7: Slices through the graph of $F(x,y) = x(1-x)^2 + y^2(1-y)$ through $x = 1/3$ and $y = 2/3$. (1/3,2/3,8/27) is a local maximum of F.

Definition 13.2.1 Critical points. A point (a,b) of the domain D of a function of two variables, F, is an *interior point* of the domain if there is a circle centered at (a,b) whose interior is a subset of D. Points of D that are not interior points are called *boundary points.*

An interior point (a,b) of F is a *critical point* of F means that

$$F_1(a,b) = 0 \qquad \text{and} \qquad F_2(a,b) = 0$$

or that one of $F_1(a,b)$ or $F_2(a,b)$ fails to exist.

Boundary points of D are also critical points of F.

There are four critical points of $F(x,y) = x(1-x)^2 + y^2(1-y)$,

$$(1/3, 2/3), \qquad (1, 2/3), \qquad (1/3, 0), \qquad \text{and} \qquad (1, 0).$$

The critical point (1/3,2/3) is discussed above and shown in Figure 13.7.

The critical point (1,2/3) is illustrated in Figure 13.8 and is neither a high point nor a low point, it is a *saddle* point. The surface is convex upward in the x-direction and is concave downward in the y-direction. The tangent at (1,2/3,4/27) parallel to the y-axis is above the surface; the tangent parallel to the x-axis is below the surface and would normally not be visible.

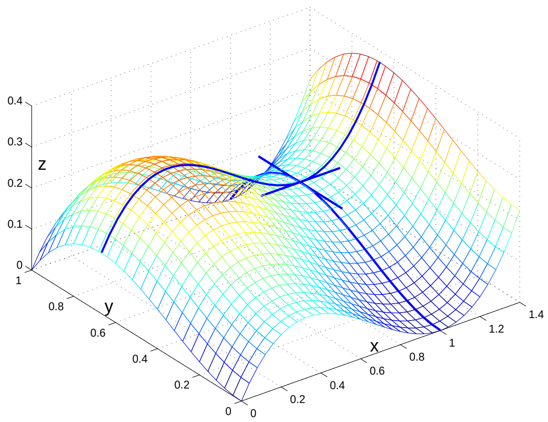


Figure 13.8: Slices through the graph of $F(x,y) = x^2(1-x) + y(1-y)^2$ through $x = 1$ and $y = 2/3$. (1,2/3,4/27) is a saddle point of F.

The critical point (1,0) is illustrated in Figure 13.9 and is a local minimum for F and is illustrated in Figure 13.9. The domain for y is [-0.4,1]; in Figures 13.7 and 13.8 the domain for x is [0,1]. Also the viewpoint is lower in Figure 13.9 than in Figures 13.7 and 13.8 in order to look underneath the graph at the low point.

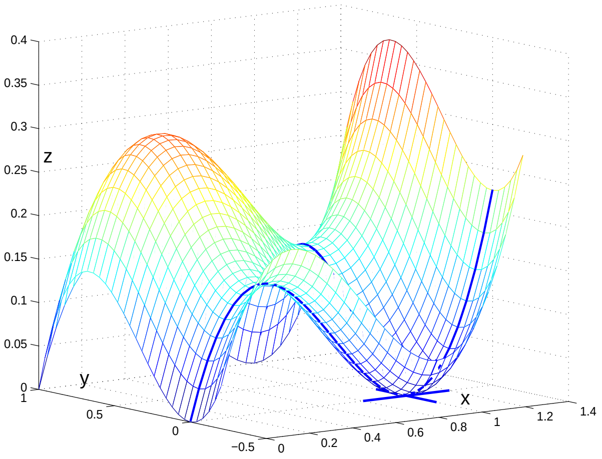


Figure 13.9: Slices through the graph of $F(x,y) = x(1-x)^2 + y^2(1-y)$ through $x = 1$ and $y = 0$. (1,0,0) is a local minimum of F.

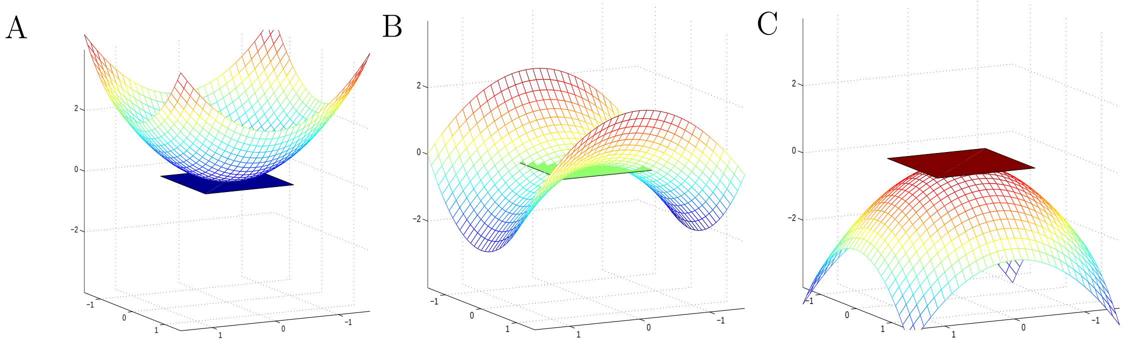


Figure 13.10: Graphs of A. $z = x^2 + y^2$, B. $z = x^2 - y^2$, and C. $z = -x^2 - y^2$. The local linear approximation to each is the horizontal plane $z = 0$.

Explore 13.2.1 Locate the critical point (1/3,0) of $F(x, y) = x(1 - x)^2 + y^2(1 - y)$ in Figure 13.9 and classify it as a local maximum, local minimum or saddle point. ■

Example 13.2.1 The graphs of $F(x, y) = x^2 + y^2$, $G(x, y) = x^2 - y^2$ and $H(x, y) = -x^2 - y^2$ shown in Figure 13.10 illustrate three important options. The origin, (0,0), is a critical point of each of the graphs and the $z = 0$ plane is the tangent plane to each of the graphs at (0,0). For F, for example,

$$\begin{array}{rclcrclcrcl} F(x,y) & = & x^2+y^2 & \quad & F_1(x,y) & = & 2x & \quad & F_2(x,y) & = & 2y \\[1em] F(0,0) & = & 0 & & F_1(0,0) & = & 0 & & F_2(0,0) & = & 0 \end{array}$$

The origin, (0,0), is a critical point of F, the linear approximation to F at (0,0) is $L(x, y) = 0$, and the tangent plane is $z = 0$, or the x, y plane. The same is true for G and H; the tangent plane at (0,0,0) is $z = 0$ for all three examples. This seemingly monotonous information is saved by the observations of the relation of the tangent plane to the graphs of the three functions. For F, (0,0,0) is the lowest point of the graph of F, and for H, (0,0,0) is the highest point of the graph of H. This is similar to the horizontal lines associated with high and low points of graphs of functions of one variables. The graph of G is different. The point (0,0,0) is called a saddle point of the graph of G. The portion of G in the x, z plane ($y = 0$) has a low point at (0,0,0) and the portion of G in the y, z plane ($x = 0$) has a high point at (0,0,0).

For a function f of a single variable and a number a for which $f'(a) = 0$, there is a simple second derivative test, Theorem 9.2.3, that distinguishes whether $(a, f(a))$ is a locally high point ($f''(a) < 0$) or a locally low point ($f''(a) > 0$). There is also a second derivative test for functions of two variables.

Definition 13.2.2 Definition of Local Maxima and Minima. If (a, b) is a point in the domain of a function F of two variables, $F(a, b)$ is a local maximum for F means that there is a number $\delta_0 > 0$ such that if (x, y) is in the domain of F and $\sqrt{(x - a)^2 + (y - b)^2} < \delta_0$ then $F(x, y) \leq F(a, b)$.

The definition of local minimum is similar.

Theorem 13.2.1 Local Maxima and Minima of functions of two variables. Suppose (a, b) is a critical point of a function F of two variables that has continuous first and second partial derivatives in a circle with center at (a, b) and

$$\Delta = F_{1,1}(a,b)F_{2,2}(a,b) - (F_{1,2}(a,b))^2.$$

Case 1. If $\Delta > 0$ and $F_{1,1}(a,b) > 0$ then $F(a,b)$ is a local minimum of F.

Case 2. If $\Delta > 0$ and $F_{1,1}(a,b) < 0$ then $F(a,b)$ is a local maximum of F.

Case 3. If $\Delta < 0$ then $(a, b, F(a,b))$ is a saddle point of F.

Case 4. If $\Delta = 0$, punt, use another supplier.

We omit the proof of Theorem 13.2.1, but illustrate its application to the functions F, G, and H of Example 13.2.1 and shown in Figure 13.10.

$$\text{For} \quad F(x,y) = x^2 + y^2, \quad F_{1,1}(0,0) = 2 > 0, \quad \Delta = 2 \times 2 - 0 = 4 > 0,$$

and the origin is a local minimum for F.

$$\text{For} \quad G(x,y) = x^2 - y^2, \quad \Delta(0,0) = 2 \times (-2) - 0 = -4 < 0,$$

and the origin is a saddle point for G. We have not defined a saddle point. For our purposes, a saddle point is a critical point for which $\Delta < 0$.

$$\text{For} \quad H(x,y) = -x^2 - y^2, \quad H_{1,1}(0,0) = -2, \quad \Delta = -2 \times (-2) - 0 = 4 > 0,$$

and the origin is a local maximum for H.

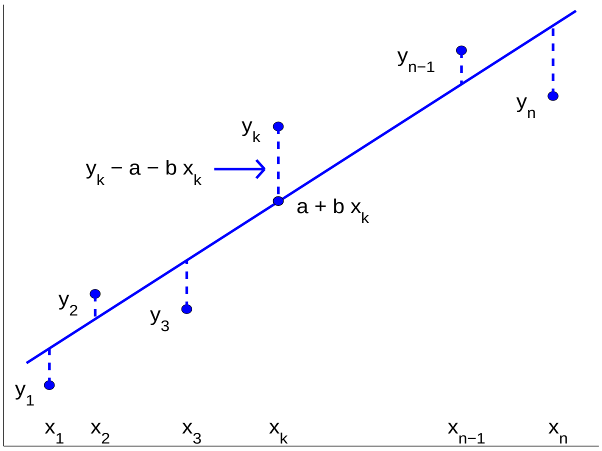
Example 13.2.2 Least squares fit of a line to data. A two variables minimization problem crucial to the sciences is the fit of a linear function to data.

Problem. Given points (x_1, y_1), (x_2, y_2), $\cdots$, (x_n, y_n), where $x_i \neq x_j$ if $i \neq j$, find numbers a_0 and b_0 so that $(a, b) = (a_0, b_0)$ minimizes

$$SS(a,b) = \sum_{k=1}^{n} (y_k - a - bx_k)^2 \tag{13.5}$$

In Figure 13.2.2.2, $SS(a,b)$ is the sum of the squares of the lengths of the dashed lines.

Figure for Example 13.2.2.2 A graph of data and a line. $SS(a,b)$ is the sum of the squares of the lengths of the dashed lines.



Solution. The critical points of SS are the solutions to the equations

$$SS_1(a,b) = \frac{\partial}{\partial a} SS = 0 \qquad SS_2(a,b) = \frac{\partial}{\partial b} SS = 0. \tag{13.6}$$

$$\begin{aligned}
SS_1(a,b) &= \frac{\partial}{\partial a}\left[\sum_{k=1}^{n} \left(y_k - a - b x_k\right)^2\right] \\
&= \sum_{k=1}^{n} \frac{\partial}{\partial a} \left(y_k - a - b x_k\right)^2 \\
&= \sum_{k=1}^{n} 2\left(y_k - a - b x_k\right)(-1) \\
&= 2\sum_{k=1}^{n} \left(a + b x_k - y_k\right) \\
&= 2\left(a \sum_{k=1}^{n} 1 + b \sum_{k=1}^{n} x_k - \sum_{k=1}^{n} y_k\right) \\
SS_2(a,b) &= \frac{\partial}{\partial b}\left[\sum_{k=1}^{n} \left(y_k - a - b x_k\right)^2\right] \\
&= \sum_{k=1}^{n} \frac{\partial}{\partial b} \left(y_k - a - b x_k\right)^2 \\
&= \sum_{k=1}^{n} 2\left(y_k - a - b x_k\right)(-\,x_k) \\
&= 2\sum_{k=1}^{n} \left(a x_k + b x_k^2 - x_k y_k\right) \\
&= 2\left(a \sum_{k=1}^{n} x_k + b \sum_{k=1}^{n} s_k^2 - \sum_{k=1}^{n} x_k y_k\right)
\end{aligned}$$

Imposing the conditions $SS_1(a,b) = 0$ and $SS_2(a,b) = 0$ and simplifying leads to the *Normal Equations*:

$$\begin{array}{rcrcl}
a\,n & + & b\,\sum_{k=1}^{n} x_k & = & \sum_{k=1}^{n} y_k \\
a\,\sum_{k=1}^{n} x_k & + & b\,\sum_{k=1}^{n} x_k^2 & = & \sum_{k=1}^{n} x_k y_k
\end{array} \tag{13.7}$$

Some notation is useful:

$$S_x = \sum_{k=1}^n x_k, \quad S_y = \sum_{k=1}^n y_k, \quad S_{xx} = \sum_{k=1}^n x_k^2, \quad S_{xy} = \sum_{k=1}^n x_k\, y_k, \quad S_{yy} = \sum_{k=1}^n y_k^2.$$

The solution to the Normal Equations 13.7 is

$$\begin{aligned} a_0 &= \frac{S_{xx}\, S_y - S_x S_{xy}}{\Delta} \\ b_0 &= \frac{nS_{xy} \; - \; S_x S_y}{\Delta} \\ \Delta &= nS_{xx} \; - \; (S_x)^2 \end{aligned} \tag{13.8}$$

It is important that $\Delta \neq 0$. The proof that Δ is actually positive involves some clever and not very intuitive algebra. Form the sum

$$S = \sum_{k=1}^n (S_{xx} - S_x x_k)^2 \;.$$

Because the x_k are distinct, at least one of $S_{xx} - S_x x_k \neq 0$ and $S > 0$.

$$\begin{aligned} S &= \sum_{k=1}^n (S_{xx} - S_x x_k)^2 \\ &= \sum_{k=1}^n ((S_{xx})^2 - 2S_{xx}S_x x_k + (S_x)^2 x_k^2) \\ &= \sum_{k=1}^n (S_{xx})^2 - 2S_{xx}S_x \sum_{k=1}^n x_k + (S_x)^2 \sum_{k=1}^n x_k^2 \\ &= n(S_{xx})^2 - 2S_{xx}S_x \times S_x + (S_x)^2 \times S_{xx} \\ &= S_{xx}(nS_{xx} - (S_x)^2) \\ &= S_{xx} \times \Delta \end{aligned}$$

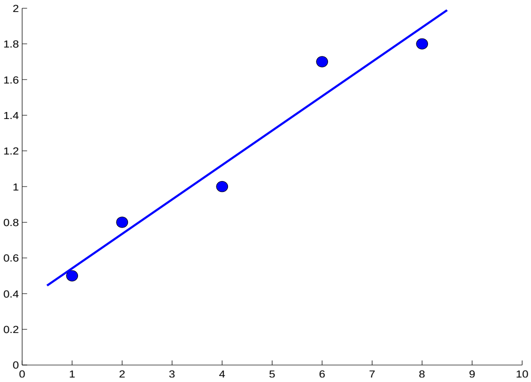
Now $S_{xx} > 0$ and $S = S_{xx} \times \Delta > 0$, so $\Delta > 0$.

Explore 13.2.2 Why is S_{xx} positive? ▬

Example 13.2.3 Fit a line to the data in Example Figure 13.2.3.3.

Figure for Example 13.2.3.3 Data, a graph of the data and a line $y = a + bx$ fit to the data:

x_t	y_t
1	0.5
2	0.8
4	1.0
6	1.7
8	1.8



$$\begin{aligned} S_x &= 1+2+4+6+8=21, \\ S_y &= 0.5+0.8+1.0+1.7+1.8=5.8 \\ S_{xx} &= 1^2+2^2+4^2+6^2+8^2=121, \\ S_{xy} &= 1\times 0.5+2\times 0.8+4\times 1.0+6\times 1.7+8\times 1.8=30.7 \end{aligned}$$

$$\begin{aligned} \Delta &= nS_{xx}\ -\ (S_x)^2 = 5\times 121-(21)^2=164 \\ a &= (S_{xx}\,S_y - S_xS_{xy})/\Delta = (121\times 5.8-21\times 30.7)/164 = 0.348 \\ b &= (nS_{xy}\ -\ S_xS_y)/\Delta = (5\times 30.7-21\times 5.8)/164 = 0.193 \end{aligned}$$

The line $y = 0.348 + 0.193x$ is the closest to the data in the sense of least squares and is drawn in Example Figure 13.2.3.3.

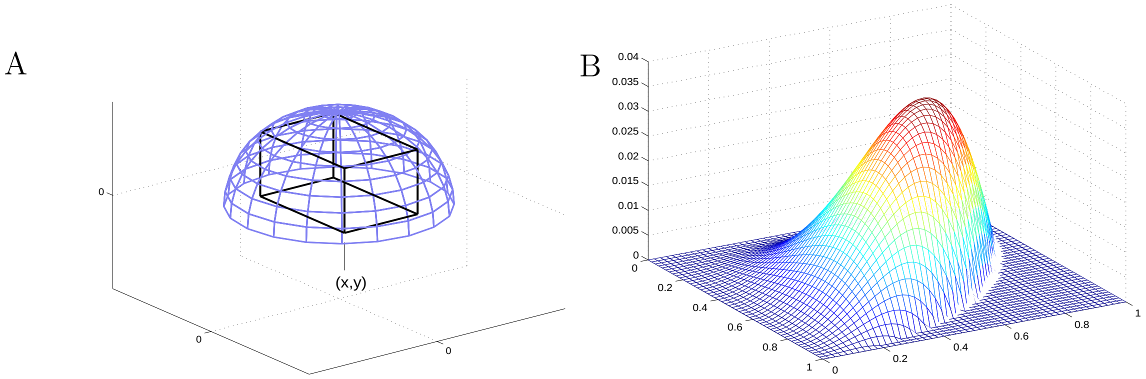
You would systematically organize the arithmetic if you fit very many lines to data as we just did. Better than that, however, is that your calculator does all of this arithmetic for you. In STAT, CALC you have LinR. Enter your data in xStat and yStat and on the screen type LinR xStat,yStat. LinR is found in STAT, CALC and xStat and yStat are found in LIST, NAMES. The following MATLAB commands do the same task for the data in Example Figure 13.2.3.3.

```
close all;clc;clear                          |    plot(x,y,'o','linewidth',2)
x=[1 2 4 6 8]; y=[0.5 0.8 1.0 1.7 1.8];      |    hold
P=polyfit(x,y,1)                             |    plot(xp,yp,'linewidth',2)
xp=[0.5 8.5];    yp=polyval(P,xp);           |
```

Example 13.2.4 *Problem:* Find the dimensions of the largest box (rectangular solid) that will fit in a hemisphere of radius R.

Solution. Assume the hemisphere is the graph of $z = \sqrt{R^2 - x^2 - y^2}$ and that the optimum box has one face in the x, y-plane and the other four corners on the hemisphere (see Figure 13.2.4.4).

Figure for Example 13.2.4.4 A. A box in a hemisphere. One corner of the box is at (x, y) in the x, y-plane. B. Graph of $W(x, y) = \max(0, x^2\, y^2\, (1 - x^2 - y^2))$.



The volume, V of the box is

$$V(x, y) = 2x \times 2y \times z = 2x \times 2y \times \sqrt{R^2 - x^2 - y^2}$$

Before launching into partial differentiation, it is perhaps clever, and certainly useful, to observe that the values of x and y for which V is a maximum are also the values for which $V^2/16$ is a maximum.

$$\frac{V^2(x, y)}{16} = W(x, y) = \frac{16x^2y^2(R^2 - x^2 - y^2)}{16} = R^2x^2y^2 - x^4y^2 - x^2y^4.$$

It is easier to analyze $W(x, y)$ than it is to analyze $V(x, y)$.

$$\begin{aligned} W_1(x, y) &= 2R^2xy^2 - 4x^3y^2 - 2xy^4 \\ &= 2xy^2(R^2 - 4x^2 - 2y^2) \\ W_2(x, y) &= 2R^2x^2y - 2x^4y^2 - 4x^2y^3 \\ &= 2x^2y(R^2 - 2x^2 - 4y^2) \end{aligned}$$

Solving for $W_1(x, y) = 0$ and$W_2(x, y) = 0$ yields $x = R/\sqrt{3}$ and $y = R/\sqrt{3}$, for which $z = R/\sqrt{3}$. The dimensions of the box are $2R/\sqrt{3}$, $2R/\sqrt{3}$, and $R/\sqrt{3}$.

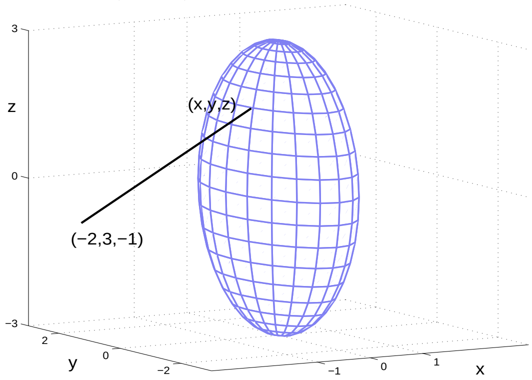
Explore 13.2.3 Are there some critical points of W other than $(\,R/\sqrt{3},\ R/\sqrt{(}3),\ R/\sqrt{3}\,)$? ■

Example 13.2.5 Warning: Obnubilation Zone. *Problem.* Find the point of the ellipsoid

$$\frac{x^2}{1} + \frac{y^2}{4} + \frac{z^2}{9} = 1 \qquad \text{that is closest to} \qquad (-2, 3, -1) \tag{13.9}$$

See Example Figure 13.2.5.5

Figure for Example 13.2.5.5 The ellipsoid $x^2/1 + y^2/4 + z^2/9 = 1$ and a line from (-2,3,-1) to a point (x,y,z) of the ellipsoid.



Solution. Claim without proof: The point (x, y, z) of the ellipsoid that is closest to $(-2, 3, -1)$ will have negative x, positive y and negative z coordinates.

The distance between (a, b, c) and (p, q, r) in 3-dimensional space is

$$\sqrt{(p-a)^2 + (q-b)^2 + (r-c)^2}.$$

The distance from (-2,3,-1) to (x, y, z) on the ellipsoid is

$$D(x, y, z) = \sqrt{(x+2)^2 + (y-3)^2 + (z+1)^2}$$

We solve for z in Equation 13.9 and write

$$z = -3\sqrt{1 - x^2 - y^2/4} \qquad z \text{ is negative.} \tag{13.10}$$

$$E(x, y) = \sqrt{(x+2)^2 + (y-3)^2 + (1 - 3\sqrt{1 - x^2 - y^2/4})^2} \tag{13.11}$$

Define $F(x, y) = (E(x, y))^2$ and write

$$\begin{aligned}
F(x, y) &= (x+2)^2 + (y-3)^2 + \left(1 - 3\sqrt{1 - x^2 - y^2/4}\right)^2 \\
F_1(x, y) &= 2(x+2) + 2\left(1 - 3\sqrt{1 - x^2 - y^2/4}\right) \frac{\partial}{\partial x}\left(1 - 3\sqrt{1 - x^2 - y^2/4}\right) \\
&= 2(x+2) + 2\left(1 - 3\sqrt{1 - x^2 - y^2/4}\right) \frac{(-3)(-2x)}{2\sqrt{1 - x^2 - y^2/4}} \\
F_2(x, y) &= 2(y-3) + 2\left(1 + 3\sqrt{1 - x^2 - y^2/4}\right) \frac{3(-2y/4)}{2\sqrt{1 - x^2 - y^2/4}}
\end{aligned}$$

Now we have a mess. We need to (and can!) solve for (x, y) in

$$\begin{aligned}
F_1(x, y) &= 2(x+2) + 6x\left(\frac{1}{\sqrt{1 - x^2 - y^2/4}} - 3\right) &= 0 \\
F_2(x, y) &= 2(y-3) + \frac{6y}{4}\left(\frac{1}{\sqrt{1 - x^2 - y^2/4}} - 3\right) &= 0
\end{aligned}$$

First we write

$$\begin{aligned} 2(x+2) &= -6x\left(\frac{1}{\sqrt{1-x^2-y^2/4}}-3\right) \qquad (13.12) \\ 2(y-3) &= -\frac{6y}{4}\left(\frac{1}{\sqrt{1-x^2-y^2/4}}-3\right), \end{aligned}$$

divide corresponding sides of the two equations, and find that

$$y = \frac{12x}{3x-2}. \qquad (13.13)$$

Substitute this expression for y in Equation 13.12, simplify, and find that

$$8x - 2 = \frac{\mp 3x(3x-2)}{\sqrt{(1-x^2)(9x^2-12x+4)-36x^2}}$$

Square both sides of this equation and clear fractions.

$$(8x-2)^2\left[(1-x^2)(9x^2-12x+4)-36x^2\right] = 9x^2(9x^2-12x+4)$$

Multiply and collect.

$$576x^6 - 1056x^5 + 2485x^4 - 380x^3 - 480x^2 + 176x - 16 = 0 \qquad (13.14)$$

Now we have it! Go to POLY on your calculator, enter order = 6, and enter the coefficients. Press SOLVE and wait 20 seconds. The two real (not complex) answers are $x = -0.482870022$ and $x = 0.165385675$. Use $x = -0.482870022$ and compute $y = 1.680224831$ and $z = -0.726205610$ from Equations 13.13 and 13.10, and the distance from (-2,3,-1) to the ellipsoid is 2.029. **Whew!**

Explore 13.2.4 For the algebraically strong, fill in the algebra omitted in Example 13.2.5

Exercises for Section 13.2 Maxima and minima of functions of two variables.

Exercise 13.2.1 Find the critical points, if any, of F.

a. $F(x,y) = 2x + 5y + 7$

b. $F(x,y) = x^2 + 4xy + 3y^2$

c. $F(x,y) = x^3(1-x) + y$

d. $F(x,y) = xy(1-xy)$

e. $F(x,y) = (x-x^2)(y-y^2)$

f. $F(x,y) = \frac{x}{y}$

g. $F(x,y) = e^{x+y}$

h. $F(x,y) = \sin(x+y)$

i. $F(x,y) = \frac{x^2}{1+y^2}$

j. $F(x,y) = \cos x \sin y$

Exercise 13.2.2 For each of the following functions, find the critical points and use Theorem 13.2.1 to determine whether they are local maxima, local minima, or saddle points or none of these.

$$\begin{array}{lrcl} \text{a.} & F(x,y) & = & -x^2+xy-y^2 \\ \text{b.} & F(x,y) & = & x^2+xy-y^2 \\ \text{c.} & F(x,y) & = & x^2+y^2-2xy+2x-2y \\ \text{d.} & F(x,y) & = & -x^2-5y^2+2xy-10x+6y+20 \end{array}$$

Exercise 13.2.3 Find C and b so that Ce^{bx} closely approximates the data

x	0	1	2	3	4
y	2.18	5.98	16.1	43.6	129.7

Observe that for $y = Ce^{bx}$, $\ln y = \ln C + bx$. Therefore, fit $a + bx$ to the number pairs, $(x, \ln y)$ using linear least squares. Then $\ln y_k \doteq a + bx_k$, and

$$y_k \doteq e^{a+bx_k} = e^a \times e^{bx_k} = Ce^{bx_k}, \quad \text{where} \quad C = e^a.$$

Exercise 13.2.4 a. Find a, b, and c so that $y = a + bx + cx^2$ is the least squares approximation to data, (x_1, y_1), $x_2, y_2)$, $\cdots$, (x_n, y_n). To do so you will need to minimize

$$SS = \sum_{k=1}^{n} \Big(y_k - (a + bx_k + cx_k^2)\Big)^2 .$$

This is a three-variable minimization problem. The solution will be similar to the least squares line approximation to data of Example 13.2.2. b. In Exercise Table 13.2.4 are data showing the height of a ball falling in air above a Texas Instruments CBL motion detector. Find the parabola that is the least squares fit to the data.
c. Check your answer by using P2reg on your calculator.

Table for Exercise 13.2.4 Height of a ball falling in aire above a Texas Instruments CBL motion detector.

Time$_i$ sec	0.232	0.333	0.435	0.537	0.638	0.739	0.840
Position$_i$ cm	245.9	235.8	214.3	184.1	146.0	99.3	45.5

Exercise 13.2.5 Find a and b so that $\sin(ax + b)$ closely approximates the data

x	0	1	2	3	4
y	0.97	0.70	0.26	-0.26	-0.5

Observe that for $y = \sin(ax + b)$, $\arcsin y = ax + b$. Therefore, fit $ax + b$ to the number pairs, $(x, \arcsin y)$ using linear least squares.

Exercise 13.2.6 Interpret the real root $x = 0.165385675$ of Equation 13.14 related to the ellipsoid example.

Exercise 13.2.7 Find the largest box that will fit in the positive octant ($x \geq 0$, $y \geq 0$, and $z \geq 0$) and underneath the plane $z = 12 - 2x - 3y$.

Exercise 13.2.8 Find the largest box that will fit in the positive octant and underneath the hemisphere $z = \sqrt{25 - x^2 - y^2}$.

Exercise 13.2.9 Find the point of the plane $z = 2x + 3y - 12$ that is

1. closest to the origin.
2. closest to (4,5,6)

Exercise 13.2.10 Find the point of the sphere $x^2 + y^2 + z^2 = 25$ that is closest to (3,4,5).

Exercise 13.2.11 Find the point of the ellipsoid of Equation 13.9

$$\frac{x^2}{1} + \frac{y^2}{4} + \frac{z^2}{9} = 1 \qquad \text{that is farthest from} \qquad (-2, 3, -1) \tag{13.15}$$

Exercise 13.2.12 In Exercise 8.3.3[1] we found that the average power over a whole cycle of a bounding flight of a bird should be

$$\overline{P} = (1 - x)P_{\text{folded}} + xP_{\text{flapping}} = A_b u^3 + xA_w u^3 + B\frac{m^2 g^2}{xu}$$

where A_b and $A_b + A_w$ are, respectively, drag coefficients of the bird without wings extended and with wings extended and flapping, B is a constant, u is the speed of flight, m is the mass of the bird, g is the acceleration of gravity, and x is the fraction of a flight cycle during which the wings are flapping. You were to find the fraction, x, for which the required power is a minimum and the fraction, x, for which the required energy is a minimum, with $E = \overline{P}/u$. You may have found that $x = \sqrt{A_w/B}mg/u^2$ for both answers.

Find, if there is one, the combination of flight speed, u, and fraction x for which the power $\overline{P}$ is a minimum.

Find, if there is one, the combination of flight speed, u, and fraction x for which the energy E is a minimum.

---

[1] Analysis based on R. McNeill Alexander, "Optima for Animals" Princeton University Press, Princeton, NJ, 1996, Section 3.1, pp 45-48.

13.3 The diffusion equation $u_t(x,t) = c^2\, u_{xx}(x,t)$.

Partial derivatives may appear in equations and when they do the equations are called partial differential equations. This is a vast field of study. We give one very important example, the diffusion equation, and suggest a numerical method for its solution.

The diffusion equation in one space dimension is

$$u_t(x,t) \quad = \quad c^2 u_{xx}(x,t), \qquad a \leq x \leq b, \qquad 0 \leq t. \tag{13.16}$$

Embryonic digit development On the limb bud of the developing vertebrate embryo, there is a "zone of polarizing activity" (ZMP) on the "pinky" side of the limb bud that causes the pinky side to be different from the thumb side. A protein, *sonic hedgehog*, is emitted from the ZMP and diffuses across the limb bud. The concentration of sonic hedgehog decreases with the distance from the ZMP; those digits closest to ZMP with high hedgehog protein concentration develop into the pinky and ring fingers; those farthest from the ZMP with low hedgehog protein concentration develop into the index finger and thumb (see Figure 13.11). This extremely important diffusion example is too complex for us to model, and we turn to simpler problems.

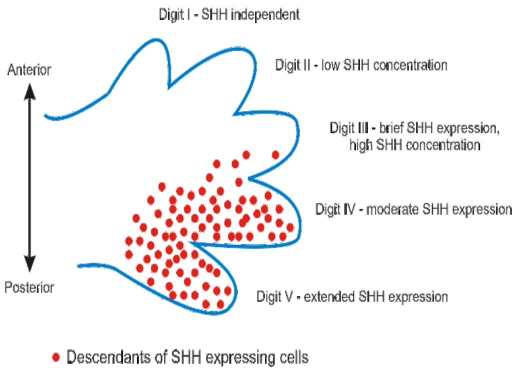


Figure 13.11: Sonic hedgehog specifies digit identity in mammalian development. From http://en.wikipedia.org/wiki/Hedgehog_signaling_pathway

You may think of a brass rod of diameter, d, that is small compared to its length, $L = b - a$, and $u(x,t)$ as the temperature in the rod at position x and at time t. If the temperature is not at equilibrium, the flow of heat in the rod will cause the temperature to equilibrate according to Equation 13.16. Alternatively, you may think of a glass tube of diameter, d, that is small compared to its length, $L = b - a$, and is filled with distilled water with a small amount of salt dissolved in it, and $u(x,t)$ as the concentration of salt at position, x, and time t. If the salt concentration is not at equilibrium, salt will flow in the rod and will cause the concentration to equilibrate according to Equation 13.16.

A single equation describes $u(x,t)$ for both of these problems and a host of other problems. Molecules diffuse in intercellular fluids during embryonic development and within intracellular fluids; diseases diffuse in a population; an invasive species diffuses over an extended range. James D. Murray has shown[2] that a linked pair of reaction-diffusion equations defined on a two-dimensional surface can nicely replicate numerous mammalian coat patterns, ranging from the giraffe, leopard, zebra, to the elephant (neutral).

In order to determine $u(x,t)$ it is necessary to know the initial state of u,

$$u(x,0) = g(x), \qquad \text{say,}$$

and the constraints on $u(x,t)$ at the ends of the rod,

$$u(a,t) = h_a(t), \qquad \text{and} \quad u(b,t) = h_b(t), \quad \text{for example.}$$

We derive the diffusion equation in terms of the diffusion of salt (or any solute). Consider a circular glass rod of cross sectional area A and length L filled with water and assume that at time $t = 0$ the concentration of the salt along the tube is $g(x)$. Let $u(x,t)$ be the salt concentration at distance x from one end of the rod at time t. Figure 13.12A. Assume that the diameter of the rod is small enough that the salt concentration at any position x along the rod depends only on x.

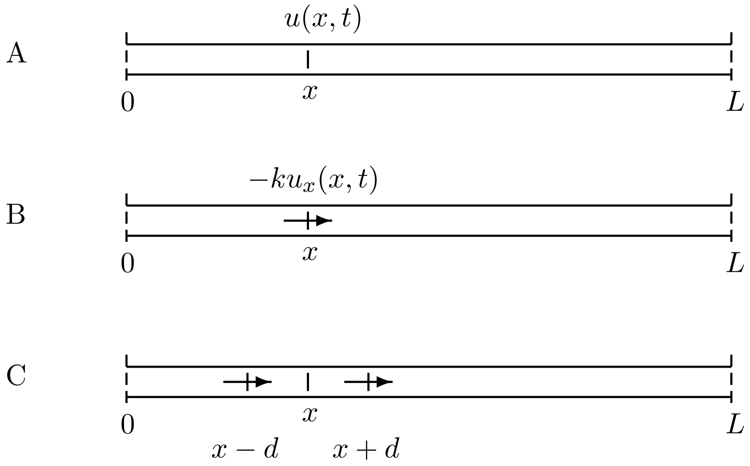


Figure 13.12: A. Glass rod, salt concentration at x at time t is $u(x,t)$. B. The rate at which salt flows past position x is proportional to $-u_x(x,t)$. C. The amount of salt between $x - d$ and $x + d$ at time t is approximately $u(x,t) \times A \times 2d$.

> **Diffusion: Mathematical model 1.** Salt tends to flow from regions of high concentration to regions of low concentration. The rate at which salt flows past position x in the direction of increasing x at time t is proportional to $-u_x(x,t)$ and to A, the cross sectional area of the rod. Figure 13.12B.

From this model we write

$$R(x,t) = -kAu_x(x,t) \tag{13.17}$$

where $R(x,t)$ is the rate at which salt diffuses past position x at time t and k is a proportionality constant that is a property of the solvent/solute system.

---

[2] James D. Murray, How the leopard gets its spots, *Scientific American,* 1988, **258**(3) pp 80-87., James D. Murray, *Mathematical Biology,* Springer-Verlag, Heidelberg, 1989.

Units. The units on u, u_x, A and R are, respectively,

$$\begin{array}{rcll} u & : & \dfrac{\text{gm}}{\text{cm}^3} & \dfrac{\text{mass}}{\text{distance}^3} \\[2ex] u_x & : & \dfrac{\text{gm}}{\text{cm}^3 \times cm} & \dfrac{\text{mass}}{\text{distance}^4} \\[2ex] A & : & \text{cm}^2 & \text{distance}^2 \\[2ex] R & : & \dfrac{\text{gm}}{\text{sec}} & \dfrac{\text{mass}}{\text{time}} \end{array}$$

In order to balance the units on Equation 13.17, the units on k must be cm^2/sec.

$$R = kAu_x, \qquad \frac{\text{gm}}{\text{sec}} = k \times \text{cm}^2 \times \frac{\text{gm}}{\text{cm}^3 \times cm}, \qquad \frac{1}{\text{sec}} = k \times \frac{1}{\text{cm}^2}, \qquad k = \frac{\text{cm}^2}{\text{sec}}.$$

Now consider a section of the rod between $x - d$ and $x + d$, Figure 13.12C. During a time interval t to $t + \delta$ salt flows past $x - d$, flows past $x + d$, and accumulates (or depletes) in the section.

> **Diffusion: Mathematical model 2.** The amount of salt in any small region of the tube is approximately the concentration of salt at some point in the region times the volume of the region.

From Mathematical Model 2 we write that the amount of salt in the section from $x - d$ to $x + d$ at time t is approximately $u(x, t) \times A \times 2d$.

> **Diffusion: Mathematical model 3.** During a time interval from t to $t + \delta$, the amount of salt that flows into a region minus the amount of salt that flows out of the region is the the accumulation within the region.

Now $R(x - d, t)$ is the rate at which salt flows past $x - d$ in the direction of increasing x. The amount of salt that flows into the section $x - d$ to $x + d$ during the time t to $t + \delta$ is approximately $R(x - d, t) \times \delta$. The amount of salt that flows out of the section during time t to $t + \delta$ is approximately $R(x + d, t) \times \delta$, Either of rates, $R(x - d, t)$ and $R(x + d, t)$, rates may be negative with corresponding flow negative.

The accumulation in the section during a time interval from t to $t + \delta$ is approximately $u(x, t + \delta) \times A \times 2d$ minus $u(x, t) \times A \times 2d$.

For Mathematical Model 3, we write

$$\begin{aligned} u(x, t + \delta) \times A \times 2d - u(x, t) \times A \times 2d &\doteq R(x - d, t) \times \delta - R(x + d, t) \times \delta \\[3ex] &= -kAu_x(x - d, t) \times \delta - (-kAu_x(x + d, t)) \times \delta \\[3ex] \frac{u(x, t + \delta) - u(x, t)}{\delta} &\doteq k\frac{u_x(x + d, t) - u_x(x - d, t)}{2d} \qquad (13.18) \end{aligned}$$

Using the Mean Value Theorem[3] twice, there are numbers τ in $(t, t+\delta)$ and ξ in $(x-d, x+d)$ such that

$$\frac{u(x,t+\delta)-u(x,t)}{\delta} = u_t(x,\tau) \qquad \text{and} \qquad \frac{u_x(x+d,t)-u_x(x-d,t)}{2d} = u_{xx}(\xi,t)$$

Then

$$u_t(x,t) \doteq u_t(x,\tau) \doteq k u_{x,x}(\xi,t) \doteq k u_{xx}(x,t)$$

As $\delta \to 0$ and $d \to 0$, all of the errors reduce (we suppose to zero) and we write

$$u_t(x,t) = k u_{xx}(x,t) \qquad \text{Diffusion equation.} \tag{13.19}$$

The proportionality constant k is positive and is usually written as c^2 to signal this and to simplify analytical solutions. As noted above, the units on k are cm^2/sec. The size of k reflects how rapidly the salt moves in water or the heat moves in a rod or a disease spreads in a population or generally how rapidly a substance diffuses in its medium. If k is large, $u(x,t)$ changes rapidly; if k is small, $u(x,t)$ changes slowly. For example,

$$u(x,t) = e^{-kt}\sin x, \qquad 0 \le x \le \pi, \qquad 0 \le t$$

is a solution to Equation 13.19:

$$u_t(x,t) = -ke^{-kt}\sin x, \qquad u_x(x,t) = e^{-kt}\cos x, \qquad u_{xx}(x,t) = -e^{-kt}\sin x,$$

$$u_t(x,t) = k u_{xx}(x,t).$$

Graphs of $e^{-t}\sin x$ and $e^{-0.5t}\sin x$ appear in Figure 13.13A and B respectively. It can be seen that the graph in B with smaller k changes more slowly than the graph in A.

---

[3] Theorem 9.1.1, Mean Value Theorem: If F is a continuous function defined on an interval $[a,b]$ and F' is continuous on (a,b), then there is a number c in (a,b) such that $F'(c) = (F(b)-F(a))/(b-a)$.

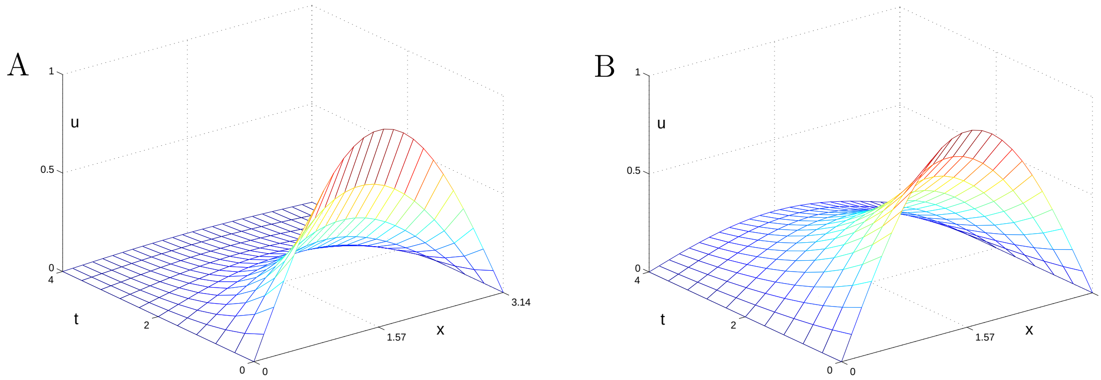


Figure 13.13: A. Graph in 3-dimensional space of $u(x,t) = e^{-t}\sin x$. B. Graph of $u(x,t) = e^{-0.5t}\sin x$.

Explore 13.3.1 For any salt concentration, $u(x,t)$, the amount of salt in the tube at time t is $\int_0^L u(x,t)\,dx$. Show that for $u(x,t) = e^{-kt}\sin x$, $0 \leq x \leq \pi$, the amount, of salt in the tube at time t is $2\,e^{-kt}$. In both cases, $\int_0^\pi u(x,t)\,dx$ is initially 2 and decreases with time; the amount of salt in the tube is decreasing and salt must be leaking out the ends.

The initial concentration of salt in the rod is required in order to compute the concentration at later times. Assume that there is a known function g such that at time $t = 0$ the concentration at position x is $g(x)$. Then

$$u(x,0) = g(x) \qquad 0 < x < L \qquad \text{Initial condition.} \tag{13.20}$$

Finally we need some knowledge about the ends of the rod, referred to as *boundary conditions.* The ends may be sealed so that no salt diffuses past either end. This is expressed as

$$u_x(0,t) = u_x(L,t) = 0, \qquad 0 \leq t, \qquad \text{Insulated boundary conditions.} \tag{13.21}$$

Alternatively, we might assume the rod connects two reservoirs in which the salt concentration is constant, salt may diffuse into or out of the reservoir, but there is no flow of solvent into or out of the rod. Then there will be two concentrations, C_0 and C_L, such that

$$u(0,t) = C_0, \qquad U(L,t) = C_L, \qquad 0 \leq t \qquad \text{Fixed boundary conditions.} \tag{13.22}$$

Explore 13.3.2 Suppose the $x = 0$ end of the tube is attached to a reservoir with salt concentration $c_0 = 1$ and the $x = L$ end of the tube is sealed. What would be the boundary conditions? What would $u(x,t)$ be for 'large' values of t?

Equation 13.19 for which $u(x,t)$ is concentration of salt also describes the temperature, $u(x,t)$, in a rod of length L that is insulated along its sides. In this case, the initial condition $g(x)$ would be the temperature distribution along the rod at time $t = 0$. The rod may also be insulated at each end and boundary condition 13.21 would apply, and this accounts for the name 'Insulated boundary condition.' Alternatively, one end of the rod may be exposed to, say, steam and the other end exposed to ice water, and Fixed Boundary Condition 13.22 would apply.

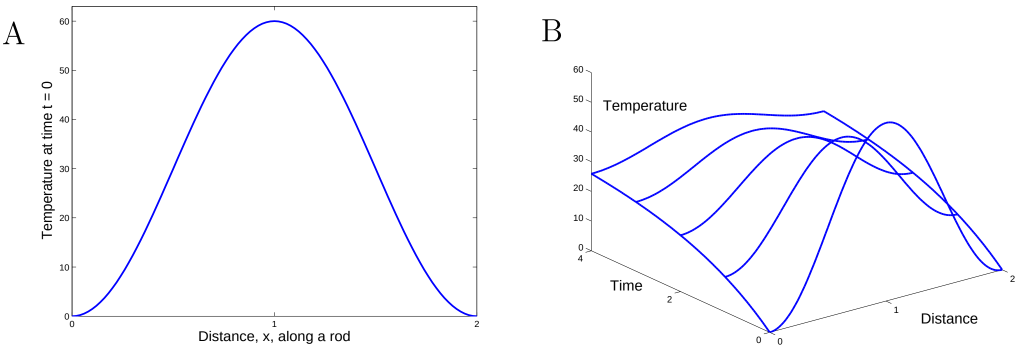


Figure 13.14: Partial graphs of Equation 13.23. A. Graph of temperature at time $t = 0$. B. Graphs of the temperature at times 0, 0.5, 1.0, 1.5 and 2.0. The graphs of temperature at the ends $x = 0$ and $x = 2$ are included.

Explore 13.3.3 Suppose on a flat sandy beach the temperature at a depth of 2 meters is constant, equal to 20°C, and the temperature at the surface of the beach is $27 + 4\sin((2\pi/24)t)$ °C. Suppose the sand temperature varies only vertically. What equations would you like to solve if you were interested in a nest of turtle eggs buried 80 cm?

There are analytical solutions to the diffusion equation 13.19 with initial condition 13.20 with either of the boundary conditions 13.21 or 13.22. Only a few of them are simple enough for our use. We describe one example and include two examples in Exercises 13.3.5 and 13.3.10.

Example 13.3.1 *Problem.* Let

$$u(x,t) = 30 * (1 - e^{-t}\cos\pi * x) \qquad 0 \le x \le 2 \qquad 0 \le t. \tag{13.23}$$

Show that

$$u_t(x,t) = \frac{1}{\pi^2}u_{xx}(x,t), \quad u(x,0) = 30 * (1 - \cos\pi x), \quad \text{and} \quad \begin{array}{rcl} u_x(0,t) & = & 0 \\ u_x(2,t) & = & 0. \end{array} \tag{13.24}$$

Assuming Equations 13.24 are correct, $u(x,t)$ would describe the temperature in a rod of length 2 that is perfectly insulated along its side, had an initial temperature of $30 * (1 - \cos\pi x)$ at position x, and was perfectly insulated on each end. The diffusion coefficient of the material in the rod is $k = 1/\pi^2$. Alternatively, the equations would describe the salt concentration in a tube closed at each end when the initial salt concentration at position x was $30 * (1 - \cos\pi x)$.

Explore 13.3.4 What will be the 'eventual' temperature distribution (or salt concentration) in the rod (tube)?

A graph of the initial temperature distribution appears in Figure 13.14A, and graphs of the temperature at times 0, 0.5, 1, 1.5 and 2 appear in Figure 13.14B.

Solution. First compute some partial derivatives.

$$u(x,t) \quad = \quad 30(1 - e^{-t}\cos\pi x) \tag{13.25}$$

$$u_t(x,t) = 30(0-(e^{-t})(-1)\cos\pi x) = 30e^{-t}\cos(\pi x) \tag{13.26}$$

$$u_x(x,t) = 30(0-e^{-t}(-\sin\pi x)(\pi) = 30\pi e^{-t}\sin\pi x \tag{13.27}$$

$$u_{xx}(x,t) = 30\pi^2 e^{-t}\cos\pi x \tag{13.28}$$

From Equations 13.26 and 13.28

$$u_t(x,t) = 30e^{-t}\cos(\pi x) = \frac{1}{\pi^2}30\pi^2 e^{-t}\cos(\pi x) = \frac{1}{\pi^2}u_{xx}.$$

From Equation 13.23

$$u(x,0) = 30*(1-e^{-t}\cos\pi * x)\Big|_{t=0} = 30(1-\cos\pi x).$$

From Equation 13.27,

$$u_x(0,t) = 30\pi e^{-t}\sin\pi x\Big|_{x=0} = 0 \qquad \text{and} \qquad u_x(2,t) = 30\pi e^{-t}\sin\pi x\Big|_{x=2} = 0$$

Thus all of Equations 13.24 are satisfied.

13.3.1 Numerical solutions to the diffusion equation.

Finding analytic solutions to the diffusion equation is beyond the scope of this text. However, a numerical scheme for approximating a solution is well within reach.

Partition the tube into n intervals of length $d = L/n$, and partition time into intervals of length δ, as shown in Figure 13.15 for $n = 5$ and the space time grid shown above the tube.

Begin with Equation 13.19,

$$u_t(x,t) = ku_{xx}(x,t)$$

From Equations 9.22 and 9.24

$$u_t(x,t) \doteq \frac{u(x,t+\delta)-u(x,t)}{\delta} \qquad \text{and} \qquad u_{xx}(x,t) \doteq \frac{u(x+d,t)-2u(x,t)+u(x-d,t)}{d^2}.$$

Using these in the previous equation leads to

$$\frac{u(x,t+\delta)-u(x,t)}{\delta} \doteq k\frac{u(x+d,t)-2u(x,t)+u(x-d,t)}{d^2}. \tag{13.29}$$

Now we write an exact equation

$$\frac{v_{i,j+1}-v_{i,j}}{\delta} = k\frac{v_{i+1,j}-2v_{i,j}+v_{i-1,j}}{d^2}. \tag{13.30}$$

where $v_{i,j}$ is an approximation to $u(i*d, j*\delta)$. Equations 13.30 can be solved as is illustrated in the next example. Using this notation in Equation 13.30 and rearranging leads to

$$v_{i,j+1} = v_{i,j} + \hat{k}\,(v_{i-1,j}-2v_{i,j}+v_{i+1,j}) \qquad \text{where} \qquad \hat{k} = \frac{k\times\delta}{d^2}. \tag{13.31}$$

Equation 13.31 defines $v_{i,j+1}$ (at time value $j+1$)in terms of $v_{i-1,j}$, $v_{i,j}$ and $v_{i+,j}$, values of v at the immediately preceding time value j. See Figure 13.16. The computation is started at time $j = 0$ with values of $v_{i,0}$ equal to values of the initial condition $g(x)$ and progresses 'upward' in time, one layer at a time.

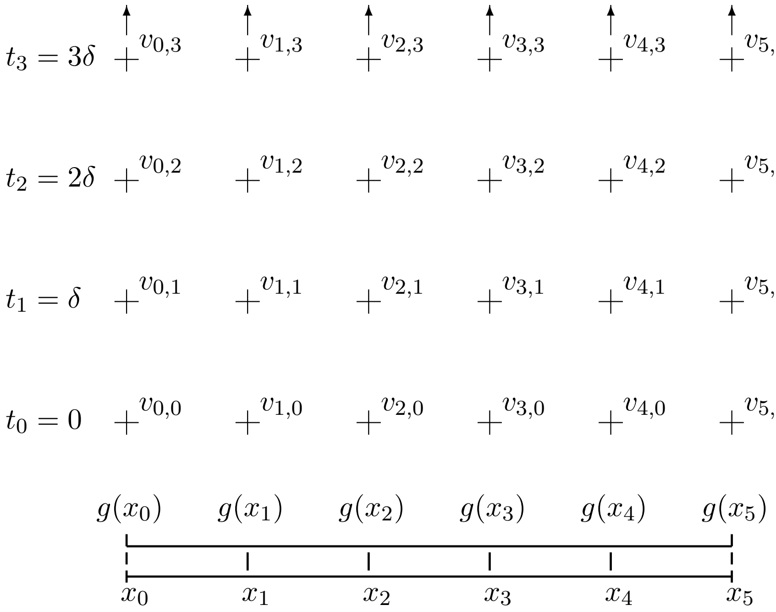


Figure 13.15: Glass tube and grid for diffusion computation.

Example 13.3.2 Assume there is a glass tube of length $L = 1$ meter and of cross sectional area $A = 1$ cm filled with water; initially there is no salt in the tube; the left end is attached to a salt water reservoir with salt concentration = 1 and the right end is attached to a pure water reservoir. Let $u(x,t)$ be the salt concentration in the tube at position x and time t.

Explore 13.3.5 What do you expect the 'eventual' salt concentration along the tube to be?

The diffusion equation is $u_t(x,t) = k u_{xx}(x,t)$. Because initially there is no salt in the tube, $g(x) = 0$ for $0 < x < 1$, and the reservoirs at the ends of the tube imply the fixed boundary conditions 13.22, $u(0,t) = 1$ and $u(1,t) = 0$, $0 \leq t$.

Partition the tube into 5 equal intervals. There is in Figure 13.15 an array of points horizontally distributed with position x along the tube and distributed vertically in time t.

In this example,

$$v_{i,j} \doteq u(i \times 1/5, j \times \delta), \qquad i = 1, 5, \qquad j = 1, \cdots .$$

and from Equation 13.30

$$v_{i,j+1} = v_{i,j} + \hat{k}\,(v_{i-1,j} - 2v_{i,j} + v_{i+1,j}) \qquad \text{where} \qquad \hat{k} = \frac{\delta \times k}{d^2}. \tag{13.32}$$

The boundary conditions 13.22 with $u(0,t) = 1$ and $u(1,t) = 0$ lead to

$$v_{0,j} = 1 \qquad\qquad v_{5,j} = 0 \tag{13.33}$$

The initial condition and equations 13.32 determine the $v_{i,j}$ one horizontal layer at a time for the interior grid points, $1 < i < 5$.

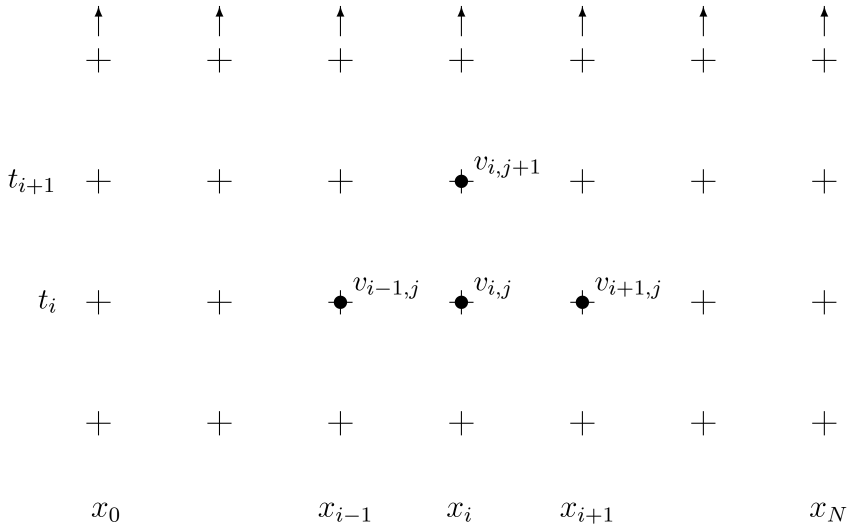


Figure 13.16: Geometric arrangement of the grid points of Equation 13.31.

Begin with the initial condition, Equation 13.20:

$$v_{i,0} = g(x_i) = 0, \qquad i = 1, 4 \tag{13.34}$$

Then for the bottom layer of the grid in Figure 13.15

$$v_{0,0} = 1 \qquad \text{and} \qquad v_{j,0} = 0 \qquad j = 1, \cdots, 5.$$

Then compute the next layer up for $t = \delta$:

$$\begin{aligned}
v_{1,1} &= v_{1,0} + \hat{k}(v_{0,0} - 2v_{1,0} + v_{2,0}) \\
v_{2,1} &= v_{2,0} + \hat{k}(v_{1,0} - 2v_{2,0} + v_{3,0}) \\
v_{3,1} &= v_{3,0} + \hat{k}(v_{2,0} - 2v_{3,0} + v_{4,0}) \\
v_{4,1} &= v_{4,0} + \hat{k}(v_{3,0} - 2v_{4,0} + v_{5,0}) \\
v_{0,1} &= v_{1,1} \\
v_{5,1} &= v_{4,1}
\end{aligned}$$

In a similar way as many layers as necessary can be computed. The computations for $\hat{k} = 0.2$ are shown below. Calculator and MATLAB programs that produce the computations appear in Table 13.1.

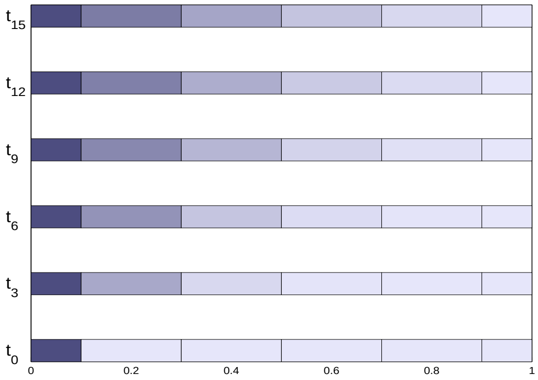
Remember that $\hat{k} = \dfrac{\delta \times k}{d^2}$ incorporates the time step, δ, and dimension step, d, as well as the diffusion constant, k.

Table 13.1: A calculator program and MATLAB program and approximations computed from Equations 13.32-13.34 with $\hat{k} = 0.2$

```
:Fix 3
:6->dimL V
:6->dimL VN
:For(i,1,6)
:0->V(I)
:End
:For(J,1,15)
:For(I,2,5)
:V(I)+.2*(V(I+1)-2*V(
I)+V(I-1))->VN(I)
:End
:1->VN(1)
:0->VN(6)
:Pause VN
:For(I,1,6)
:VN(I)->V(I)
:End
:End
:Fix 9
```

```
MATLAB
close all;clc;clear
for j = 1:6
  v(1,j) = 0;
end
v(1,1)=1;
for i = 2:15
   v(1,i) = 1;
   v(6,i) = 0;
   for j = 2:5
      dd=v(i-1,j+1)-2*v(i-1,j)+v(i-1,j-1);
      v(i,j) = v(i-1,j)+0.2*dd;
      v(i,j) = v(i-1,j)+0.2*dd;
    end
end
v
```

t_{15}	1.000	0.688	0.421	0.224	0.092	0.000
t_{14}	1.000	0.678	0.406	0.209	0.084	0.000
t_{13}	1.000	0.667	0.389	0.194	0.075	0.000
t_{12}	1.000	0.655	0.371	0.178	0.066	0.000
t_{11}	1.000	0.642	0.351	0.160	0.056	0.000
t_{10}	1.000	0.626	0.329	0.142	0.047	0.000
t_9	1.000	0.609	0.305	0.123	0.037	0.000
t_8	1.000	0.588	0.278	0.102	0.028	0.000
t_7	1.000	0.565	0.248	0.082	0.019	0.000
t_6	1.000	0.536	0.214	0.061	0.011	0.000
t_5	1.000	0.501	0.177	0.041	0.005	0.000
t_4	1.000	0.458	0.134	0.022	0.002	0.000
t_3	1.000	0.400	0.088	0.008	0.000	0.000
t_2	1.000	0.320	0.040	0.000	0.000	0.000
t_1	1.000	0.200	0.000	0.000	0.000	0.000
t_0	1.000	0.000	0.000	0.000	0.000	0.000
	x_0	x_1	x_2	x_3	x_4	x_5



These computations may be visualized as salt (gray color) migrating to the right in a tube as illustrated in the graphic. ▬

There is a severe constraint on $\hat{k} = \frac{k \times \delta}{d^2}$ in order that Equations 13.31 yield values of $v_{i,j}$ that reasonably approximate the target function $u(x,t)$. We must have

$$\hat{k} = \frac{k \times \delta}{d^2} < \frac{1}{2}. \tag{13.35}$$

The consequence of $\hat{k} > 1/2$ is illustrated in Exercise 13.3.1b, where $\hat{k} = 0.6$. In Example 13.3.2 the tube partition is $d = 0.2$ meters. If one wanted a smaller partition in order to more accurately approximate the salt concentration, say $d = 0.01$, one centimeter or 1/20th of the space step of the example, then the time step δ would have to be 1/400th of the first time dimension. It would take 400 iterations of the resulting equations in order to move one of the original time steps. There is an interesting alternate procedure that is not so constrained that will be found in numerical analysis books.

Exercises for Section 13.3, The diffusion equation.

Exercise 13.3.1

1. Enter the program of Table 13.1 into your calculator, run it and confirm the computations shown in Table 13.1, or do the same computations with the MATLAB program.
2. The program is written for $\hat{k} = 0.2$. Alter the program so that $\hat{k} = 0.6$ and run it. Do the computed approximations match what you think will be the salt concentrations?

Exercise 13.3.2

a. Enter the calculator or MATLAB program of Table 13.1 into your calculator and alter it to accommodate 10 intervals in [0,1] for x. Retain $\hat{k} = 0.2$.

b. Observe that the x-interval $d = 0.1$, is now one-half of the previous value of 0.2, and that

$$\hat{k} = \frac{\delta \times k}{d^2}.$$

Assume that the conductance coefficient, k, has not changed. How must the time increment, δ, change in order that $\hat{k}$ and k be the same as before and d be one-half of its previous value?

c. Run your program and report the result.

Exercise 13.3.3 Solve Equations 13.32, 13.33, and 13.34 using an Excel spread sheet.

Exercise 13.3.4 A number of diffusion simulator programs can be found with Google, "diffusion simulator". Most seemed to be blocked to public access, however, and most will require your computer to be able to run Java Applets. Try to find access to a diffusion simulator and run some experiments on it.

Exercise 13.3.5 a. Show that

$$u(x,t) = 20e^{-t}\sin\pi x, \qquad 0 \le x \le 1, \qquad 0 \le t \tag{13.36}$$

solves

$$u_t(x,t) = \frac{1}{\pi^2}u_{xx}(x,t), \quad u(x,0) = 20\sin\pi x, \quad \text{and} \quad u(0,t) = u(1,t) = 0 \tag{13.37}$$

b. Describe a physical problem for which this is a solution.

c. What is the 'eventual' value of $u(x,t)$ (what is $\lim_{t\to\infty} u(x,t)$)?

d. At what time, t, will the maximum value of $u(x,t)$ for $0 \le x \le 1$ be 20?

Exercise 13.3.6 In Example 13.3.2, what is

$$\lim_{t\to\infty} u(x,t)\,?$$

Alternatively, what is

$$\lim_{j\to\infty} v_{i,j}\,?$$

The columns of Table 13.1 may suggest an answer.

Exercise 13.3.7 a. How can the calculator program or MATLAB program in Table 13.1 be modified if initially the salt concentration in the tube were 0.5?

b. If initially the salt concentration at position x in the tube is x?

Exercise 13.3.8 For the insulated ends boundary condition 13.22

$$u_x(0,t) = 0 \qquad \text{and} \qquad u_x(L,t) = 0,$$

you might approximate these partial derivatives with the difference quotients

$$u_x(0,t) \doteq \frac{u(0+d,t) - u(0,t)}{d} \qquad \text{and} \qquad u_x(L,t) \doteq \frac{u(L,t) - u(L-d,t)}{d}.$$

Then $u_x(0,t) = 0$ and $u_x(L,t) = 0$ would lead to

$$u(0,t) = u(0+d,t) \qquad \text{and} \qquad u(L,t) = u(L-d,t).$$

Suppose the initial condition is

$$u(x,0) = x$$

a. Modify the calculator or MATLAB program in Table 13.1 (better, modify the calculator or MATLAB program of Exercise 13.3.2) to use this boundary condition and initial condition.

b. Run the program and report the result.

c. What do you expect

$$\lim_{t\to\infty} u(x,t) \qquad \text{to be?}$$

Exercise 13.3.9 a. For any $g(x)$, how much salt is in the rod (of length 1 meter and cross section 1cm^2) at time $t = 0$? Consider some special cases such as $g(x) \equiv 1\text{g/cm}^3$ and $g(x) = x\text{g/cm}^3$. Then for general g, think, approximately how much salt is in the first 10 cm of the rod, the second 10 cm of the rod, $\cdots$, the last 10 cm of the rod, and add those quantities.

b. For the insulated end boundary condition 13.22 and any initial condition $g(x)$ and rod length L what is

$$\lim_{t\to\infty} u(x,t)\,?$$

Exercise 13.3.10 Suppose there is an infinitely long tube containing water lying along the X-axis from $-\infty$ to ∞ and at time $t = 0$ a bolus injection of one gram of salt is made at the origin. Let $u(x,t)$ be the concentration of salt at position x in the tube at time t.

Considering $t = 0$ is a bit of stressful: $u(x,0) = 0$ for $x \neq 0$; but the bolus injection of one gm at the origin causes the concentration at $x = 0$ and $t = 0$ to be rather large; $u(0,0) = \infty$.

Moving on, for $t > 0$ we may assume that

$$u_t(x,t) = k u_{xx}(x,t) \tag{13.38}$$

where the diffusion coefficient, k, describes the rate at which salt diffuses in water.

a. Show that

$$u(x,t) = \frac{1}{\sqrt{4\pi k t}} e^{-x^2/(4kt)} \tag{13.39}$$

is a solution to Equation 13.38.

b. Suppose $k = 1/4$. Sketch the graphs of $u(x,1)$, $u(x,4)$, and $u(x,8)$.

c. Suppose $k = 1/4$. Sketch the graphs of $u(x,1)$, $u(x,1/2)$, and $u(x,1/4)$.

d. Estimate the areas under the previous curves. For any time, t_0, what do you expect to be the area under the curve of $u(x,t_0)$, $\infty < x < \infty$.

Exercise 13.3.11 Diffusion in two dimensions is similar to that in one dimension. The two-dimensional diffusion equation is

$$u_t(x,y,t) = k(u_{xx}(x,y,t) + u_{yy}(x,y,t)). \tag{13.40}$$

1. Suppose a square thin copper plate is embedded in perfect thermal insulation with only one edge exposed. Initially the plate is at 0°C. Then 100°C steam passes over the exposed edge. Describe how you might approximate the temperature distribution within the plate as time progresses.
2. A single instance of a highly contagious influenza occurs at the center of a square city and diffuses through the uniformly distributed population according to Equation 13.40, with $u(x, y, t)$ being the fraction of the population at location (x, y) that is infected at time t. Describe how you may approximate the progress of the disease as a function of time.

13.4 Chapter 13 Suggested Solutions

Explore 13.1.1. A. The graph is discontinuous for all points (x, y) for which $y = x^2$.

Exercise 13.1.1 See Figures 13.17- 13.20.

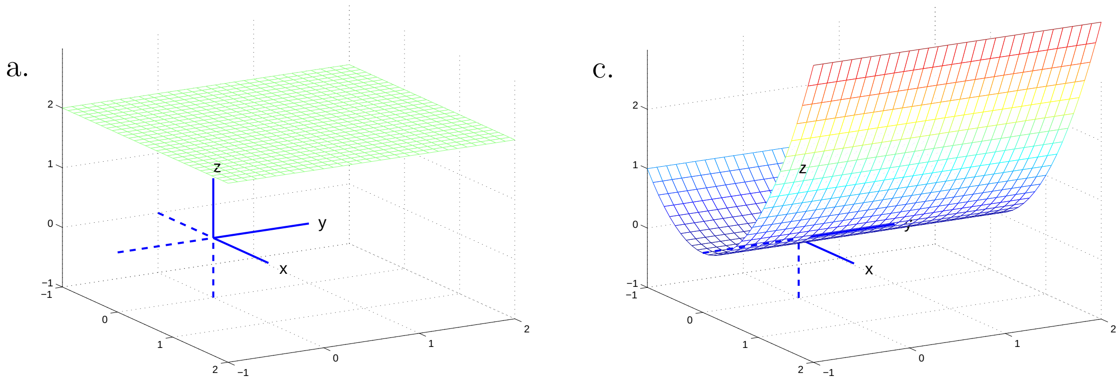


Figure 13.17: **Exercise 13.1.1** a. $z = 2$, c. $z = x^2$.

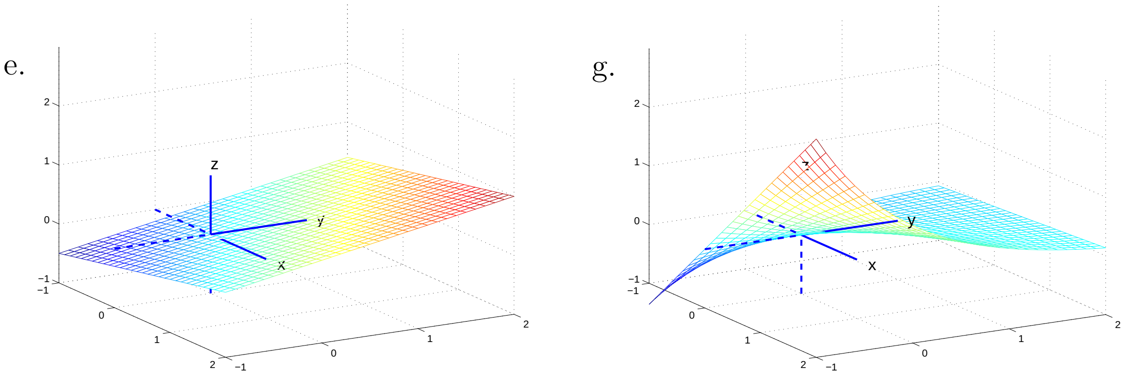


Figure 13.18: **Exercise 13.1.1** e. $z = 0.2\,x + 0.3\,y$, g. $z = 0.5\,xe^{-y}$.

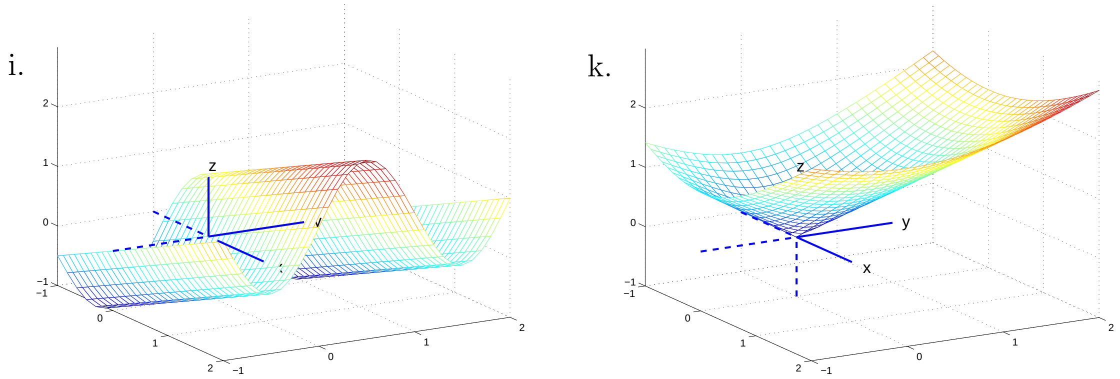


Figure 13.19: **Exercise 13.1.1** i. $z = x/2 + \sin \pi y$, k. $z = \sqrt{x^2 + y^2}$.

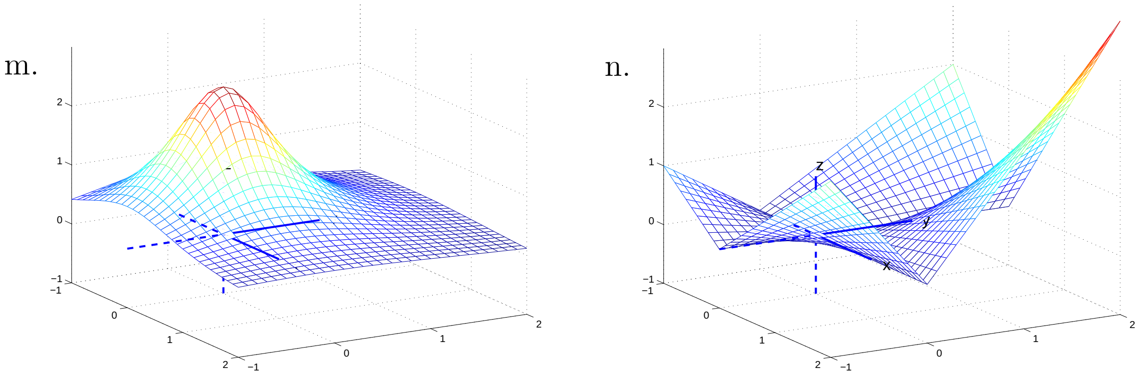


Figure 13.20: **Exercise 13.1.1** m. $z = 1/(0.4 + x^2 + y^2)$, o. $z = |x\,y|$.

Exercise 13.1.2

$F(x, y)$	$F_1(x, y)$	$F_2(x, y)$	$F_{1,1}(x, y)$	$F_{1,2}(x, y)$ $= F_{2,1}(x, y)$	$F_{2,2}(x, y)$
a. $3x - 5y + 7$	3	-5	0	0	0
c. x^3y^5	$3x^2y^5$	$5x^2y^4$	$6xy^5$	$10x^2y^4$	$20x^3y^3$
e. $\ln x + \ln y$	$\frac{1}{x}$	$\frac{1}{y}$	$-\frac{1}{x^2}$	0	$-\frac{1}{y^2}$
g. $e^x e^y$	$e^x e^y$	$e^x e^y$	$e^x e^y$	$e^x e^y$	$e^x e^y$
i. $\sin(2x + 3y)$ $= \sin A$	$2\cos(A)$	$3\cos(A)$	$-4\sin(A)$	$-6\sin(A)$	$-9\sin(A)$

Exercise 13.1.4

$F(x, y)$	(a, b)	$F_1(x, y)$	$F_1(a, b)$	$F_2(x, y)$	$F_2(a, b)$
a. $\frac{x}{1+y^2}$	$(1, 0)$	$\frac{1}{1+y^2}$	$\frac{1}{2}$	$\frac{2xy}{1+y^2}$	0
c. e^{-xy}	$(0, 0)$	$-ye^{-xy}$	0	$-xe^{-xy}$	0

Exercise 13.1.5

$$\begin{aligned}
\text{a. } F(x,y) &= 4x+7y-16 \qquad\qquad\qquad\qquad (a,b)=(3,2)\\
L(x,y) &= 10+4(x-3)+7(x-2) = 4x+7y-16\\
& \left.\frac{|F(x,y)-L(x,y)|}{\sqrt{(x-a)^2+(y-b)^2}}\right|_{(x,y)=(a+0.1,b)}^{(a,b)=(3,2)}\\
&= \frac{|4\times 3.1+7\times 2-16-(4\times 3.1+7\times 2-16)|}{\sqrt{(3.1-3)^2+(2-2)^2}}\\
&= 0\\
& \left.\frac{|F(x,y)-L(x,y)|}{\sqrt{(x-a)^2+(y-b)^2}}\right|_{(x,y)=(a+0.01,b+.01)}^{(a,b)=(3,2)}\\
&= \frac{|4\times 3.01+7\times 2.01-16-(4\times 3.01+7\times 2.01-16)|}{\sqrt{(3.01-3)^2+(2.01-2)^2}}\\
&= 0
\end{aligned}$$

$$\begin{aligned}
\text{c.}\quad F(x,y) &= \frac{x}{y+1} \qquad\qquad\qquad (a,b)=(1,0)\\[2ex]
L(x,y) &= 1+1\times(x-1)-1\times(y-0)=x-y\\[2ex]
&\left.\frac{|F(x,y)-L(x,y)|}{\sqrt{(x-a)^2+(y-b)^2}}\right|_{(x,y)=(1.1,0)}^{(a,b)=(1,0)}\\[2ex]
&= \frac{|1.1/(0+1)-(1.1-0)|}{\sqrt{(1.1-1)^2+(0-0)^2}}\\[2ex]
&= 0\\[2ex]
&\left.\frac{|F(x,y)-L(x,y)|}{\sqrt{(x-a)^2+(y-b)^2}}\right|_{(x,y)=(1.01,0.01)}^{(a,b)=(1,0)}\\[2ex]
&= \frac{|1.01/(0.01+1)-(1.01-0.01)|}{\sqrt{(1.01-1)^2+(0.01-0)^2}}\\[2ex]
&= 0\\[2ex]
\text{e.}\quad F(x,y) &= \sin\pi(x+y) \qquad\qquad (a,b)=(1/2,1/4)\\[2ex]
L(x,y) &= \frac{\sqrt{2}}{2}-\frac{\pi\sqrt{2}}{2}\times(x-1/2)+(y-1/4)\\
&= \frac{\sqrt{2}}{2}\,(1-\pi(x+y-3/4))
\end{aligned}$$

Exercise 13.1.6

$$P=\frac{nRT}{V} \qquad\qquad \text{a.}\quad \frac{\partial}{\partial V}P=-\frac{nRT}{V^2}<0.$$

For fixed T, P decreases as V increases.

Exercise 13.1.7 Graphs of a, c, e, and g. are shown in Figure 13.21.

a. $F(x,y)=0.5x+y+1$ $\quad(a,b)=(0.5,1)$
$F_x=0.5$ $\quad F_y=1$
$F_x(0.5,1)=0.5$ $\quad F_y(0.5,1)=1$ $\quad F(0.5,1)=2.25$
$Z=0.5(x-0.5)+1(y-1)_+2.25=0.5x+y+1$

c. $F(x,y)=(x^2+y^2)/2$ $\quad(a,b)=(1,0.5)$
$F_x=x$ $\quad F_y=y$
$F_x(1,0.5)=1$ $\quad F_y(1,0.5)=0.5$ $\quad F(1,0.5)=5/8$
$Z=1(x-1)-0.5(y-0.5)_+5/8=x-y/2-1/8$

e. $F(x,y) = \sqrt{9-x^2-y^2} \qquad (a,b) = (1,0.2)$
$F_x = -x/\sqrt{9-x^2-y^2} \qquad F_y = -y\sqrt{9-x^2-y^2}$
$F_x(1,0.2) = -1/\sqrt{7.96} \qquad F_y(1,0.2) = -0.2/\sqrt{7.96} \qquad F(1,0.2) = \sqrt{7.96}$
$Z = -(x-1) - 0.2(y-0.2) + 7.96 = -x - 0.2y + 9.16$

g. $F(x,y) = 2/(1+x^2+y^2) \qquad (a,b) = (1,1)$
$F_x = -4x/(1+x^2+y^2)^2 \qquad F_y = -4y/(1+x^2+y^2)^2$
$F_x(1,1) = -4/9 \qquad F_y(1,1) = -4/9 \qquad F(1,1) = 2/3$
$Z = (-4/9)((x-1) + (y-1) - 3/2)$

A MATLAB program to graph c. follows

```
close all;clc;clear
%% c. F(x,y) = (x^2 + y^2)/2  (a,b)= (1,0.5)

x = [-1:0.1:2.00];
y = [-1:0.1:2.00];

for i = 1:length(x)
    for j=1:length(y)
       Z(i,j) = (x(j)^2+y(i)^2)/2;   %% c.
    end
end

mesh(x,y,Z)
hold

T=ones(21);
[xx,yy,zz]=sphere;
r=0.05;
surf(1*T+r*xx,0.5*T+r*yy,5/8+1.3*r*zz)    %% c.
text(1.5,0.6,+0.8,’(1, 0.5, 1.25)’,’fontsize’,16) %% c.

xt=[-0.5:1:0.5]; yt=[-0.5:1:0.5];
for i = 1:length(xt)
    for j = 1:length(yt)
        zt(i,j)= 5/8 + 1*(xt(j))+0.5*(yt(i));  %% c.          %% c.
        ct(i,j)=2.8;     %% c.
    end
end

surf(1+xt,0.5+yt,zt,ct)   %% c.
```

```
hidden off

j0=21,i0=16;         %% c. x(21)=1.0, y(16)=0.5
for j=1:length(x)    %% slice parallel to y
    slx(j) = x(j);
    sly(j) = y(i0);
    slz(j) = Z(i0,j);
end
plot3(slx,sly,slz,'k','linewidth',2)
for i=1:length(y)    %% slice parallel to x
    slx(i) = x(j0);
    sly(i) = y(i);
    slz(i) = Z(i,j0);
end
plot3(slx,sly,slz,'k','linewidth',2)

plot3([0 1],[0 0],[0 0],'linewidth',2)
plot3([0 0],[0 1],[0 0],'linewidth',2)
plot3([0 0],[0 0],[0 1.5],'linewidth',2)
plot3([-1 0],[0 0],[0 0],'--','linewidth',2)
plot3([0 0],[-1 0],[0 0],'--','linewidth',2)
plot3([0 0],[0 0],[-1 0],'--','linewidth',2)
text(0,1.1,0,'y','fontsize',16)
text(1.2,0,0,'x','fontsize',16)
text(0.0,0.0,1.7,'z','fontsize',16)

set(gca,'xtick',[-1 0 1 2])
set(gca,'ytick',[-1 0 1 2])
set(gca,'ztick',[-1 0 1 2])

view(60,20)
axis([-1 2 -1 2 -0.2 2.2])    %% c, d
```

Exercise 13.1.8 See the graph in Figure 13.22

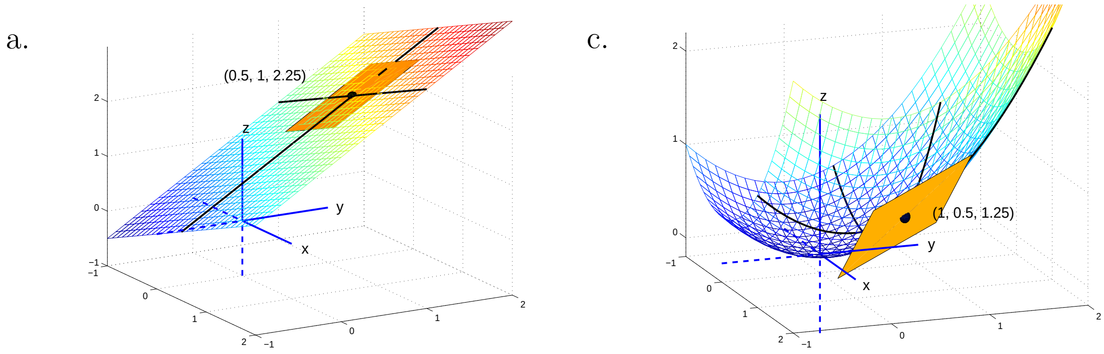


Figure 13.21: **Exercise 13.1.7** a. $F(x,y) = 0.5x + y + 1, \quad (a,b) = (0.5,1)$. The tangent plane lies in the graph. c. $F(x,y) = (x^2 + y^2)/2, \quad (a,b) = (1,0.5)$.

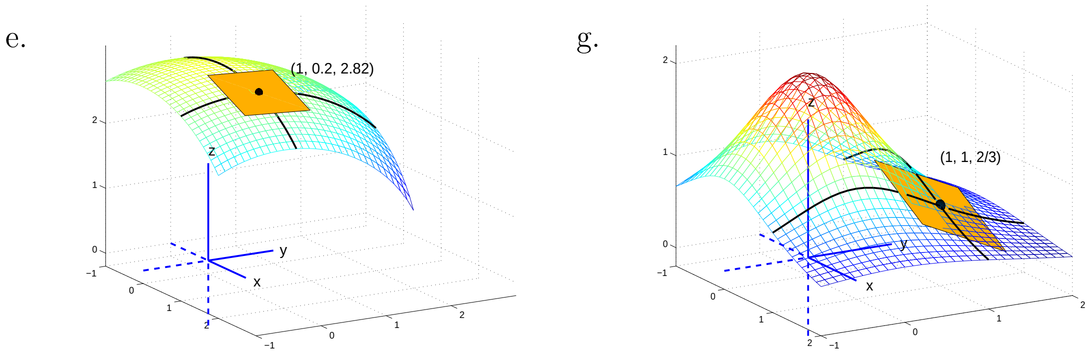


Figure 13.21: Continued. **Exercise 13.1.7** e. $F(x,y) = \sqrt{9 - x^2 - y^2}, \quad (a,b) = (1,0.2)$. g. $F(x,y) = 1/(1 + x^2 + y^2), \quad (a,b) = (1,0.5)$.

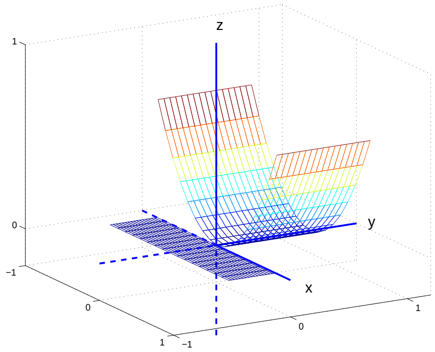


Figure 13.22: **Exercise 13.1.8** Graph of $F(x,y) = x^2 \quad \text{for} \quad y > 0 \quad \text{and} \quad = 0 \quad \text{for} \quad y \leq 0$.

Exercise 13.2.1.

	$F_1(x,y)$	$F_2(x,y)$	Critical points
a.	2	5	No critical point.
c.	$2x + 4y$	$4x + 6y$	$(0,0)$
e.	$(1-2x)(y-y^2)$	$(x-x^2)(1-2y)$	$(0,0), \quad (0,1), \quad (1,0), \quad (1,1), \quad (1/2,1/2)$
g.	e^{x+y}	e^{x+y}	No critical point.

Exercise 13.2.2.

	Critical point	$F_{1,1}$	$F_{1,1}F_{2,2} - F_{1,2}$	
a.	$(0,0)$	-2	$(-2)(-2) - 1 > 0$	Local Maximum
c.	$(a, a+2)$	2	$(2)(2) - (-2) > 0$	Local Minimum, any a.

Exercise 13.2.3.

x	0	1	2	3	4
y	2.18	5.98	16.1	43.6	129.7
$\ln y$	0.7793	1.7884	2.7788	3.7750	4.8652

$$\begin{aligned}
S_x &= 0+1+2+3+4 = 10, \\
S_y &= 0.7793 + 1.7884 + 2.7788 + 3.7750 + 4.8652 = 13.9868 \\
S_{xx} &= \underline{\qquad\qquad\qquad\qquad}? = 30, \\
S_{xy} &= 0 \times 0.7793 + 1 \times 1.7884 + 2 \times 2.7788 + 3 \times 3.7750 + 4 \times 4.8652 = 38.1318 \\
\Delta &= nS_{xx} - (S_x)^2 = 5 \times 30 - (10)^2 = 50 \\
a &= (S_{xx}\, S_y - S_x S_{xy})/\Delta = (30 \times 13.9868 - 10 \times 38.1318)/50 = 0.7657 \\
b &= (nS_{xy} - S_x S_y)/\Delta = (5 \times 38.1318 - 10 \times 13.9868)/50 = 1.0158
\end{aligned}$$

$$\ln y_k \doteq 0.7657 + 1.0158 x_k, \quad y_k \doteq e^{0.7657 + 1.0158 x_k} = e^{0.7657} \times e^{1.0158 x_k} = 2.1504 e^{1.0158 x_k}.$$

Exercise 13.2.4.

$$\begin{aligned}
SS &= \sum_{k=1}^{n} \Big(y_k - (a + bx_k + cx_k^2)\Big) \\
\frac{\partial SS}{\partial a} &= \sum_{k=1}^{n} 2\Big(y_k - (a + bx_k + cx_k^2)\Big)(-1) \\
\frac{\partial SS}{\partial b} &= \sum_{k=1}^{n} 2\Big(y_k - (a + bx_k + cx_k^2)\Big)(-x_k) \\
\frac{\partial SS}{\partial c} &= \underline{\qquad\qquad\qquad\qquad}?
\end{aligned}$$

Setting

$$\frac{\partial SS}{\partial a} = 0, \quad \frac{\partial SS}{\partial b} = 0 \quad \text{and} \frac{\partial SS}{\partial c} = 0$$

leads to

$$a \sum_{k=1}^{n} 1 + b \sum_{k=1}^{n} x_k + c \sum_{k=1}^{n} x_k^2 \;=\; \sum_{k=1}^{n} y_k$$

Write the second equation.

$$a\sum_{k=1}^{n} x_k^2 + b\sum_{k=1}^{n} x_k^3 + c\sum_{k=1}^{n} x_k^4 \quad = \quad \sum_{k=1}^{n} x_k^2 y_k.$$

Rewrite the equations with implied notation:

$$\begin{aligned} an + bS_x + cS_{xx} &= S_y \\ aS_x + bS_{xx} + cS_{xxx} &= S_{xy} \end{aligned}$$

Write the third equation.

Bolt of Lightning! The solutions of these equations are:

$$\begin{aligned}
\Delta &= nS_{xx}S_{xxxx} - nS_{xxx}S_{xxx} - S_xS_xS_{xxxx} + S_xS_{xxx}S_{xxx} + S_{xx}S_xS_{xxx} - S_{xx}S_{xx}S_{xx} \\
a &= \frac{S_yS_{xx}S_{xxxx} - S_yS_{xxx}S_{xxx} - S_{xy}S_xS_{xxxx} + S_{xy}S_{xxx}S_{xxx} + S_{xxy}S_xS_{xxx} - S_{xxy}S_{xx}S_{xx}}{\Delta} \\
b &= \frac{nS_{xy}S_{xxxx} - nS_{xxy}S_{xxx} - S_xS_{xy}S_{xxxx} + S_xS_{xxy}S_{xxx} + S_{xx}S_{xy}S_{xxx} - S_{xx}S_{xy}S_{xx}}{\Delta} \\
c &= \frac{nS_{xx}S_{xxy} - nS_{xxx}S_{xy} - S_xS_xS_{xxy} + S_xS_{xxx}S_{xy} + S_{xx}S_xS_{xy} - S_{xx}S_{xx}S_y}{\Delta}
\end{aligned}$$

b. For the ball drop data:

$$\Delta = 0.01784, \quad a = 241.8, \quad b = 118.8 \quad \text{and} \quad c = -420.5.$$

A plot of the Ball Drop data and the parabola, $P(t) = 241.8 + 118.8 - 420.5t^2$ is shown in Figure 13.23.

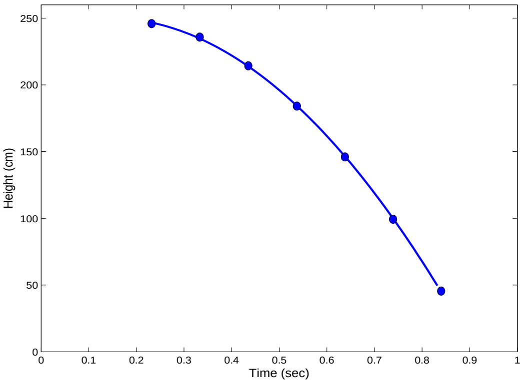


Figure 13.23: Graphs of the Ball Drop data and the parabola $P(t) = 241.8 + 118.8t - 420.5t^2$.

On the TI-86 calculator,

```
{0.232,0.333,0.435,0.537,0.638,0.739,0.840}->T
```

```
{245.9,235.8,214.3,184.1,146.0,99.3,45.5}->H
```

```
P2reg (T,H)  <Ret>
```

yields

```
QuadraticReg
y=ax^2+bx+c
n=7.00000
PRegC=
{-420.5137 118.8343 241.7755}
```

which are the same coefficients obtained with very little effort.

Exercise 13.2.7. $V = xyz$, $z = 12 - 2x - 3y$, $V = xy(12 - 2x - 3y) = 12xy - 2x^2y - 3xy^2$.

$$\begin{aligned} V_x &= 12y - 4xy - 3y^2 = y(12 - 4x - 3y) \\ v_y &= 12x - 2x^2 - 6xy = x(12 - 2x - 6y) \end{aligned}$$

$V_x = V_y = 0$ implies that $(x, y) =$ (0,0), (0,4), (6,0), or (2,4/3). Maximum volume occurs when $(x, y) = (2, 4/3)$ and is $2 \times 4/3 \times 4 = 32/3$.

Exercise 13.2.9. The distance from (4,5,6) to $z = 2x + 3y - 12$ is $D = \sqrt{(4-x)^2 + (5-y)^2 + (6-z)^2} = \sqrt{(4-x)^2 + (5-y)^2 + (6-(2x+3y-12))^2}$. Let $E = D^2 = (4-x)^2 + (5-y)^2 + (6-(2x+3y-12))^2 = (4-x)^2 + (5-y)^2 + (18-2x-3y)^2$.

$$\begin{aligned} E_x &= -2(4-x) - 4(18-2x-3y) = 10x + 8y - 80 \\ E_y &= -2(5-y) - 6(18-2x-3y) = 12x + 20y - 118 \end{aligned}$$

$E_x = E_y = 0$ implies that $(x, y) = (164/26, 55/26)$. Then $z = 2x + 3y - 12 = 181/26)$. The point (164/26,55/26,181/26) is the point of the plane $z = 2x + 3y - 12$ closest to (4,5,6).

Exercise 13.2.11. (0.165385675, -1.319707523, 2.198900733).

Explore 13.3.3. You would be interested in $u(0.8, t)$ where

$$\begin{aligned} u_t(x,t) &= k^2 u(x,t) && x \text{ is distance in meters below the surface and } t \text{ is time in hours} \\ u(x,0) &= 27 - 3.5x && \text{Rather arbitrary, linear, initial condition} \\ u(0,t) &= \underline{\hspace{3cm}}? && \text{Surface boundary condition} \\ u(2,t) &= \underline{\hspace{3cm}}? && \text{Boundary condition at 2 m depth} \end{aligned}$$

Exercise 13.3.1. With $\hat{k} = 0.6$ in the program of Table 13.1

```
:V(I)+.2*(V(I+1)-2*V(        becomes      :V(I)+.6*(V(I+1)-2*V(
I)+V(I-1))->VN(I)                          I)+V(I-1))->VN(I)
```

and the output becomes:

```
1      0      0       0       0  0
1     .6      0       0       0  0
1    .48      0       0       0  0
1    .72   .216    .216       0  0
1   .586   .518    .086    .130  0
1   .794   .300    .372    .026  0
1   .621   .639    .121    .218  0
1   .859   .317    .490    .029  0
1   .618   .746    .110    .288  0
1   .924   .288    .599    .358  0
1   .588   .856    .058    .358  0
1   .996   .216    .717  -0.037  0
1   .530   .984  -0.036   0.437  0
1  1.085   .100    .860   -.109  0
1   .443  1.147  -0.177    .538  0
```

The negative concentrations and concentrations bigger than 1.0 are clearly not correct. The oscillations (for example, V(3) being .216, .518, .300, .639, etc) are also not representative of real salt concentrations.

Exercise 13.3.2.

```
[a.]  Original program                    New commands
      :Fix 3
      :6->dimL V                          :11->dimL V
      :6->dimL VN                         :11->dimL VN
      :For(i,1,6)                         :For(i,1,11)
      :0->V(I)
      :End
      :For(J,1,15)
      :For(I,2,5)                         :For(i,2,10)
      :V(I)+.2*(V(I+1)-2*V(
      I)+V(I-1))->VN(I)
      :End
      :1->VN(1)
      :0->VN(6)                           :0->VN(11)
      :Pause VN
      :For(I,1,6)                         :For(I,1,11)
      :VN(I)->V(I)
      :End
      :End
      :Fix 9
```

[b.] δ is now 1/4 of the previous δ.

```
[c.] 1.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000
     1.000  .200 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000
     1.000  .320  .040 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000
     1.000  .400  .088  .008 0.000 0.000 0.000 0.000 0.000 0.000 0.000
     1.000* .458  .134* .022  .002* .000 0.000 0.000 0.000 0.000 0.000
     1.000  .501  .177  .041  .005  .003  .000  .000  .000  .000  .000
     1.000  .536  .214  .061  .011  .001  .00006 .000 .000  .000  .000
     1.000  .565  .248  .082  .019  .003  .0003  .00001 .00 .000  .000
     1.000* .588  .278* .102  .029* .006  .0008* .00007 .00 .000  .000
```

The entries marked with * are to be compared with the numbers

```
     1.000        0.200        0.000
     1.000        0.320        0.004
```

in the Table 13.1.

Exercise 13.3.5.

a.

$$\begin{aligned}
u(x,t) &= 20e^{-t}\sin\pi x, \qquad 0\le x\le 1, \qquad 0\le t\\
u_t(x,t) &= -20e^{-t}\sin\pi x\\
u_x(x,t) &= 20\pi e^{-t}\cos\pi x\\
u_{xx}(x,t) &= -20\pi^2 e^{-t}\sin\pi x
\end{aligned}$$

$$\begin{aligned}
u_t(x,t) &= -20e^{-t}\sin\pi x = \frac{1}{\pi^2}\left(-20\pi^2e^{-t}\sin\pi x\right) = \frac{1}{\pi^2}u_{xx}(x,t)\\
u(x,0) &= 20e^0\sin\pi x = 20e^o\sin\pi x\\
u(0,t) &= 20e^t\sin\pi 0 = 0\\
u(1,t) &= 20e^t\sin\pi q = 0
\end{aligned}$$

b. The temperature of an insulated rod with ends inserted in a 0° reservoir and an initial temperature along the rod of 20sin πx.

c. $\lim_{t\to\infty} u(x,t) = 0$ for all x. The temperature in the rod decreases to zero.

d. Only at $t = 0$ will the temperature anywhere in the rod be 20°, and this will occur at $t = 0$, $x = 0.5$.

Exercise 13.3.8.

```
Original program (with 10 intervals)     New commands
:Fix 3
:11->dimL V
:11->dimL VN
:For(i,1,11)
:0->V(I)                                 :(I-1)/10->V(I)
:End
:For(J,1,15)
:For(I,2,10)
:V(I)+.2*(V(I+1)-2*V(
I)+V(I-1))->VN(I)
:End
:1->VN(1)                                :VN(2)->VN(1)
:0->VN(11)                               :VN(10)->VN(11)
:Pause VN
:For(I,1,11)
:VN(I)->V(I)
:End
:End
:Fix 9
```

Exercise 13.3.9. The solution depends on the general concept that when the concentration of salt in a region R of volume V is a constant, C, the total amount of salt in R is VC.

If the concentration of salt in a cylinder of length L and cross sectional area, A,is not constant, the amount of salt in the cylinder is

$$A \times \int_0^L g(x)\,dx,$$

but that requires Chapter 10. This exercise is a good introduction to that chapter.

For the case, $g(x) = 1$ for $0 \leq x \leq L$, the amount of salt is $1 \times L \times A$.

Now suppose $g(x) = x$ g/cm^3, L = 1 meter and A=1cm^2. Consider the sections of the rod [12,14] and [76,78] cm. In [12,14] the concentration is approximately 13/100, the concentration at the midpoint of the interval, and in [76,78] the concentration is approximately [77/100]. The sum of these concentrations is 1.0 and the average concentration of the two intervals is approximately (actually exactly) 0.5. By considering pairs of such intervals that fill [0,1] one may decide that the average concentration is approximately (and actually) 0.5. So the total salt in the rod is 0.5.

For the general concentration, $g(x)$, consider 1-cm intervals from [0,1] to [99,100]. **Assume** that the salt concentration in the interval $[x, x+1]$ is approximately $g(x+1/2)$. Then the total salt in the rod would be approximately

$$0.01Ag(0.5) + 0.01Ag(1.5) + 0.01Ag(2.5) + \cdots 0.01Ag(98.5) + 0.01Ag(99.5)$$

An even closer approximation could be computed by considering 1-mm intervals.

Exercise 13.3.10.

a.

$$\begin{aligned}
u(x,t) &= \frac{1}{\sqrt{4\pi kt}}e^{-x^2/(4kt)} \\
u_x(x,t) &= \frac{1}{\sqrt{4\pi kt}}e^{-x^2/(4kt)}\frac{-2x}{4kt} \\
&= \frac{-1}{2kt\sqrt{4\pi kt}}xe^{-x^2/(4kt)} \\
u_{xx}(x,t) &= \frac{-1}{2kt\sqrt{4\pi kt}}\left(e^{-x^2/(4kt)} + xe^{-x^2/(4kt)}\frac{-2x}{4kt}\right) \\
&= \frac{-e^{-x^2/(4kt)}}{2kt\sqrt{4\pi kt}}\left(1 - \frac{x^2}{2kt}\right) \\
u_t(x,t) &= \frac{1}{\sqrt{4\pi k}}\frac{-1}{2}t^{-3/2}e^{-x^2/(4kt)} + \frac{1}{\sqrt{4\pi kt}}e^{-x^2/(4kt)}\frac{-x^2}{4k}\frac{-1}{t^2} \\
&= \frac{-e^{-x^2/(4kt)}}{2t\sqrt{4\pi kt}}\left(1 - \frac{x^2}{2kt}\right) \\
u_t(x,t) &= ku_{xx}(x,t)
\end{aligned}$$

b.

c. See Figure 13.24.

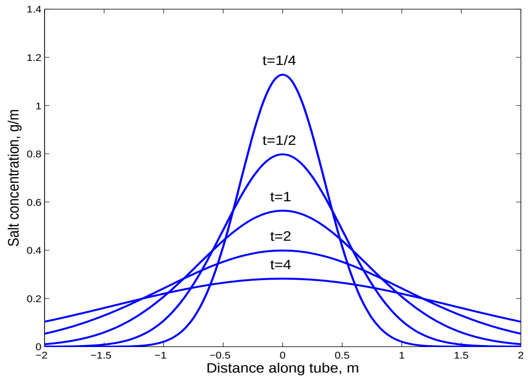


Figure 13.24: **Exercise 13.3.10.** Graphs of concentration, $u(x,t)$ of salt in an infinite tube following an initial infusion of a single gram at the origin for the times indicated.

d. The area under the curve is the total amount of salt in the tube which is 1 for all time.

Exercise 13.3.11.

a. Solve the equations

$$\begin{aligned}
u_t(x,y,t) &= k(u_{xx}(x,y,t)+u_{yy}(x,y,t)) \quad 0 \le x \le 1, \quad 0 \le y \le 1, \quad 0 \le t \\
u(x,y,0) &= 0 \\
u(x,0,t) &= 100 \\
u(x,1,t) &= 0 \\
u(0,y,t) &= 0 \\
u(1,y,t) &= 0
\end{aligned}$$

A discrete approximation to the two dimensional diffusion equation is

$$\frac{v_{i,j,k+1}-v_{i,j,k}}{\delta} = k\left(\frac{v_{i+1,j,k}-2v_{i,j,k}+v_{i-1,j,k}}{d^2}+\frac{v_{i,j+1,k}-2v_{i,j,k}+v_{i,j-1,k}}{d^2}\right).$$

b. Solve the equations

$$\begin{aligned}
u_t(x,y,t) &= k(u_{xx}(x,y,t)+u_{yy}(x,y,t)) \quad -1 \le x \le 1, \quad -1 \le y \le 1, \quad 0 \le t \\
u(0,0,0) &= 1 \\
u(x,y,0) &= 0, \qquad \text{for} \qquad x \ne 0, \quad \text{or} \quad y \ne 0 \\
u_y(x,1,t) &= 0 \\
u_y(x,-1,t) &= 0 \\
u_x(1,y,t) &= 0 \\
u_x(-1,y,t) &= 0
\end{aligned}$$

Chapter 14

First Order Difference Equation Models of Populations

Where are we going?

The difference equation models of population growth and penicillin clearance introduced in Chapter **??** are refined to account for additional factors (limited population environments and continuous penicillin infusion). Formulas for solutions to linear systems are developed. Qualitative analysis of nonlinear systems shows important biological and mathematical concepts of **equilibrium point** and **asymptotically stable** and **unstable** equilibria.

14.1 Difference Equations and Solutions.

The equation

$$\begin{array}{lccccc} \text{Initial} & \text{Change per} & & \text{Removal} & & \text{Environmental} \\ \text{condition} & \text{time interval} & & \text{rate} & & \text{input} \\ & & & & & \\ P_0 = 0 & P_{t+1} - P_t & = & -0.23 P_t & + & 50 \quad \text{mg} \end{array} \tag{14.1}$$

might well represent continuous infusion of penicillin into a patient. The penicillin kinetics suggested by Equation 14.1 are that each 5 minutes, 23% of the penicillin in the serum is removed and 50 milligrams of penicillin are added[1].

We found in Section **??** that the solution to the difference equation

$$P_{t+1} - P_t = r\,P_t \,+\, b \tag{14.2}$$

is

$$P_t = -\frac{b}{r} + \left(P_0 + \frac{b}{r}\right)(1+r)^t. \tag{14.3}$$

---

[1] You should doubt that removal of 23% of serum penicillin each 5 minutes is realistic, and we ask you to read the footnote on page **??**.

Thus the response to continuous penicillin infusion, Equation 14.1, is

$$P_t = \frac{50}{0.23} + \left(0 - \frac{50}{0.23}\right)\,(1 - 0.23)^t = 217 - 217 \times 0.77^t.$$

Important to the analysis (and to the nurse) is the equilibrium state $E = -b/r$ $(= 217$ mg). After two hours $(t = 24)$ the amount of penicillin in the patient will be 217 mg and will continue at that level. The nurse must decide whether 217 mg in the patient will prevent infection, or perhaps be toxic. It depends, of course, on the size of the patient.

In the *Report of the International Whaling Commission (1978)*, J. R. Beddington refers to the following model of Sei whale populations.

$$N_{t+1} - N_t = -0.06N_t + N_{t-8}\left[0.06 + 0.0567\left\{1 - \left(\frac{N_{t-8}}{N_*}\right)^{2.39}\right\}\right] - 0.94C_t \qquad (14.4)$$

N_t, N_{t+1} and N_{t-8} represent the adult female whale population subjected to whale harvesting in years t, $t+1$, and $t-8$, respectively. C_t is the number of female whales harvested in year t. There is an assumption that whales become subject to harvesting the same year that they reach sexual maturity and are able to reproduce, at eight years of age. The whales of age less than 8 years are not included in N_t. N_* is the number of female whales that the environment would support with no harvesting taking place.

Explore 14.1.1 Explain the term $-0.06N_t$ in Equation 14.4. ■

In order to solve for progressive values of the whale population, it is necessary to know the first nine values of N_t, for $t = 0, 1, 2, \cdots 8$, and the number, C_t. of whales that would be caught. However, once those values are known, N_t can be computed for any integer value of t. It is a chore to compute very many values of N_t using only paper and pencil, but is rather easy to compute even a thousand values using a calculator program. *There is no formula for the solution to the whale equation as there is in the penicillin example, however.* Our method will be to give qualitative descriptions of the solutions — what happens to the whale population if harvested at a certain intensity for a long period of time.

These are two examples of *difference equations* with *initial data.* They give approximate values of the underlying real world penicillin kinetics or populations. Usually when difference equations arise, there is a property of the system (population, light intensity, chemical concentration, etc.) that can be measured as a quantity, Q, and one of the obvious things that one can say about Q is how the change, $Q_{t+1} - Q_t$, depends on time, t, or on Q_t, or perhaps on other values of Q such as Q_{t-8}.

Difference equations are by nature based on discrete independent variables. The discrete characteristic is natural for the whale model because of the annual breeding cycle of whales. It is a minor limitation, however for penicillin kinetics which is a continuous process, and does not proceed in five minute jumps. Division of bacteria is in fact a discrete process, but because of the large number of bacteria (of the order 10^8/ml) and the variety of bacterial ages in the population, the population can be better modeled with a continuous model.

The following equations are, respectively, examples of first order, second order and third order autonomous (explained below) difference equations with initial conditions.

$$\begin{array}{llcll} & & & & \text{Order} \\ Q_0 = 4 & Q_{t+1} - Q_t & = & 0.05\sqrt{Q_t} & 1 \\ Q_0 = 1 \quad Q_1 = 2 & Q_{t+1} - Q_t & = & -0.5Q_t + 0.5Q_{t-1} & 2 \\ Q_0 = 2 \quad Q_1 = 1 \quad Q_2 = 1 & Q_{t+1} - Q_t & = & 0.2Q_t - 0.4Q_{t-1}Q_{t-2} & 3 \end{array} \tag{14.5}$$

The first order and third order equations are nonlinear because of the square root and product terms. The second order equation is linear because each term involves only one value of Q_k to the first power. All three equations are *autonomous* because the change, $Q_{t+1} - Q_t$ depends only on values of Q_k.

The following difference equation is not autonomous

$$Q_{t+1} - Q_t = -0.5Q_t + 0.5Q_{t-1} + t^2$$

because of the term t^2. The relation between Q-values changes with time.

The whale equation

$$N_{t+1} = 0.94N_t + N_{t-8}\left[0.06 + 0.0567\left\{1 - \left(\frac{N_{t-8}}{N_*}\right)^{2.39}\right\}\right] - 0.94C_t$$

is not autonomous because of the term C_t, representing the catch in year t which may change from year to year. If the catch were set to be 2 percent of the population, $C_t = 0.02N_t$, however, the resulting equation would be autonomous.

A solution to a difference equation is a sequence, $Q = \{Q_0, Q_1, Q_2, \cdots\}$, of numbers that satisfies the initial conditions and the condition imposed by the difference equation. Usually one can compute at least some of the first terms of the sequence. Consider

$$Q_0 \;=\; 2 \qquad Q_{t+1} - Q_t \;=\; Q_t + t$$

First, convert to iteration form

$$Q_0 \;=\; 0 \qquad Q_{t+1} \;=\; 2 \times Q_t + t$$

Then

$$\begin{array}{lcccccccc} t = 0 & Q_{0+1} & = & 2 \times Q_0 + 0 & = & 2 \times 0 + 0 & = & 0 & = & Q_1 \\ t = 1 & Q_{1+1} & = & 2 \times Q_1 + 1 & = & 2 \times 0 + 1 & = & 1 & = & Q_2 \\ t = 2 & Q_{2+1} & = & 2 \times Q_2 + 2 & = & 2 \times 1 + 2 & = & 4 & = & Q_3 \end{array}$$

Additional values of Q_t can be computed as needed.

Exceptions. It may happen that only a finite sequence of values may be computed from the difference equation or that the values computed become unrealistic (negative, for example, when the numbers in the sequence represent population size). Consider the initial condition and difference equation

$$P_0 \;=\; 1 \qquad P_{t+1} - P_t \;=\; \sqrt{4 - P_t}$$

One can compute (using iteration form, $P_{t+1} = P_t + \sqrt{4 - P_t}$)

$$\begin{array}{llllll}
t = 0 & P_1 = & P_0 + \sqrt{4 - P_0} & = 1 + \sqrt{4 - 1} & = & 2.732\cdots \\
t = 1 & P_2 = & P_1 + \sqrt{4 - P_1} & = 2.732 + \sqrt{1.268} & = & 3.858 \\
t = 2 & P_3 = & P_2 + \sqrt{4 - P_2} & = 3.858 + \sqrt{0.142} & = & 4.235 \\
t = 3 & P_4 = & P_3 + \sqrt{4 - P_3} & = 4.235 + \sqrt{-0.235} & & \textbf{Punt!}
\end{array}$$

Thus P_4 requires the square root of a negative number and is outside our domain of real numbers. We stop with P_3.

Uniqueness of solutions. Difference equations with specified initial conditions of the form

$$Q_{t+1} - Q_t = F(t, Q_t) \qquad Q_0 = Q_{init} \qquad \text{where } Q_{init} \text{ is specified}$$

have unique solutions. Because F is a function; for each pair, (t, Q_t), F assigns a unique number, $F(t, Q_t)$, to the pair. Once Q_0 is known, $Q_1 - Q_0 = F(0, Q_0)$, $Q_1 = Q_0 + F(0, Q_0)$, so that Q_1 is known uniquely, and the rest of the sequence is similarly determined.

As has been noted, some pair (t, Q_t) may fall outside the domain of F so that the computation terminates after a finite number of terms. The point here is that there is no ambiguity in the terms computed, they are unique.

The second order difference equation,

$$Q_{t+2} - Q_{t+1} = Q_t + 0.06\, Q_{t+1} \times (1 - Q_t), \qquad Q_{t+2} = Q_{t+1} + Q_t + 0.06\, Q_{t+1} \times (1 - Q_t)$$

states specifically that

$$Q_2 = Q_1 + Q_0 + 0.06\, Q_1 \times (1 - Q_0).$$

In order to compute Q_2, both Q_0 and Q_1 are required, but once Q_0 and Q_1 are specified, Q_2 is determined. And Q_3 is determined by Q_1 and Q_2. All of the terms Q_2, Q_3, Q_4, $\cdots$ are determined by Q_0, Q_1, and the equation $Q_{t+2} - Q_{t+1} = Q_t + 0.06 Q_{t+1} \times (1 - Q_t)$.

For a kth-order difference equation,

$$Q_{t+1} - Q_t = F(t, Q_t, Q_{t-1}, \cdots Q_{t-k+1}),$$

k initial conditions, Q_0, Q_1, $\cdots$, Q_{k-1}, are required to determine Q_k, Q_{k+1}, $\cdots$.

On most calculators it is easy to compute successive iterates of an autonomous first order recursion equation using the 'Previous Answer' key, ANS. For example, to compute iterates of

$$Q_0 = 0.5 \qquad\qquad Q_{t+1} = 1.5 Q_t\,(1 - Q_t) \tag{14.6}$$

on your calculator type

0.5, ENTER, 1.5 × ANS × (1 - ANS), ENTER, ENTER, $\cdots$

The following numbers appear on your screen:

0.50000	0.37500	0.35156	0.34194	$\cdots$
$= Q_0$	$= Q_1$	$= Q_2$	$= Q_3$	$\cdots$

Looking ahead. Equation 14.6, is an instance ($\rho = 1.5$) of a renown equation,

$$Q_0 \quad \text{given}, \qquad Q_{t+1} = \rho\, Q_t\,(1 - Q_t). \tag{14.7}$$

which is an important introduction to the exotic subjects of chaos and fractals. We use it in examples and exercises before the thorough description in Section 14.6 of its source, the Verhulst's population growth Equation 14.29,

$$P_{t+1} - P_t = R\,P_t \times \left(1 - \frac{P_t}{M}\right),$$

where P_t is population size at time, t, R is population growth rate in an unlimited environment, and M is the carrying capacity of the environment. The relation between the two equations is

$$\begin{aligned}
P_{t+1} - P_t &= R\,P_t \times \left(1 - \frac{P_t}{M}\right) \\
P_{t+1} &= P_t + R\,P_t - R\,\frac{P_t^2}{M} \\
&= (1+R)P_t\left(1 - \frac{R\,P_t}{M(1+R)}\right) \\
\frac{P_{t+1}}{M(1+R)/R} &= (1+R)\frac{P_t}{M(1+R)/R}\left(1 - \frac{P_t}{M(1+R)/R}\right) \\
Q_{t+1} &= \rho\, Q_t\,(1 - Q_t)
\end{aligned}$$

where

$$\rho = 1 + R, \qquad \text{and} \qquad Q_t = \frac{P_t}{M(1+R)/R}.$$

14.1.1 Formulas for solutions.

Claim:

$$P_t = \frac{1}{3}\,2^t + \frac{2}{3} + t \tag{14.8}$$

is a formula for the solution to

$$P_0 = 1, \qquad P_1 = \frac{7}{3}, \qquad P_{t+2} - 5P_{t+1} + 6P_t = 2t. \tag{14.9}$$

Proof of claim. We must:

1. Check to see whether formula 14.8 yields $P_0 = 1$ and $P_1 = 7/3$. By the formula

$$\begin{array}{llllll} t=0 & P_0 & = & \frac{1}{3}\,2^0 + \frac{2}{3} + 0 & = & 1 \\[2ex] t=1 & P_1 & = & \frac{1}{3}\,2^1 + \frac{2}{3} + 1 & = & \frac{7}{3} \qquad \textbf{Initial conditions are satisfied.} \end{array}$$

2. Show that $P_{t+2} - 5P_{t+1} + 6P_t = 2t$. To compute P_{t+1} substitute $t+1$ for each t that appears in the solution, $P_t = \frac{1}{3}\,2^t + \frac{2}{3} + t$, and to compute P_{t+2} substitute $t+2$ for each t that appears in the solution.

$$\begin{array}{llll} t & P_t & = & \frac{1}{3}\,2^t + \frac{2}{3} + t \\[2ex] t+1 & P_{t+1} & = & \frac{1}{3}\,2^{t+1} + \frac{2}{3} + (t+1) \\[2ex] t+2 & P_{t+2} & = & \frac{1}{3}\,2^{t+2} + \frac{2}{3} + (t+2) \end{array}$$

Next compute

$$\begin{aligned} P_{t+2} - 5P_{t+1} + 6P_t &= \left(\frac{1}{3}\,2^{t+2} + \frac{2}{3} + (t+2)\right) - 5\left(\frac{1}{3}\,2^{t+1} + \frac{2}{3} + (t+1)\right) \\ &\quad +6\left(\frac{1}{3}\,2^t + \frac{2}{3} + t\right) \\[1ex] &= \left(\frac{1}{3}\,2^{t+2} - \frac{5}{3}\,2^{t+1} + \frac{6}{3}\,2^t\right) + (t - 5t + 6t) + \\ &\quad \left(\frac{2}{3} + 2 - 5\times\frac{2}{3} - 5 + 6\times\frac{2}{3}\right) \\[1ex] &= \left(\frac{4}{3}\,2^t - \frac{10}{3}\,2^t + \frac{6}{3}\,2^t\right) + (2t) + (0) \\[1ex] &= 2t \qquad \textbf{Difference Equation is satisfied.} \end{aligned}$$

To check that a proposed formula actually defines the solution of a difference equation, use the two steps of the previous example. It is more difficult to find such a formula. You are asked to find solutions to first order linear difference equations with constant coefficients in Exercises 14.1.15 and 14.1.16. The same procedure works for second order linear difference equations with constant coefficients. Part of the procedure is illustrated by the following two examples.

Problem. Find numbers b and c such that

$$P_t = b\,t + c \qquad \text{is a solution to} \qquad P_{t+2} - 5P_{t+1} + 6P_t = 2 + t$$

Solution. Substitute $P_t = b\,t + c$ into the difference equation.

$$\begin{aligned} b\,(t+2) + c - 5\,(b\,(t+1) + c) + 6\,(b\,t + c) &= 2 + t \\ c + 2b - 5c - 5b + 6c + (b - 5b + 6b)\,t &= 2 + t \\ 2c - 3b + 2bt &= 2 + t \end{aligned}$$

Match coefficients. That is, require the constant terms and coefficients of t on the two sides of the equation to be the same:

$$2c - 3b = 2, \qquad \text{and} \qquad 2b = 1.$$

Then $b = 1/2$ and $c = 7/4$, and $P_t = (1/2)t + 7/4$ solves $P_{t+2} - 5P_{t+1} + 6P_t = 2 + t$. There may be initial conditions that are not satisfied by $P_t = (1/2)t + 7/4$, however, and additional terms (**Lightning Bolt:** of the form $C_1 2^t + C_2 3^t$) may be required

Problem. Find numbers C and R such that

$$P_t = C \times R^t \qquad \text{is a solution to} \qquad P_{t+2} - 5P_{t+1} + 6P_t = 3 \times 5^t$$

Solution. Substitute $P_t = C \times R^t$ into the difference equation.

$$\begin{aligned} C \times R^{t+2} - 5C \times R^{t+1} + 6C \times R^t &= 3 \times 5^t \\ C \times R^2 \times R^t - 5C \times R \times R^t + 6C \times R^t &= 3 \times 5^t \\ \Big(C \times R^2 - 5C \times R + 6C\Big) \times R^t &= 3 \times 5^t \end{aligned}$$

By inspection, choose $R = 5$ and $C = 3/(5^2 - 5 \times 5 + 6) = 1/2$. Then $P_t = (1/2)\,5^t$ solves $P_{t+2} - 5P_{t+1} + 6P_t = 3 \times 5^t$.

Exercises for Section 14.1, Difference Equations and Solutions.

Exercise 14.1.1 Write the Difference Equations 14.5 in iteration form and compute three terms after the initial conditions.

Exercise 14.1.2 Compute Q_2, Q_3, Q_4 and Q_5 for

$$\begin{array}{llllllrcl}
\text{a.} & Q_0 & = & 0.1 & Q_1 & = & 0.1 & Q_{t+2} & = & Q_{t+1} + 0.06 \times (1 - Q_t) \\
\text{b.} & Q_0 & = & 10 & Q_1 & = & 12 & Q_{t+2} - Q_{t+1} & = & 0.2 \times Q_t \times \left(1 - \frac{Q_t}{100}\right) \\
\text{c.} & Q_0 & = & 0 & Q_1 & = & 1 & Q_{t+2} & = & Q_{t+1} + Q_t \\
\text{d.} & Q_0 & = & 0.3 & Q_1 & = & 0.7 & Q_{t+2} & = & 5 \times Q_{t+1} - 6 \times Q_t \\
\text{e.} & & & & & & & Q_t & = & 0.2 \times 2^t + 0.1 \times 3^t
\end{array}$$

Exercise 14.1.3 Peroxidase catalyzes the reaction

$$2H_2O_2 \rightarrow 2H_2O + 2O$$

The rate of the reactions is proportional to the concentration of peroxidase times the concentration of hydrogen peroxide, H_2O_2. Because the enzyme peroxidase recycles in the reaction, suppose the concentration of enzyme is constant, $= E$. Assume time is measured in in 0.1 second intervals, and let w_t denote the concentration of H_2O_2 at time t.

1. Assume the proportionality constant for the reaction is k. Write a difference equation showing the change in $H2O_2$ between time t and time $t + 1$.
2. Assume the concentration of H_2O_2 at time $t = 0$ is 0.2 molar. Write an equation for w_t in terms of t.

Exercise 14.1.4 Use your calculator and the 'Previous Answer' key or computer to compute Q_1, $\cdots$ Q_{50} for

$$Q_0 = 10 \qquad\qquad Q_{t+1} = Q_t + 0.2\,Q_t\left(1 - \frac{Q_t}{50}\right)$$

(Type 10, ENTER, ANS + 0.2 × ANS × (1 – ANS/50), ENTER (50 times))

You should find $Q_{50} = 49.99583$. Approximately what will be the values of Q_{51}, $\cdots$ Q_{100}?

Exercise 14.1.5 Use your calculator or computer to compute Q_1, $\cdots$ Q_{20} for

a. $$\begin{aligned} Q_0 &= 5 \\ Q_{t+1} &= Q_t + 0.1\,Q_t\left(1-\frac{Q_t}{20}\right) \end{aligned}$$

b. $$\begin{aligned} Q_0 &= 5 \\ Q_{t+1} &= 1.1\,Q_t\left(1-\frac{Q_t}{20}\right) \end{aligned}$$

c. $$\begin{aligned} Q_0 &= 5 \\ Q_{t+1} &= Q_t + 0.1\,Q_t + 0.02 * Q_t^2 \end{aligned}$$

d. $$\begin{aligned} Q_0 &= 0.5 \\ Q_{t+1} &= \cos(Q_t) \end{aligned}$$

e. $$\begin{aligned} Q_0 &= 0.8 \\ Q_{t+1} &= Q_t\,(2-Q_t) \end{aligned}$$

f. $$\begin{aligned} Q_0 &= 0.8 \\ Q_{t+1} &= Q_t\,(3-Q_t) \end{aligned}$$

g. $$\begin{aligned} Q_0 &= 0.5 \\ Q_{t+1} &= Q_t\,(3.5-Q_t) \end{aligned}$$

h. $$\begin{aligned} Q_0 &= 0.6 \\ Q_{t+1} &= Q_t\,(3.5-Q_t) \end{aligned}$$

i. $$\begin{aligned} Q_0 &= 0.8 \\ Q_{t+1} &= e^{-Q_t} \end{aligned}$$

j. $$\begin{aligned} Q_0 &= 0.8 \\ Q_{t+1} &= \left(\sqrt{Q_t}\right)\;\times\; e^{-Q_t} \end{aligned}$$

k. $$\begin{aligned} Q_0 &= 0.8 \\ Q_{t+1} &= \frac{Q_n + 2/Q_n}{2} \end{aligned}$$

l. $$\begin{aligned} Q_0 &= 0.8 \\ Q_{t+1} &= \frac{Q_n + 3/Q_n}{2} \end{aligned}$$

Exercise 14.1.6 Find the first five terms of the solutions to

a. $Q_0 = 1$ $\quad Q_1 = 0$ $\qquad Q_{t+1} - Q_t = Q_{t-1}$

b. $Q_0 = 1$ $\quad Q_1 = 0$ $\qquad Q_t - Q_{t-1} = Q_{t-2}$

c. $Q_0 = 1$ $\quad Q_1 = 1$ $\qquad Q_{t+1} - Q_t = Q_{t-1}$

d. $Q_0 = 1$ $\quad Q_1 = 0$ $\qquad Q_{t+1} = -Q_{t-1}$

e. $Q_0 = 1$ $\quad Q_1 = 1$ $\qquad Q_t = -Q_{t-2}$

f. $Q_0 = 1$ $\quad Q_1 = 0$ $\quad Q_2 = 2$ $\qquad Q_{t+1} - Q_t = Q_t - Q_{t-2}$

g. $Q_0 = 1$ $\quad Q_1 = 1$ $\quad Q_2 = 1$ $\qquad Q_{t+1} = Q_t - Q_{t-1} + Q_{t-2}$

h. $Q_0 = 0.6$ $\quad Q_1 = 0.7$ $\quad Q_2 = 0.5$ $\qquad$ $$\begin{aligned} Q_{t+1} - Q_t &= 0.6\,Q_{t-2}(1-Q_{t-2}) \\ &\quad -0.2\,Q_{t-1} \end{aligned}$$

Exercise 14.1.7 Compute Q_2, Q_3, and Q_4 for

(a) $Q_0 = 1 \qquad Q_1 = 1 \qquad Q_{t+2} - 2\,Q_{t-1} + Q_t = 0$

(b) $Q_0 = 10 \qquad Q_1 = 5 \qquad Q_{t+2} - 1.2\,Q_{t+1} + 0.32\,Q_t = 0$

Exercise 14.1.8 Enter the following program on your calculator or computer.

```
Prgm, Edit WH3
:0.5->XM1
:0.6->XM0
:For(I,1,6)
:XN=XM0 + 0.06*XM1*(1-XM1)- 0.02*XM1
:XM0->XM1
:XN->XM0
:Pause Disp XM0
:End
```

```
MATLAB
xm1 = 0.5;
xm0 = 0.6
for i = 1:6
    xn=xm0+0.06*xm1*(1-xm1)-0.2*xm1;
    xm1=xm0;
    xm0=xn
end
```

Exit Prgm. Type WH3 and ENTER, ENTER, ENTER, ENTER, ENTER, ENTER, ENTER.
Interpret the results.
Alternatively, run the MATLAB program and interpret the results.

Exercise 14.1.9 For each equation, modify one of the programs in Exercise 14.1.8 to compute P_2, P_3, $\cdots$ P_7.

a. $P_0 = 0.3$ $\quad P_1 = 0.4$ $\quad P_{t+1} = P_t + 0.6\,P_{t-1}(1 - P_{t-1}) - 0.02P_{t-1}$

b. $P_0 = 0.8$ $\quad P_1 = 0.7$ $\quad P_{t+1} = P_t + 0.6\,P_{t-1}(1 - P_{t-1}) - 0.02P_{t-1}$

c. $P_0 = 0.5$ $\quad P_1 = 0.6$ $\quad P_t = 2\,P_{t-1} - 0.96\,P_{t-2}$

d. $P_0 = 1$ $\quad P_1 = 2$ $\quad P_{t+2} = \dfrac{2P_{t+1}}{1 + P_t/50}$

e. $P_0 = 1$ $\quad P_1 = 2$ $\quad P_{t+2} = 1.2\,P_{t+1}\left(1 - \frac{P_t}{10}\right) - 0.1$

f. $P_0 = 1$ $\quad P_1 = 2$, $P_2 = 1$ $\quad P_{t+3} = P_{t+2} + P_t\left(1 - \frac{P_t}{50}\right) - 0.1 * P_{t+1}$

Exercise 14.1.10 Compute solutions until they become negative or imaginary for the systems:

$$\begin{array}{llll}
\text{a.} & \begin{aligned} P_0 &= \pi/4 \\ P_{t+1} &= \ln(\tan(P_t)) \end{aligned} & \text{b.} & \begin{aligned} P_0 &= 5 \\ P_{t+1} &= P_t - 1 \end{aligned} \\
\text{c.} & \begin{aligned} P_0 &= 0.76 \\ P_{t+1} &= 2\sqrt{P_t}\,(1 - P_t^2) \end{aligned} & \text{d.} & \begin{aligned} P_0 &= 2 \\ P_{t+1} &= 0.9P_t - 0.1 \end{aligned}
\end{array}$$

Exercise 14.1.11 The following difference equations, initial data, and solutions have been scrambled.

1. Match each solution to a correct initial condition and difference equation that it satisfies.
2. Compute Q_{50} using the solution.
3. Show algebraically that the solution satisfies the proposed difference equation.

Solutions	Initial Data	Difference Equations
$S_1 \quad P_t = 2 + 4t$	$ID_1 \quad P_0 = 0$	$DE_1 \quad P_{t+1} - P_t = 4t + 8$
$S_2 \quad P_t = 3 \times 5^t$	$ID_2 \quad P_0 = 1$	$DE_2 \quad P_{t+1} - P_t = 0$
$S_3 \quad P_t = 5 \times 3^t$	$ID_3 \quad P_0 = 2$	$DE_3 \quad P_{t+1} - P_t = 2t + 1$
$S_4 \quad P_t = 2t^2 + 6t$	$ID_4 \quad P_0 = 3$	$DE_4 \quad P_{t+1} - P_t = 2 \times P_t$
$S_5 \quad P_t = 4 + t^2$	$ID_5 \quad P_0 = 4$	$DE_5 \quad P_{t+1} - P_t = 4$
$S_6 \quad P_t = 1$	$ID_6 \quad P_0 = 5$	$DE_6 \quad P_{t+1} - P_t = 4 \times P_t$

Exercise 14.1.12 For each equation find a number E such that $P_t = E$ is a solution.

$$\begin{array}{llll}
\text{a.} & P_{t+1} - 0.9P_t = 2 & \text{b.} & P_{t+1} + 0.9P_t = 2 \\
\text{c.} & P_{t+1} - 0.5P_t = 3 & \text{d.} & P_{t+2} - 0.2P_{t+1} - 0.6P_t = 2 \\
\text{e.} & P_{t+1} + 0.9P_t = -5 & \text{f.} & P_{t+2} + 0.2P_{t+1} - 0.4P_t = 2
\end{array}$$

Exercise 14.1.13 For each equation find a numbers b and c such that $P_t = b\,t + c$ is a solution.

$$\begin{array}{llll}
\text{a.} & P_{t+1} - 0.9\,P_t = t & \text{b.} & P_{t+1} - 0.5\,P_t = 3 + 2t \\
\text{c.} & P_{t+1} + 0.9\,P_t = 1 + 2t & \text{d.} & P_{t+2} - 0.2\,P_{t+1} - 0.6\,P_t = 5 + 4t \\
\text{e.} & P_{t+1} + 0.9\,P_t = 3 - t & \text{f.} & P_{t+2} + 0.2\,P_{t+1} - 0.4\,P_t = 2 - 3t
\end{array}$$

Exercise 14.1.14 For each equation find a numbers C and R such that $P_t = C \times R^t$ is a solution.

$$\begin{array}{llclll}
\text{a.} & P_{t+1} - 0.9P_t & = & 3 \times 2^t & \text{b.} & P_{t+1} + 0.9P_t = 5 \times 0.5^t \\
\text{c.} & P_{t+1} - 0.5P_t & = & 0.8^t & \text{d.} & P_{t+2} - 0.2P_{t+1} - 0.6P_t = 2 \times 0.9^t \\
\text{e.} & P_{t+1} + 0.9P_t & = & -0.9^t & \text{f.} & P_{t+2} + 0.2P_{t+1} - 0.4P_t = 2 \times 1.1^t
\end{array}$$

Exercise 14.1.15 How to solve

$$P_0 \quad \text{known} \qquad P_{t+1} - P_t = rP_t + b + c\,t. \tag{14.10}$$

a. Compare this equation with Equation 14.2.

b. Find an 'equilibrium' *linear* sequence $E_t = p + q\,t$ such that

$$\begin{aligned}
E_{t+1} - E_t &= rE_t + b + c\,t \\
p + q(t+1) - (p + q\,t) &= r(p + q\,t) + b + c\,t \\
q - r\,p + -r\,q\,t &= b + c\,t
\end{aligned}$$

Equate coefficients (equate the constant terms and coefficients of t on the two sides of the equation):

$$q - r\,p = b \qquad -r\,q = c$$

and solve for p and q.

c. Conclude that $E_t = -b/r - (c/r^2) - (c/r)\,t$.

d. Subtract the two equations:

$$\begin{aligned}
P_{t+1} - P_t &= rP_t + b + c\,t \\
E_{t+1} - E_t &= rE_t + b + c\,t
\end{aligned}$$

to get

$$P_{t+1} - E_{t+1} = (1 + r)(P_t - E_t)$$

Argue that $P_t - E_t = (P_0 - E_0)(1 + r)^t$.

e. Conclude that

$$P_t = -\left(\frac{b}{r} + \frac{c}{r^2} + \frac{c}{r}t\right) + \left(P_0 + \frac{b}{r} + \frac{c}{r^2}\right)(1 + r)^t \tag{14.11}$$

f. Compare this solution with Equation 14.3.

Exercise 14.1.16 How to solve

$$P_0 \quad \text{known} \qquad P_{t+1} - P_t = rP_t + De^{kt}. \tag{14.12}$$

a. Find an 'equilibrium' *exponential* sequence $E_t = Ce^{kt}$ such that

$$\begin{aligned} E_{t+1} - E_t &= rE_t + De^{kt} \\ \\ Ce^{k(t+1)} - Ce^{kt} &= rCe^{kt} + De^{kt} \\ \\ (Ce^k - C - rC)e^{kt} &= De^{kt} \end{aligned}$$

Choose $C = D/(e^k - 1 - r)$.

b. Subtract the two equations:

$$\begin{aligned} P_{t+1} - P_t &= rP_t + De^{kt} \\ E_{t+1} - E_t &= rE_t + De^{kt} \end{aligned}$$

to get

$$P_{t+1} - E_{t+1} = (1 + r)(P_t - E_t)$$

Argue that $P_t - E_t = (P_0 - E_0)(1 + r)^t$.

c. Conclude that

$$P_t = \frac{D}{e^k - 1 - r}e^{kt} + \left(P_0 - \frac{D}{e^k - 1 - r}\right)(1 + r)^t \tag{14.13}$$

Exercise 14.1.17 Suppose there is a lake of volume 100,000 cubic meters and a stream that runs into the lake and out of the lake at a rate of 2,000 meters cube per day. Suppose further a mining operation is developed in the drainage area to the lake and one kilogram of mercury leaches into the lake each day and is mixed uniformly into the water. How much mercury is in the lake after 1000 days?

Exercise 14.1.18 Use the following formulas to find solutions to the subsequent equations.

DifferenceEquation	Solution
$P_{t+1} - P_t = r\,P_t + b$	$P_t = -\frac{b}{r} + \left(P_0 + \frac{b}{r}\right)(1 + r)^t.$
$P_{t+1} - P_t = rP_t + b + ct$	$P_t = -\left(\frac{b}{r} + \frac{c}{r^2} + \frac{c}{r}t\right) + \left(P_0 + \frac{b}{r} + \frac{c}{r^2}\right)(1 + r)^t$
$P_{t+1} - P_t = rP_t + De^{kt}$	$P_t = \frac{D}{e^k - 1 - r}e^{kt} + \left(P_0 - \frac{D}{e^k - 1 - r}\right)(1 + r)^t$

a. $P_0 = 2 \quad P_{t+1} - 0.8P_t = 0$ b. $P_0 = 2 \quad P_{t+1} - 0.8P_t = 1$

c. $P_0 = 2 \quad P_{t+1} - 1.2P_t = 0$ d. $P_0 = 2 \quad P_{t+1} - 1.2P_t = 2$

e. $P_0 = 2 \quad P_{t+1} - 0.8P_t = 3 + 2t$ f. $P_0 = 2 \quad P_{t+1} - 0.8P_t = 3e^t$

g. $P_0 = 2 \quad P_{t+1} - 1.2P_t = -1 + 4t$ h. $P_0 = 2 \quad P_{t+1} - 1.2P_t = 2e^{-t}$

14.2 Graphical methods for difference equations.

The difference equations for which we have found solution formulas have all been linear difference equations. Each of them is one of the forms

$$P_{t+1} - P_t = rP_t + G_t \qquad \text{or} \qquad P_{t+2} + a_1 P_{t+1} + a_0 P_t = G_t$$

The terms involving the unknown sequence P_t are all of first power. No term is P_{t+1}^2 or $\ln P_t$ or $\frac{1}{P_t}$ or $P_{t+2} \times P_t$. The sequence G_t, can involve linear and exponential terms in t, and even polynomials in t (but do not push this – solutions may be hard to write), but is independent of P_t.

It is usually hard to find formulas for solutions to nonlinear equations. There are no formulas that describe the solutions to

$$\begin{aligned} P_0 &= 4 & P_{t+1} - P_t &= 0.2 \times e^{P_t/50} \\ P_0 &= 10 & P_{t+1} - P_t &= 0.2 \times P_t \times \left(1 - \tfrac{P_t}{500}\right) \end{aligned}$$

$$\left.\begin{aligned} P_0 &= 10 \\ P_1 &= 8 \\ P_2 &= 12 \end{aligned}\right\} \qquad P_{t+1} - P_t = -0.2 \times P_t + P_{t-2}\left[0.2 + 0.3\left(1 - \frac{P_{t-2}}{500}\right)\right]$$

But we can write solutions to the nonlinear equation

$$P_0 = 2 \qquad P_{t+1} = \frac{1.2P_t}{1 + P_t/50}$$

(This is actually a linear equation in disguise.)

The previous nonlinear equations all have reasonable population model interpretations, and we look for graphical methods that will show the qualitative behavior of solutions as a good replacement for having a formula for the solution.

For the equation,

$$P_{t+1} - P_t = 0.2 \times P_t \times \left(1 - \frac{P_t}{500}\right)$$

we can select various values for P_0 and compute subsequent values of P_1, P_2, $\cdots$, as is illustrated in the table and graph in Figure 14.1.

It is fairly apparent that at least these three solutions have the same horizontal asymptote, at about P= 500. We will find that the number 0.2 regulates the rate at which P_t moves towards the asymptote.

Time	P_t	P_t	P_t
t	$P_0 = 100$	$P_0 = 400$	$P_0 = 650$
0.0	100.0	400.0	650.0
1.0	116.0	416.0	611.0
2.0	133.8	430.0	583.9
3.0	153.4	442.0	564.3
4.0	174.7	452.3	549.8
5.0	197.4	460.9	538.8
6.0	221.3	468.1	530.5
7.0	246.0	474.1	524.0
8.0	271.0	479.0	519.0
9.0	295.8	483.0	515.0
...	...	...	...

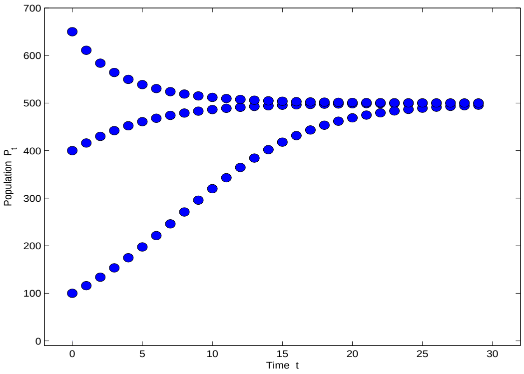


Figure 14.1: Table and graph for $P_{t+1} - P_t = 0.2 \times P_t \times \left(1 - \frac{P_t}{500}\right)$

For

$$P_{t+1} - P_t = -0.2 \times P_t + P_{t-2}\left[0.2 + 0.3\left(1 - \frac{P_{t-2}}{500}\right)\right]$$

we have to select values for P_0, P_1 and P_2 in order to initiate the computation, and then additional values can be computed and graphed, as shown in Figure 14.2

Time	P_t	P_t	P_t
t	$P_0 = 100$	$P_0 = 400$	$P_0 = 650$
	$P_1 = 100$	$P_1 = 400$	$P_1 = 650$
	$P_2 = 100$	$P_2 = 400$	$P_2 = 650$
0.0	100.0	400.0	650.0
1.0	100.0	400.0	650.0
2.0	100.0	400.0	650.0
3.0	124.0	424.0	591.5
4.0	143.2	443.2	544.7
5.0	158.6	458.6	507.3
6.0	179.6	471.0	491.6
7.0	203.0	480.5	487.6
8.0	226.6	487.5	489.4
9.0	251.7	492.4	492.3
...	...	...	...

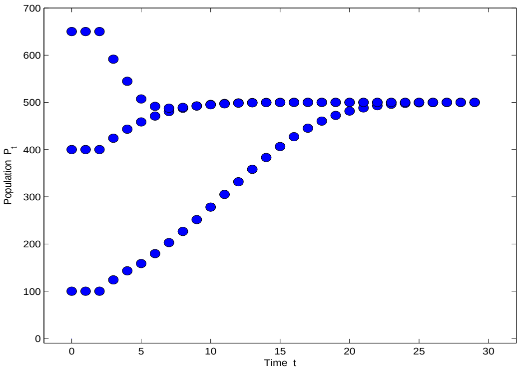


Figure 14.2: Table and graph for $P_{t+1} - P_t = -0.2 \times P_t + P_{t-2}\left[0.2 + 0.3\left(1 - \frac{P_{t-2}}{500}\right)\right]$.

Exercises for Section 14.2, Graphical methods for difference equations.

Exercise 14.2.1 Using the graphs in Figures 14.1 and 14.2 as guides, draw graphs of the solutions to

$$\begin{array}{lrclrcl}
\text{a.} & P_0 & = & 100 & P_{t+1}-P_t & = & 0.1P_t \times \left(1-\frac{P_t}{500}\right) \\
\text{b.} & P_0 & = & 100 & P_{t+1}-P_t & = & 0.2P_t \times \left(1-\frac{P_t}{200}\right) \\
\text{c.} & P_0 & = & 100 & & & \\
 & P_1 & = & 100 & P_{t+1}-P_t & = & -0.2\times P_t + P_{t-3}\left[0.2+0.3\left(1-\frac{P_t}{1000}\right)\right] \\
 & P_2 & = & 100 & & &
\end{array}$$

Exercise 14.2.2 Plot the graph of

$$w_0 = 2, \qquad w_{t+1} = 5.1 \times \frac{w_t}{5+w_t}$$

Solution 1. Use the 'Previous Answer' key on your calculator:

2.0, ENTER, 5.1× ANS/(5 + ANS), ENTER, ENTER, $\cdots$.

Your display will show 2.0, 1.45713, 1.50885, 0.954255, $\cdots$, which are the values of w_0, w_1, w_2, w_3, $\cdots$. They can be plotted against the index, 0, 1, 2, 3, $\cdots$ to yield the graph.

Solution 2. Enter the following program on your calculator:

```
PROGRAM:ITER
:21-> dimL T
:21-> dimL W
:0 -> T(1)
:2 -> W(1)
:For(K,1,20)
:K+1 -> T(K+1)
:5.1*W(K)/(5+W(K)) -> W(K+1)
:End
:min(T)->xMin
:max(T)->xMax
:(xMax-xMin)/5->xScl
:min(W)->yMin
:max(W)->yMax
:(yMax-yMin)/5->yScl
:Disp ''yMin'', yMin
:Disp ''yMax'', yMax
:Pause
:Scatter T,W
:Stop
```

'Unselect' all functions in GRAPH. Run ITER. It will display yMin 0.27726 yMax 2.00000 and pause. Press ENTER and it will display the graph. You can remove the menu from the bottom of the screen with CLEAR. To finish, press EXIT. You need yMin and yMax to draw the graph on paper.

Solution 3. Enter and run the MATLAB program:

```
close all;clc;clear
w(1) = 2;
for k = 1:20
    w(k+1) = 5.1*w(k)/(5+w(k));
end
plot(w,'x','linewidth',2)
```

Exercise 14.2.3 Plot graphs of solutions to

$$\begin{array}{lllcl}
\text{a.} & w_0 = 2 & w_{t+1} & = & 1.2 \times \dfrac{w_t}{0.5 + w_t} \\[2ex]
\text{b.} & w_0 = 0.2 & w_{t+1} & = & 1.2 \times \dfrac{w_t}{0.5 + w_t} \\[2ex]
\text{c.} & w_0 = 2 & w_{t+1} & = & 1.2 \times w_t \times e^{-w_t/10} \\[2ex]
\text{d.} & w_0 = 0.1 & w_{t+1} & = & 1.2w_t \times \cos(w_t) \\[2ex]
\text{e.} & w_0 = 0.001 & w_{t+1} & = & w_t + \sin w_t \\[2ex]
\text{f.} & w_0 = 0 & w_{t+1} & = & w_t + \sin w_t \\[2ex]
\text{g.} & w_0 = 0 & w_{t+1} & = & w_t + 1 \\[2ex]
\text{h.} & \begin{array}{rcl} w_0 & = & 0 \\ w_1 & = & 1 \end{array} & w_{t+2} & = & w_{t+1} - w_t
\end{array}$$

Exercise 14.2.4 For each equation use $W_0 = 0.2$ and plot N iterates to the equations:

$$\begin{array}{llclc}
\text{a.} & w_{t+1} & = & 2.8w_t(1 - w_t) & N = 20 \\[1.5ex]
\text{b.} & w_{t+1} & = & 3.2w_t(1 - w_t) & N = 20 \\[1.5ex]
\text{c.} & w_{t+1} & = & 3.5w_t(1 - w_t) & N = 100 \\[1.5ex]
\text{d.} & w_{t+1} & = & 3.56w_t(1 - w_t) & N = 200 \\[1.5ex]
\text{e.} & w_{t+1} & = & 3.58w_t(1 - w_t) & N = 400 \\[1.5ex]
\text{f.} & w_{t+1} & = & 3.5825w_t(1 - w_t) & N = 400
\end{array}$$

Exercise 14.2.5 For the iteration equation,

$$w_0 = 0.2, \qquad w_{t+1} = Rw_t(1 - w_t)$$

and for each number, $R = 2.5 + k \times 0.01$, $k = 0, \cdots, 150$, compute 1000 iterates of the equation and plot (R, w_k) for $k = 950, \cdots, 1000$.

14.3 Equilibrium Points, Stable and Nonstable

In This Section:

The first crucial question about difference equations that describe populations is whether the solution progresses to a finite nonzero equilibrium value. The alternative is that the population decreases to extinction or grows beyond environmental bounds or oscillates repeatedly, possibly without periodic repetition.

The second question is, given that the population size is at a nonzero equilibrium value, if the population experiences a small perturbation (due to an extremely cold winter, for example), will the size return to the equilibrium value?

Methods are given for answering these questions by examination of the difference equation, even without a solution to the difference equation.

Definition 14.3.1 That the number, E, is an equilibrium value of the difference equation

$$P_{t+1} - P_t = F(P_t) \qquad \text{means that} \qquad F(E) = 0$$

Explore 14.3.1 Show that if E is an equilibrium value of $P_{t+1} - P_t = F(P_t)$ and $P_0 = E$ then for all t, $P_t = E$. ■

The whale equation

$$N_{t+1} = 0.94N_t + N_{t-8}\left[0.06 + 0.0567\left\{1 - \left(\frac{N_{t-8}}{N_*}\right)^{2.39}\right\}\right] - 0.94C_t$$

may be rescaled by dividing each term by N_*, the number of females that would be present without harvesting. Define

$$Y_t = N_t/N_* \qquad \text{and} \qquad D_t = C_t/N_*,$$

and write

$$Y_{t+1} = 0.94Y_t + Y_{t-8}\left[0.06 + 0.0567\left\{1 - Y_{t-8}^{2.39}\right\}\right] - 0.94D_t$$

We set the harvest $D_t = 0.02$ (so that $C_t = 0.02N_*$) which means that 2% of the equilibrium in an unharvested population is harvested without regard to the present population size.

Solutions to

$$Y_{t+1} = 0.94Y_t + Y_{t-8}\left[0.06 + 0.0567\left\{1 - Y_{t-8}^{2.39}\right\}\right] - 0.94 \times 0.02 \tag{14.14}$$

are plotted in Figure 14.3. The harvest is set to be 0.02 (last term in the equation), which means that 2% of the equilibrium in an unharvested population is harvested without regard to the present population size. The initial conditions are set at A: 1.0, B: 0.6, and C: 0.2 of the unharvested equilibrium population.

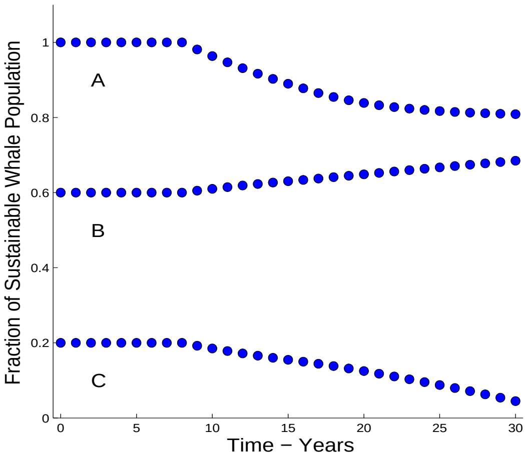


Figure 14.3: Solutions to the whale equation, Equation 14.14, with harvest set at 2 percent of the equilibrium population in the absence of harvest and three sets of initial conditions, A: $Y_i = 1.0, \quad i = 0, 8$, B: $Y_i = 0.6, \quad i = 0, 8$, and C: $Y_i = 0.2, \quad i = 0, 8$.

For both cases A and B it appears that the whale population is moving toward about 0.8, meaning that with harvest of 2% of the equilibrium unharvested population, the harvested population will stabilize at about 80% of the unharvested equilibrium. However, if the initial whale population is only 20% of the unharvested equilibrium population, when 2% harvest is initiated the population will disappear.

In order to explore the interaction of whale growth and harvest more thoroughly we introduce a modification in the model and a new way of graphing the relation. First we rewrite the model to be

$$Y_{t+1} = 0.94Y_t + Y_t\left[0.06 + 0.0567\left\{1 - Y_t^{2.39}\right\}\right] - 0.94 \times 0.02 \tag{14.15}$$

We have replaced the Y_{t-8} with Y_t which effectively assumes the whales reach sexual maturity in one year rather than in eight years. To compute data we need only a single initial value. A plot of the new whale numbers is shown in Figure 14.4 It can be seen that the dynamics are similar to those in the previous figure, except that the whale population in case C crashes more quickly, in about 16 years.

A new way of visualizing the whale growth is to examine $\mathbf{Y_{t+1}}$ as a function of $\mathbf{Y_t}$.

Thus from

$$Y_{t+1} = 0.94Y_t + Y_t\left[0.06 + 0.0567\left\{1 - Y_t^{2.39}\right\}\right] - 0.94 \times 0.02$$

we write a functional form

$$Y_{t+1} = F(Y_t)$$

where

$$F(Y) = 0.94Y + Y\left[0.06 + 0.0567\left\{1 - Y^{2.39}\right\}\right] - 0.94 \times 0.02$$

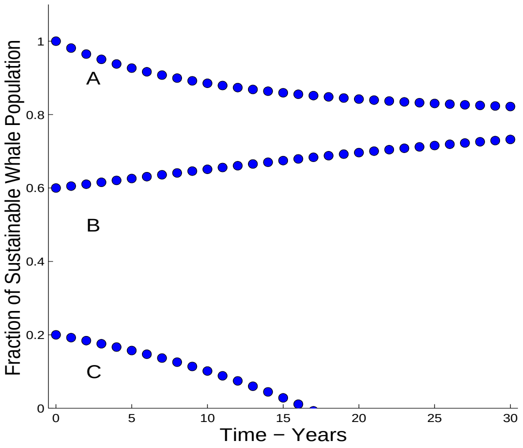


Figure 14.4: Solutions to the modified whale equation, Equation 14.15, with harvest set at 2 percent of the equilibrium population in the absence of harvest and three sets of initial conditions, A, $Y_0 = 1.0$, B, $Y_0 = 0.6$, and C, $Y_0 = 0.2$.

which simplifies to

$$F(Y) = Y + 0.0567Y\left(1 - Y^{2.39}\right) - 0.0188$$

For illustration, we first examine the equation

$$F(Y) = Y + 5 \times \left(0.0567Y\left(1 - Y^{2.39}\right) - 0.0188\right) \tag{14.16}$$

that corresponds to a model with low density birth rate and harvest five times that of the previous model. A graph of $y = F(Y)$ is shown in Figure 14.5 together with the graph of $y = x$.

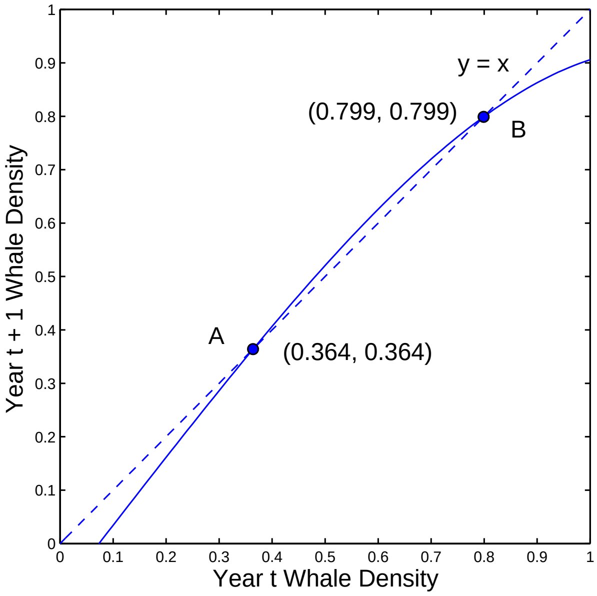


Figure 14.5: Graphs of Equation 14.16 and $y = x$.

Remember that if Y is the population in year t, then $y = F(Y)$ is the population in year $t + 1$. That gives special meaning to the two points, **A** (0.364, 0.364) and **B** (0.799, 0.799), where the graph of $y = F(Y)$ crosses the line $y = x$. If the population in year t is 0.364, then the population in year $t + 1$ is also 0.364. By the same reason, the population in year $t + 2$ is also 0.364, and according to this model, the population will stay at 0.364. Similarly, if the population is ever 0.799, by this model it will always be at 0.799.

The points **A** and **B** are called **equilibrium points** for the iteration $Y_{t+1} = F(Y_t)$. Alternatively, the common coordinates, 0.364 of **A** and 0.799 of **B**, are called **equilibrium values** of $N_{t+1} = F(N_t)$.

Equilibrium points of the iteration

$Y_{t+1} = F(Y_t)$ are found by solving for E in $E = F(E)$

Explore 14.3.2 Do this. Equation 14.16 can be simplified to

$$Y_{t+1} = 1.2835Y_t - 0.2835Y_t^{3.39} - 0.094.$$

a. Begin with $Y_0 = 0.799$ and compute Y_1, Y_2, $\cdots$, Y_{20}. You should get $Y_{20} = 0.79909107$.)

b. Repeat the calculation with $Y_0 = 0.364$. You should observe that the iterations drift downward from 0.364 and that $Y_{20} = 0.359$. In fact, Y_{43} is negative.

c. Repeat the calculation with $Y_0 = 0.365$. You should find that $Y_{20} = 0.395$.

■

You just found that for $Y_0 = 0.799$, Y_t remains close to 0.799, but for $Y_0 = 0.364$, Y_t moves away from 0.364. Neither 0.364 nor 0.799 are the exact values of the points of intersection. Closer approximations are 0.36412683047 and 0.79909120051, but neither of these is exact either. The exact values are solutions to

$$y = x \qquad \text{and} \qquad y = 1.2835x - 0.2835x^{3.39} - 0.094$$

or

$$E = 1.2835E - 0.2835E^{3.39} - 0.094 \qquad \text{or} \qquad 0.2835E - 0.2835E^{3.39} - 0.094 = 0$$

The portions of the graph of $y = F(Y)$ that lie below the graph of $y = x$ are informative. Observe the point **C**, (0.260, 0.237) in Figure 14.6. If $Y_t = 0.260$, then in the next generation, $Y_{t+1} = 0.237$ which is less than Y_t.

The portions of the graph of $y = F(x)$ below the diagonal, $y = x$, where $F(x) < x$ and $Y_{t+1} = F(Y_t) < Y_t$, are associated with decreasing population numbers. Similarly, the portions of the graph of $y = F(x)$

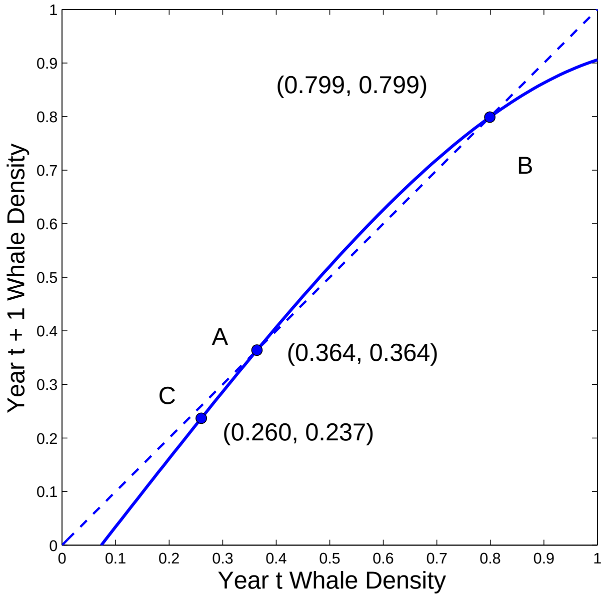


Figure 14.6: The graphs of $F(Y) = 1.2835Y - 0.2835Y^{3.39} - 0.094$ and $y = x$. The points A and B are equilibrium points of the iteration $Y_{t+1} = F(Y_t)$. The point C illustrates that if $Y_t = 0.260$, then $Y_{t+1} = 0.237$ is less than Y_t.

above the diagonal, $y = x$, are associated with increasing population numbers, $Y_{t+1} = F(Y_t) > Y_t$. These properties confer quite different characteristics on the two equilibrium points, **A** and **B**.

Briefly, iterates move away from **A** and toward **B**. Let **A** be (a, a) exactly ($a \doteq 0.364$). If Y_0 is just less than a then Y_1 is less than Y_0, Y_2 is less than Y_1, and the values Y_0, Y_1, Y_2, $\cdots$ decrease away from a.

Alternatively, if Y_0 is slightly greater than a, then Y_1 is greater than Y_0, Y_2 is greater than Y_1, and the values Y_0, Y_1, Y_2, $\cdots$ increase away from a. The number, a, is said to be a **nonstable equilibrium** for the iteration, $Y_{t+1} = F(Y_t)$.

By similar reasoning it can be seen that for $\mathbf{B} = (b, b)$, b is a **stable equilibrium** for the iteration $Y_{t+1} = F(Y_t)$. If Y_0 is greater than b Y_0, Y_1, Y_2, $\cdots$ decrease toward b and if Y_0 is slightly less than b Y_0, Y_1, Y_2, $\cdots$ increase toward b.

This discussion is repeated and expanded in the next section, but a formal definition of stable and nonstable equilibrium points follows.

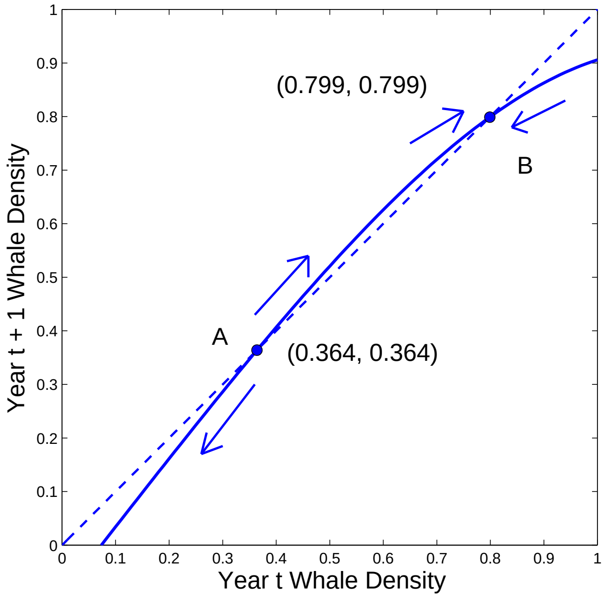


Figure 14.7: Graphs of $F(Y) = 1.2835Y - 0.2835Y^{3.39} - 0.094$ and $y = x$ showing the iteration movement of $Y_{t+1} = F(Y_t)$. Population numbers move away from **A** and toward **B**. **A** is a nonstable equilibrium, **B** is a stable equilibrium.

Definition 14.3.2 Stable and nonstable equilibrium points. Suppose F is a function defined on $[0,1]$ and for every x in $[0,1]$ $F(x)$ is in $[0,1]$.

An equilibrium point of F is a number a_e in $[0,1]$ such that $F(a_e) = a_e$.

An equilibrium point a_e of F is *locally stable* means that there is an interval (p,q) that contains a_e and if x_0 is any point in (p,q), a_e is the limit of the sequence x_0, x_1, x_2, $\cdots$ defined by $x_{n+1} = F(x_n)$ for $n = 0, 1, 2, \cdots$.

An equilibrium point is *nonstable* if it is not locally stable.

To return to the original whale equation, Equation 14.14, the same analysis applies as for Equation 14.15. The difference is that the graph of the iteration function is hardly distinct from the graph of the diagonal, $y = x$, as shown in Figure 14.8. The equilibrium points are the same and have the same stability character. The case considered simply gives easier graphs to examine.

Exercises for Section 14.3, Equilibrium Points, Stable and Nonstable.

Exercise 14.3.1 Do Explore 14.3.2.

Exercise 14.3.2 Compute the equilibrium value of the whale equation, Equation 14.4, under the condition that there is no harvest.

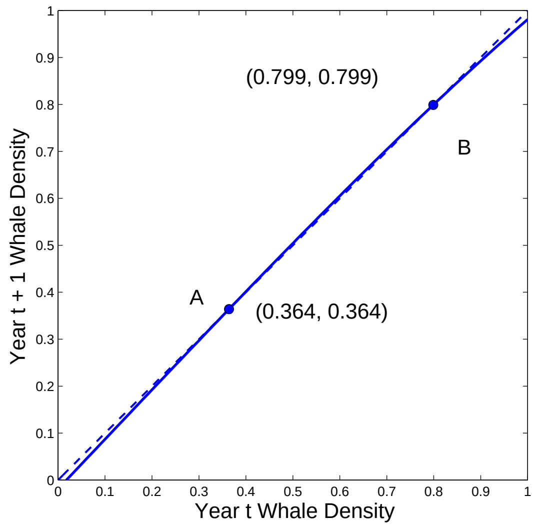


Figure 14.8: The graph of the original 1-year whale iteration. It is hardly distinct from the graph of $y = x$. The equilibrium points are the same as in Figure 14.5.

Exercise 14.3.3 Compute $x_{n+1} = F(x_n)$ for $n = 1, \cdots 20$ for each of the values of x_0. Stop if $x_n < 0$. Either list the values or plot the points (n, x_n) for $n = 0, \cdots 20$ (or the last such point if for some n $x_n < 0$), and describe the trend of each sequence.

a. $F(x) = 0.7x + 0.2$ $\qquad x_0 = 0.5 \quad x_0 = 0.8$

b. $F(x) = 1.1x - 0.05$ $\qquad x_0 = 0.5 \quad x_0 = 0.8$

c. $F(x) = x^2 + 0.1$ $\qquad x_0 = 0.1 \quad x_0 = 0.2$
$x_0 = 0.7 \quad x_0 = 0.9$

d. $F(x) = \sqrt{x} - 0.2$ $\qquad x_0 = 0.07 \quad x_0 = 0.08$
$x_0 = 0.4 \quad x_0 = 0.7$

e. $F(x) = -0.9x^2 + 2x - 0.2$ $\qquad x_0 = 0.1 \quad x_0 = 0.2$
$x_0 = 0.4 \quad x_0 = 0.7$

f. $F(x) = -0.9x^2 + 2x - 0.1$ $\qquad x_0 = 0.1 \quad x_0 = 0.2$
$x_0 = 0.4 \quad x_0 = 0.7$

g. $F(x) = x^3 + 0.2$ $\qquad x_0 = 0.1 \quad x_0 = 0.5$
$x_0 = 0.8 \quad x_0 = 0.9$

h. $F(x) = 8x^3 - 12x^2 + 6x - 1/2$ $\qquad x_0 = 0.1 \quad x_0 = 0.2$
$x_0 = 0.8 \quad x_0 = 0.9$

Exercise 14.3.4 Draw the graph of F and the graph of $y = x$ on a single axes with $0 \le x \le 1$ and $0 \le y \le 1$. The points of intersection of the graphs of F and $y = x$ are listed with F, correct to 3 decimal places. They are the equilibrium points of the iteration $x_{n+1} = F(x_n)$. For each such point, determine whether it is a locally stable equilibrium or an unstable equilibrium.

$$\begin{array}{lrcl ll}
\text{a.} & F(x) & = & 0.7x + 0.2 & (0.667, 0.667) & \\
\text{b.} & F(x) & = & 1.1x - 0.05 & (0.500, 0.500) & \\
\text{c.} & F(x) & = & x^2 + 0.1 & (0.113, 0.113) & (0.887, 0.887) \\
\text{d.} & F(x) & = & \sqrt{x} - 0.2 & (0.076, 0.076) & (0.524, 0.524) \\
\text{e.} & F(x) & = & -0.9x^2 + 2x - 0.2 & (0.262, 0.262) & (0.850, 0.850) \\
\text{f.} & F(x) & = & -0.9x^2 + 2x - 0.1 & (0.111, 0.111) & (1.000, 1.000) \\
\text{g.} & F(x) & = & x^3 + 0.2 & (0.210, 0.210) & (0.879, 0.879) \\
\text{h.} & F(x) & = & 8x^3 - 12x^2 + 6x - 1/2 & (0.146, 0.146) & (0.500, 0.500) \\
 & & & & (0.854, 0.854) &
\end{array}$$

Exercise 14.3.5 Ricker's equation[2] for recruits into a fish population is

$$\text{Recruits in period 0} \qquad = \alpha \times A_0 e^{A_0/\beta}$$

where A_0 is the adult spawning population in year 0. Think of salmon that spawn in a fresh water stream and die. The offspring that survive are recruits that form an adult population, A_1, for the cohort hatched in year 0; the adults will go to sea and return to that same fresh water stream to spawn again in, say, four years. It is reasonable to assume that the annual populations, A_2, A_3, A_4, and A_5 for that cohort satisfy,

$$A_2 = F_1\, A_1, \quad A_3 = F_2\, A_2, \quad A_4 = F_2\, A_3, \quad \text{and} \quad A_5 = F_4\, A_4,$$

$$\text{where} \qquad F_i < 1, \quad i = 1, 2, 3, 4$$

and that A_5 is the next spawning population for that cohort. Then

$$A_5 = F_4\, F_3\, F_2\, F_1 A_1 = F\, A_1 = F \cdot \alpha \times A_0 e^{A_0/\beta}$$

Let $P_0 = A_0, \quad P_1 = A_5, \quad P_2 = A_{10}, \quad \cdots$ and write

$$P_{t+1} = F \cdot \alpha \times P_t \times e^{-P_t/\beta} \tag{14.17}$$

a. Find two equilibrium values for Ricker's Equation 14.17.

b. What condition will insure a positive equilibrium value?

c. What does the condition that insures a positive equilibrium value imply about the size of the recruit population (about the relation between α and F).

---

[2] RICKER, W.E. [1954]. ”Stock and recruitment,” J. Fisheries Res. Board Can., 11, 559-623. Ricker, W. E. 1975. Computation and interpretation of biological statistics of fish populations. Fisheries Research Board of Canada, Bulletin 191. Ottawa, Canada.

14.4 Cobwebbing.

Stability of an equilibrium point of an iteration $P_{t=1} = F(P_t)$ is readily determined by **cobwebbing** the iteration graph. To explain cobwebbing we use an example difference equation

$$P_{t+1} - P_t = 0.8\,P_t\left(1 - \frac{P_t}{100}\right) - 10$$

representative of a population that is limited to size 100 by the environment, has a low density growth rate of 0.8 (high for illustrative purpose), and from which 10 units are removed (harvested) every time interval. We normalize the equation by dividing by 100.

$$\frac{P_{t+1}}{100} - \frac{P_t}{100} = 0.8\,\frac{P_t}{100}\left(1 - \frac{P_t}{100}\right) - \frac{10}{100}$$

Let $p_t = \frac{P_t}{100}$ and write

$$p_{t+1} - p_t = 0.8\,p_t\,(1 - p_t) - 0.1$$

$$p_{t+1} = p_t + 0.8\,p_t\,(1 - p_t) - 0.1$$

Define

$$F(p) = p + 0.8p(1 - p) - 0.1$$

so that the iteration is

$$p_{t+1} = F(p_t)$$

The graph of F is shown in Figure 14.9, together with the graph of $y = x$. The equilibrium points are found by solving

$$E = E + 0.8E(1 - E) - 0.1 \qquad -0.8E^2 + 0.8E - 0.1 = 0$$

$$E \quad = \quad \frac{2 \pm \sqrt{2}}{4} \quad \doteq \quad 0.146 \quad \text{or} \quad 0.853$$

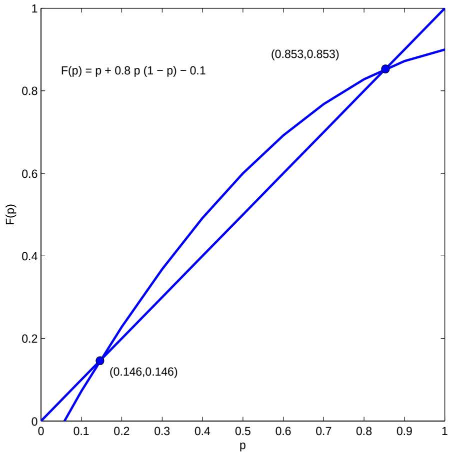


Figure 14.9: Graphs of $F(p) = p + 0.8p(1 - p) - 0.1$ and $y = x$.

Suppose we start with $p_0 = 0.5$ for which $p_1 = 0.6$, $p_2 = 0.692$, $p_3 = 0.763$, $\cdots$. The arrows in Figure 14.10A illustrate these computations.

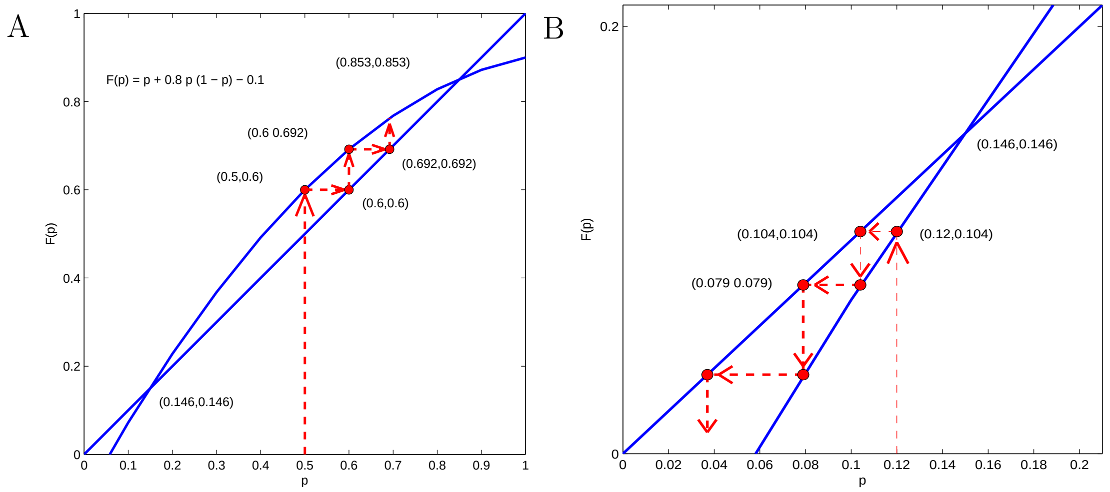


Figure 14.10: A. Cobweb path for $F(p) = p + 0.8p(1-p) - 0.1$. The path begins at $p_0 = 0.5$ and progress to $p_1 = 0.6$, $p_2 = 0.692$, $\cdots$ toward 0.853. B. Enlarged view of the region of Figure 14.10 around the non-stable equilibrium (0.146,0.146). The cobweb path begins at $p_0 = 0.12$ and progresses to $p_1 = 0.104$, $p_2 = 0.079$, $\cdots$ away from 0.146.

Start from 0.5 on the horizontal axis and move **vertically to** (0.5, 0.6) on the **graph**.
Move **horizontally to** (0.6, 0.6) on the **diagonal**.
Move **vertically to** (0.6, 0.692) on the **graph**.
Continue

It is apparent that this path (called a **cobweb**) is channeled between the graph of F and the diagonal line and that it will go to the equilibrium point (0.853, 0.853). Furthermore, any path that begins with p_0 between 0.146 and 0.853 will lead to (0.853, 0.853).

Explore 14.4.1 Start at $p_0 = 0.95$ on the horizontal axis in Figure 14.10A. Move vertically to the graph of F. Move horizontally to the diagonal. Move vertically to the graph of F. Where does this path lead?

The number $\frac{2+\sqrt{2}}{4} \doteq 0.853$ is a locally stable equilibrium. Any sequence with $p_0 > \frac{2-\sqrt{2}}{4} \doteq 0.146$ will go to (0.853, 0.853).

A similar geometric analysis shows that $\frac{2-\sqrt{2}}{4} \doteq 0.146$ is not a locally stable equilibrium point. Shown in Figure 14.10B is an enlarged picture of the region around (0.146,0.146).

If we **Start with $\mathbf{p_0} <$ 0.146** and move **vertically to the graph** of F. Then move **horizontally to the diagonal.** and move **vertically to the graph.** and continue we move to the left and eventually compute a negative number.

Explore 14.4.2 In Figure 14.10B Choose a point p_0 on the X-axis to the right of 0.146 and cobweb the sequence p_1, p_2, p_3. Are you moving away from 0.146?

We seek a general quality of F that will signal whether or not E is a locally stable equilibrium point of the iteration $p_{t+1} = F(p_t)$. Four examples in which the function $F(p)$ is linear will motivate our conclusion. Shown in Figure 14.11A is the graph of $F(p) = 0.7p + 0.2$, a line with slope 0.7, less than 1, and the equilibrium point at (2/3,2/3) is locally stable. In Figure 14.11B is the graph of $F(p) = 1.2p - 2/15$, a line with slope 1.2, greater than 1, and the equilibrium point at (2/3,2/3) is not stable.

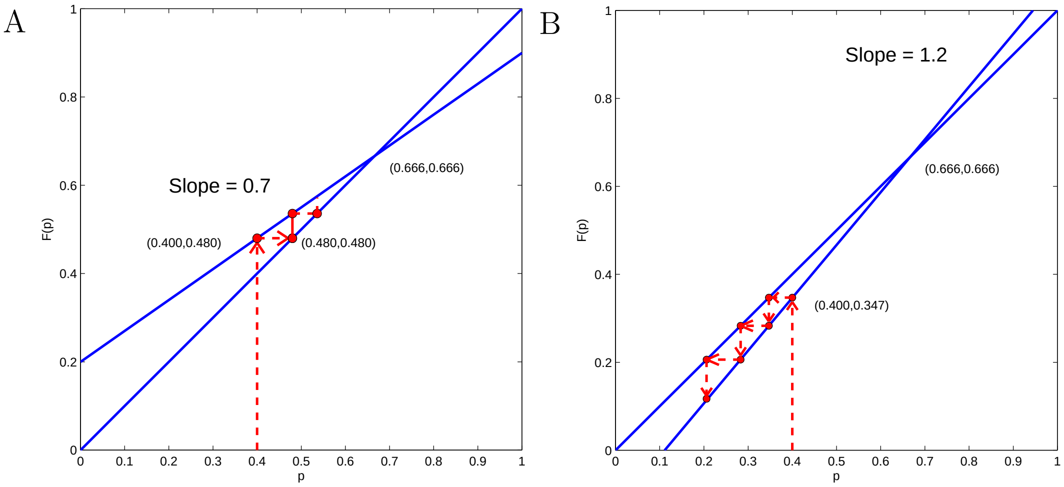


Figure 14.11: Cobwebs for linear iteration graphs. A. The graph of of $F(p) = 0.7p + 0.2$, a line with slope 0.7, less than 1, and the equilibrium point at (2/3,2/3) is locally stable. B. The graph of $F(p) = 1.2p - 2/15$, a line with slope 1.2, greater than 1, and the equilibrium point at (2/3,2/3) is not stable.

From the graphs in Figure 14.11 it appears that E is a locally stable equilibrium point if the graph of F has a slope less than one at E. From Figure 14.12 it appears that we also have to require that the slope of F at E be greater than minus one. The general result is:

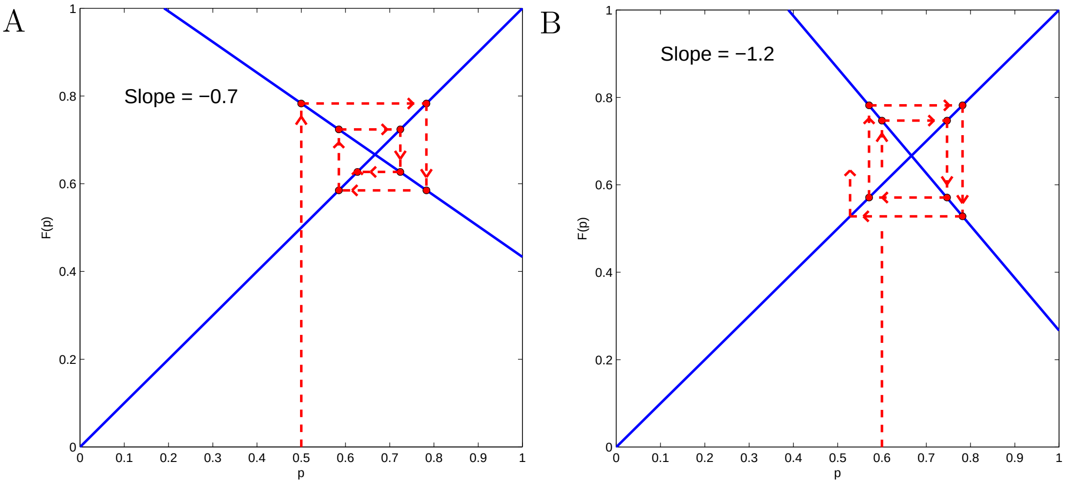


Figure 14.12: Cobwebs with linear $F(p)$. A. $F(p) = -0.7x + 1.133$ with slope = -0.7; the equilibrium point is locally stable. B. $F(p) = -1.2x + 1.467$ with slope of -1.2; the equilibrium point is not stable.

Theorem 14.4.1 If F and F' are continuous on $[0,1]$ and for x in $[0,1]$, $F(x)$ is in $[0,1]$, and E is a number in $[0,1]$ for which $F(E)=E$. Then the equilibrium point E is locally stable for the iteration $P_{t+1}=F(P_t)$ if $\mid F'(E)\mid < 1$.

Proof. The proof of Theorem 14.4.1 makes good use of the Mean Value Theorem, Theorem 9.1.1 in Volume I.

Suppose F satisfies the hypothesis of Theorem 14.4.1. Let $R=(F'(E)+1)/2<1$. Because F' is continuous there is an subinterval (a,b) of (0,1) with midpoint E for which

$$|F'(x)|<R \qquad \text{for all} \qquad a<x<b \qquad \text{(Similar to Theorem 4.1.4 in Volume I.)}$$

Suppose p_0 is in (a,b), and p_0, p_1, p_2, $\cdots$ is the iteration sequence defined by $p_{t+1}=F(p_t)$. We will show that

$$\lim_{t\to\infty} p_t \quad = \quad E$$

We first show by induction that every point of p_0, p_1, p_2, $\cdots$ is in (a,b). By hypothesis p_0 is in (a,b). Suppose p_t is in (a,b). Then

$$\begin{aligned} E &= F(E) && E \text{ is an equilibrium point.} \\ p_{t+1} &= F(p_t) && \text{Iteration equation.} \\ p_{t+1}-E &= F(p_t)-F(E) && \text{Subtraction.} \end{aligned}$$

Then there is a number c between E and p_t such that

$$\begin{aligned} p_{t+1}-E &= F'(c)(p_t-E) && && \text{Mean Value Theorem.} \\ |p_{t+1}-E| &= |F'(c)|\,|p_t-E| &<& \quad R\,|p_t-E| && |F'(c)|<R \end{aligned}$$

The last inequality asserts that p_{t+1} is closer to E than is p_t by a factor of $R<1$. Because E is the midpoint of (a,b) it follows that p_{t+1} is also in (a,b). By induction all of p_0, p_1, p_2, $\cdots$ are in (a,b).

The inequalities

$$|p_{t+1}-E|\le R\,|p_t-E|\,, \qquad t=0,1,\cdots$$

can be cascaded to find that

$$|p_t-E|<R^t\,|p_0-E|\,.$$

Because $0<R<1$, $R^t\to 0$ as $t\to\infty$, and $p_t\to E$ as $t\to\infty$. It follows that E is locally stable. End of proof.

Example 14.4.1 In the early versions of the very large Cray computers, the basis for division was the iteration

$$x_{n+1}=x_n\times(2-a\times x_n) \qquad \text{where} \qquad \frac{1}{2}<a\le 1,$$

for which $1/a$ is a the only locally stable equilibrium. The computer was wired to add, subtract and multiply, but in order to divide, for example B/A, it computed $1/A$ using the iteration (which only requires subtraction and multiplication) and multiplied $B \times 1/A$. The Cray used **24 bit, binary representation of numbers**. Every number, A, stored is written in the form

$$A = \pm\, a \times 2^{\alpha} \qquad \text{where} \qquad \frac{1}{2} < a \leq 1 \qquad \text{and} \qquad \alpha \qquad \text{is an integer}$$

The number a is a 24 digit sequence of 0's and 1's, the first term of which is 1. For example, π might be stored as $110010010000111111011010 \times 2^2$. To get the decimal equivalent of this, compute

$$\begin{aligned} \pi &\doteq \left[\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^5} + \frac{1}{2^8} + \frac{1}{2^{13}} + + \frac{1}{2^{14}} + \frac{1}{2^{15}} + \frac{1}{2^{16}} + \frac{1}{2^{17}} + \frac{1}{2^{18}} + \frac{1}{2^{20}} + \frac{1}{2^{21}} + \frac{1}{2^{23}}\right] \times 2^2 \\ &\doteq [\, 0.5 + 0.25 + 0.00390625 + \cdots + 0.0000004768372 + 0.00000011920193\,] \times 4 \\ &\doteq 0.785398 \times 4 = 3.141592 \end{aligned}$$

Explore 14.4.3 Find the decimal equivalent of the 12 bit binary number 101101010000×2^1. Does it look familiar? ■

If $A = \pm\, a\ \times 2^{\alpha}$,

$$\text{then} \qquad \frac{1}{A} = \frac{1}{\pm a \times 2^{\alpha}} = \pm \frac{1}{a} 2^{-\alpha}$$

so that to compute $\frac{1}{A}$, only $\frac{1}{a}$ where $\frac{1}{2} < a < 1$ must be computed.

We show that the number $1/a$ is the only locally stable equilibrium point of the iteration function

$$F(x) = x \times (2 - a \times x)$$

for the iteration $x_{n+1} = x_n \times (2 - a * x_n)$.

The equilibrium points, E, are

$$E = F(E) \qquad E = E \times (2 - a \times E), \qquad E = 0 \quad \text{or} \quad E = \frac{1}{a}$$

To examine stability using Theorem 14.4.1 we compute

$$F'(x) = [x \times (2 - a \times x)]' = \left[2x - a \times x^2)\right]' = 2 - 2 \times a \times x$$

$$\text{For} \quad E = 0, \quad F'(0) = 2 \qquad \text{and } E = 0 \text{ is a nonstable equilibrium.}$$

$$\text{For} \quad E = \frac{1}{a}, \quad F'\left(\frac{1}{a}\right) = 2 - 2 \times a \times \frac{1}{a} = 0 \qquad \text{and } E = \tfrac{1}{a} \text{ is a locally stable equilibrium.}$$

■

Example 14.4.2 Cray computation of $1/\pi$ In 24 digit binary notation,

$$\pi \doteq 110010010000111111011010 \times 2^2.$$

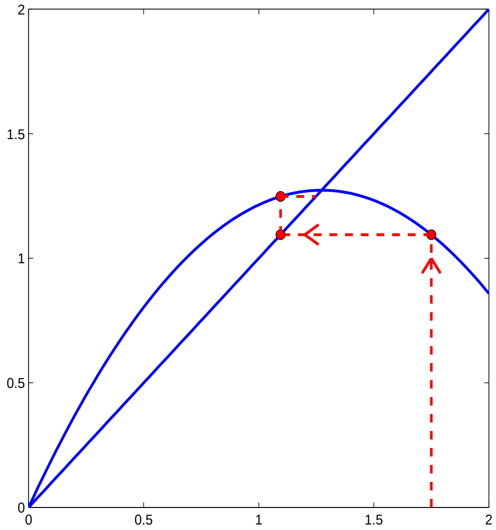


Figure 14.13: Cobweb for the iteration $x_0 = 1.75$, $x_{n+1} = x_n \times (2 - 0.785398 * x_n)$ to compute 1/0.785398.

In decimal notation,

$$\pi \doteq 0.785398 \times 2^2, \quad \text{and} \quad \frac{1}{\pi} \doteq \frac{1}{0.785398} \times 2^{-2}.$$

An iteration sequence to compute 1/0.785398 is illustrated in Figure 14.13. The sequence is $x_0 = 1.75$, $x_{n+1} = x_n(2 - 0.785398x_n)$.

Because $F'\left(\frac{1}{a}\right) = 0$, convergence is very rapid as can be seen in Figure 14.13 and by

$$x_0 = 1.75, \quad x_1 = 1.094719, \quad x_2 = 1.248209, \qquad x_3 = 1.272748,$$
$$x_4 = 1.273240, \qquad x_5 = 1.273240.$$

So $1/\pi \doteq 1.273240 \times 2^{-2} = 0.318310$

Exercises for Section 14.4, Cobwebbing.

Exercise 14.4.1 Find the locally stable equilibrium points of the following iteration functions. Draw the graphs of the iteration function $y = f(x)$ and the diagonal $y = x$. Start with $x_0 = 0.5$ and show the paths of the iterates on your graphs. With $x_0 = 0.5$ compute $x_1, \cdots, x_{10}$ (use ANS key on your calculator).

a. $x_{t+1} = x_t \times (1 - x_t)$ b. $x_{t+1} = 1.5 \times x_t \times (1 - x_t)$

c. $x_{t+1} = 2 \times x_t \times (1 - x_t)$ d. $x_{t+1} = 2.5 \times x_t \times (1 - x_t)$

e. $x_{t+1} = 3.25 \times x_t \times (1 - x_t)$ f. $x_{t+1} = 3.5 \times x_t \times (1 - x_t)$

Exercise 14.4.2 Find the locally stable equilibrium points of the following iteration functions. Draw the graphs of the iteration function $y = f(x)$ and the diagonal $y = x$. Start with $x_0 = 1.0$ and show the paths of the iterates on your graphs. With $x_0 = 1.0$ compute $x_1, \cdots, x_{10}$.

a. $x_{t+1} = \dfrac{x_t + 2/x_t}{2}$ b. $x_{t+1} = \dfrac{x_t + 5/x_t}{2}$

Exercise 14.4.3 Draw the graphs of the iteration function $y = f(x)$ and the diagonal $y = x$. For each equilibrium, choose a value of x_0 close to but distinct from that equilibrium and compute $x_1, \cdots, x_{10}$. Conclude whether the equilibrium is stable.

a. $x_{t+1} = \cos(x_t)$ b. $x_{t+1} = 2 \times \cos(x_t)$

c. $x_{t+1} = 3\ln(x_t)$ d. $x_{t+1} = e^{-x_t}$

e. $x_{t+1} = 2 \times x \times e^{-x_t}$ f. $x_{t+1} = 1.2835x_t - 0.2835x_t^{3.39} - 0.094$

Exercise 14.4.4 Continuation of Exercises D.6.1 and D.6.2.

Consider a single locus, two allele (A and a) gene in a random mating, non-overlapping generation population with the allele frequencies of A and a in the initial adult population being p and $1-p$. Then the allele frequencies in the offspring will also be p and $1-p$. If the viabilities of AA, Aa and aa are in the ratio $1+s_1 : 1 : 1+s_2$ (where s_1 and s_2 can be positive, negative or zero) then the frequencies of AA, Aa and aa in the following adult generation are $(1+s_1)p^2/F$, $2p(1-p)/F$, $(1+s_2)(1-p)^2/F$, respectively, where

$$F = (1+s_1)p^2 + 2p(1-p) + (1+s_2)(1-p)^2 = 1 + s_1p^2 + s_2(1-p)^2$$

The frequency, p^*, of the A allele in the second adult generation will be

$$p^* = \frac{(1+s_1)p^2 + p(1-p)}{F}$$

a. Find the value $\hat{p}$ of p on [0,1] for which $F(p)$ is a maximum.

b. Show that the change in allele A frequency between adult generations is

$$p^* - p = p(1-p)\frac{(s_1+s_2)p - s_2}{1 + s_1p^2 + s_2(1-p)^2}.$$

c. Find the stable equilibrium of

$$p_{n+1} = p_n + p_n(1-p_n)\frac{(s_1+s_2)p_n - s_2}{1 + s_1p_n^2 + s_2(1-p)^2} \tag{14.18}$$

for the following values of s_1 and s_2:

$s_1 = 0.2, \quad s_2 = -0.4; \qquad s_1 = -0.2, \quad s_2 = -0.4;$
$s_1 = 0.2, \quad s_2 = -0.2; \qquad s_1 = 0, \quad s_2 = -0.4;$

Exercise 14.4.5 Consider the case $s_1 = 0$, $s_2 = -1$ of Equation 14.18, referred to as a lethal recessive, the a allele is lethal if it appears on both chromosomes, but has no deleterious effect if it appears on only one chromosome. Show from Equation 14.18 that

$$p_{n+1} = \frac{1-p_n}{2-p_n}.$$

Use $p_n + q_n = 1$ to show that

$$q_{n+1} = \frac{q_n}{1+q_n}.$$

Suppose $q_0 = 0.01$ so that the frequency of aa is 0.0001, or 1 in 10,000. For what value of n will q_n be 0.005? You may want to use

$$\frac{1}{q_{n+1}} = \frac{1}{q_n} + 1.$$

Exercise 14.4.6 The equilibrium points for Ricker's equation, $P_{t+1} = F\cdot\alpha \times P_t \times e^{-P_t/\beta}$, presented in Exercise 14.3.5 are 0 and $\beta\ln(F\cdot\alpha)$. Determine the stability of 0, and the stability of $\beta\ln(F\cdot alpha)$ for $\alpha = 0.1$, $\beta = 20$ and $R = 0.05$.

Exercise 14.4.7 For each of the values of $a = 0.5$, $a = 0.625$, $a = 0.75$, and $a = 0.875$ choose $x_0 = 1.375$ and compute $x_{n+1} = x_n * (2 - a * x_n)$ until 'convergence' (repeat of the first four decimal digits). (1.375 is 1.011 in binary and is close to 4/3.) Record the number of iterations required for convergence. See Example 14.4.1

14.5 Exponential growth and L'Hospital's rule.

In 1798 and subsequent years, Malthus argued that

> The annual increase in human population is proportional to the size of the population at the beginning of the year.
>
> and
>
> The food supply increases a constant amount each year.

With P_t being the human population in year t and F_t the available food each year, the statements lead to

$$\begin{aligned} P_{t+1} - P_t &= R\,P_t \qquad \text{where} \qquad R > 1 \\ F_{t+1} - F_t &= G \qquad\quad \text{and} \qquad G > 0 \end{aligned} \tag{14.19}$$

Malthus argued that the amount of food per person,

$$\frac{F_t}{P_t} \to 0 \qquad \text{as} \qquad t \to \infty$$

except that he believed that calamity would strike long before t gets 'close to' ∞. In recent years the pressure on food supply has been even more intense than shown by simple population growth. As citizens of developing nations become aware of more attractive foods, particularly meat, they are seeking foods that require more acres per calorie produced than do the basic grain crops.

The solutions to the iterations

$$\begin{array}{lll} P_0 = 1 & P_{t+1} - P_t = 0.01\,P_t & \qquad\qquad P_t = (1.01)^t \\ & \qquad\qquad\qquad\qquad \text{are} & \qquad\qquad\qquad\qquad\qquad \text{and} \\ F_0 = 1 & F_{t+1} - F_t = 2 & \qquad\qquad F_t = 1 + 2t \end{array}$$

are shown in Figure 14.14. P_0 and F_0 are both one unit and may be considered to be the present population and the present food supply. The iteration P_t allows for 1 percent annual growth (less than present world population growth). The iteration F_t allows for a constant increase of 2 'units' per year (in year 1 food production is three times present food production; after 2 years food production is 5 times present production). It can be seen in Figure 14.14 that after 750 years the population has exceeded the available food. The exponential growth of $P_t = 1.01^t$ overtakes the linear growth of $F_t = 1 + 2t$.

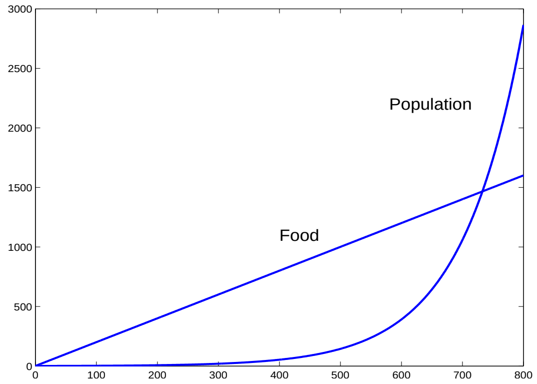


Figure 14.14: Linear growth of Food, $F_t = 1 + 2t$ and exponential growth of population, $P_t = 1.01^t$.

By cascading the iterations in Equation 14.19, we know that

$$P_t = P_0 \times (1+R)^t \qquad \text{and} \qquad F_t = F_0 + G \times t$$

Exponential functions, B^t with $B > 1$, grow faster than all polynomial functions. We show that

If $B > 1$ and n is a positive integer, then $$\lim_{t\to\infty} \frac{B^t}{t^n} = \infty$$

We first show that

$$\lim_{t\to\infty} \frac{e^t}{t} = \infty \qquad (14.20)$$

We assume it to be true that

$$\lim_{t\to\infty} e^t = \infty \qquad (14.21)$$

but have included a proof for you to complete in Problem 14.5.8.

Now

$$\frac{e^t}{t} \overset{a.}{>} \frac{e^t - e^{\frac{t}{2}}}{t} = \frac{1}{2}\frac{e^t - e^{\frac{t}{2}}}{t - \frac{t}{2}} \overset{b.}{=} \frac{1}{2}e^{c_t} \quad \text{where} \quad \frac{t}{2} < c_t < t \tag{14.22}$$

You are asked in Exercise 14.5.1 to give reasons for steps a. and b. in Equation 14.22.

$$\text{Because } \tfrac{t}{2} < c_t \quad \text{as} \quad t \to \infty, \quad c_t \to \infty \quad \text{and} \quad \frac{1}{2}e^{c_t} \to \infty$$

$$\text{Because } \frac{e^t}{t} > \frac{1}{2}e^{c_t} \text{ it follows that } \quad \lim_{t\to\infty} \frac{e^t}{t} = \infty$$

Next we show that

$$\boxed{\text{If } n \text{ is a positive integer, then} \quad \lim_{t\to\infty} \frac{e^t}{t^n} = \infty \qquad (14.23)}$$

Observe that

$$\frac{e^t}{t^n} = \frac{\left(e^{\frac{t}{n}}\right)^n}{t^n} = \frac{1}{n^n}\frac{\left(e^{\frac{t}{n}}\right)^n}{\left(\dfrac{t}{n}\right)^n} = \frac{1}{n^n}\left(\frac{e^\tau}{\tau}\right)^n \quad \text{where} \quad \tau = \frac{t}{n}$$

$$\text{As} \quad t \to \infty \quad t/2 = \tau \to \infty, \quad \text{and (Equation 14.20)} \quad \frac{e^\tau}{\tau} \to \infty.$$

Therefore (remember that n is fixed),

$$\lim_{t\to\infty} \frac{e^t}{t^n} = \lim_{\tau\to\infty} \frac{1}{n^n}\left(\frac{e^\tau}{\tau}\right)^n = \frac{1}{n^n}(\infty)^n = \infty.$$

A short digression to present L'Hôspital's Rule. A well known theorem about limits of quotients is attributed to Antoine de L'Hôspital who published the first calculus text book in 1691 [3]. One case of the theorem is

Theorem 14.5.1 L'Hospital's Theorem on $(0, \infty)$. If F and G are differentiable functions on $(0, \infty)$ and

$$G' \neq 0, \quad \lim_{t\to\infty} F(t) = \infty, \quad \lim_{t\to\infty} G(t) = \infty, \quad \text{and} \quad \lim_{t\to\infty} \frac{F'(t)}{G'(t)} = L$$

then $\displaystyle\lim_{t\to\infty} \frac{F(t)}{G(t)} = L.$ (L can be a real number or $+\infty$ or $-\infty$)

---

[3] From http://mathworld.wolfram.com/LHospitalsRule.html: " Within the book, l'Hospital thanks the Bernoulli brothers for their assistance and their discoveries. An earlier letter by John Bernoulli gives both the rule and its proof, so it seems likely that Bernoulli discovered the rule (Larson et al. 1999, p. 524)."

Larson, R.; Hostetler, R. P.; and Edwards, B. H. Calculus: Early Transcendental Functions, 2nd ed. Boston: Houghton Mifflin, 1999.

To illustrate the Theorem, we evaluate (again) $\displaystyle\lim_{t\to\infty}\frac{e^t}{t}$.

$$\text{Choose } F(t)=e^t,\ G(t)=t. \text{ Then} \qquad F'(t)=e^t, \qquad G'(t)=1, \quad \text{and}$$

$$G'=1\neq 0, \qquad \lim_{t\to\infty}F(t)=\lim_{t\to\infty}e^t=\infty, \qquad \lim_{t\to\infty}G(t)=\lim_{t\to\infty}t=\infty,$$

$$\text{and} \qquad \lim_{t\to\infty}\frac{F'(t)}{G'(t)}=\lim_{t\to\infty}\frac{e^t}{1}=\infty.$$

The hypothesis of Theorem 14.5.1 is satisfied and we can conclude

$$\lim_{t\to\infty}\frac{F(t)}{G(t)}=\lim_{t\to\infty}\frac{e^t}{t}=\infty.$$

Other forms of L'Hôspital's rule include for a a real number, (all require that $G'(t)\neq 0$)

Hypothesis 1 *Hypothesis* 2 *Hypothesis* 3

$$\begin{array}{lll} \displaystyle\lim_{t\to\infty}F(t)=0 & \displaystyle\lim_{t\to\infty}G(t)=0 & \displaystyle\lim_{t\to\infty}\frac{F'(t)}{G'(t)}=L \\ & \displaystyle\Longrightarrow \lim_{t\to\infty}\frac{F(t)}{G(t)}=L & \end{array} \tag{14.24}$$

$$\begin{array}{lll} \displaystyle\lim_{t\to a^+}F(t)=\infty & \displaystyle\lim_{t\to a^+}G(t)=\infty & \displaystyle\lim_{t\to a^+}\frac{F'(t)}{G'(t)}=L \\ & \displaystyle\Longrightarrow \lim_{t\to a^+}\frac{F(t)}{G(t)}=L & \end{array} \tag{14.25}$$

$$\begin{array}{lll} \displaystyle\lim_{t\to a^+}F(t)=0 & \displaystyle\lim_{t\to a^+}G(t)=0 & \displaystyle\lim_{t\to a^+}\frac{F'(t)}{G'(t)}=L \\ & \displaystyle\Longrightarrow \lim_{t\to a^+}\frac{F(t)}{G(t)}=L & \end{array} \tag{14.26}$$

Fundamental to proving L'Hospital's rule is to prove

Theorem 14.5.2 Extended Mean Value Theorem. If F and G are continuous functions on a closed interval $[a,b]$ and are differentiable on the open interval (a,b) and $G'(x)\neq 0$ for any x in (a,b), then there is a number c in $[a,b]$ such that

$$\frac{F(b)-F(a)}{G(b)-G(a)}=\frac{F'(c)}{G'(c)} \tag{14.27}$$

You are asked to prove Theorem 14.5.2, with suggestions, in Exercise 14.5.13.

The proofs of Theorem 14.5.1 and the implications 14.24, 14.25, and 14.26 are similar. A proof of implication 14.26 follows. The complete statement is

Theorem 14.5.3 L'Hospital's Theorem on a bounded interval. Suppose $[a, b]$ is a number interval and F and G are continuous functions defined on the half open interval $(a, b]$ and F' and G' are continuous on (a, b) and G' is nonzero on (a, b). If

$$\lim_{t\to a^+} F(t) = 0, \qquad \lim_{t\to a^+} G(t) = 0 \qquad \text{and} \qquad \lim_{t\to a^+} \frac{F'(t)}{G'(t)} = L$$

then

$$\lim_{t\to a^+} \frac{F(t)}{G(t)} = L.$$

Proof. Because $\lim_{t\to a^+} F(t) = \lim_{t\to a^+} G(t) = 0$, the domain of F and G can be extended to include a by defining $F(a) = G(a) = 0$ and the extended F and G are continuous on $[a, b]$. By the Extended Mean Value Theorem 14.5.2, for any t in (a, b) there is a number c_t between a and t such that

$$\frac{F(t)}{G(t)} = \frac{F(t) - F(a)}{G(t) - G(a)} = \frac{F'(c_t)}{G'(c_t)}.$$

As t approaches a, c_t also approaches a and

$$\lim_{t\to a^+} \frac{F(t)}{G(t)} = \lim_{t\to a^+} \frac{F'(c_t)}{G'(c_t)} = L$$

End of proof.

Theorem 14.5.3 may be used to evaluate the limit

$$\lim_{t\to 0^+} \frac{\sin t}{\ln(t+1)}$$

First we note that $\sin t$ and $\ln(t+1)$ are continuous and differentiable on (0,1] and that $\lim_{t\to 0^+} \sin t = 0$ and $\lim_{t\to 0^+} \ln(t+1) = 0$. Furthermore

$$[\sin t]' = \cos t, \quad [\ln(t+1)]' = \frac{1}{t+1}, \quad \text{and} \quad \lim_{t\to 0^+} \frac{\cos t}{1/(t+1)} = \lim_{t\to 0^+} (\cos t)(t+1) = 1.$$

By Theorem 14.5.3

$$\lim_{t\to 0^+} \frac{\sin t}{\ln(t+1)} = 1$$

Exercises for Section 14.5, Exponential growth and L'Hospital's theorem.

Exercise 14.5.1 Give reasons for steps a. and b. in Equation 14.22.

Exercise 14.5.2 Evaluate the limits using Equation 14.20.

a. $\lim_{t\to\infty} \frac{e^{0.5t}}{t}$ Note: $\frac{e^{0.5t}}{t} = 0.5\frac{e^{0.5t}}{0.5t} = 0.5\frac{e^{\tau}}{\tau}$ $\tau = 0.5t$

b. $\lim_{t\to\infty} \frac{2^t}{t}$ Note: $\frac{2^t}{t} = \frac{e^{(\ln 2)t}}{t}$

c. $\lim_{t\to\infty} \frac{e^t}{5t}$

d. $\lim_{t\to\infty} \frac{e^t}{2t+15}$ Note: $\frac{e^t}{2t+15} > \frac{e^t}{3t}$ for $t > 15$

Exercise 14.5.3 Show that if B is a number greater than 1, and n is a positive integer,

$$\lim_{t\to\infty} \frac{B^t}{t^n} = \infty$$

Exercise 14.5.4 Evaluate the limits using Equation 14.23.

a. $\lim_{t\to\infty} \frac{e^t}{t^2+15}$ b. $\lim_{t\to\infty} \frac{e^t}{t^{5,000,000}}$

c. $\lim_{t\to\infty} \frac{e^{\sqrt{t}}}{\sqrt{t}}$ d. $\lim_{t\to\infty} \frac{t}{e^t}$

e. $\lim_{t\to\infty} \frac{t}{2^t}$ f. $\lim_{t\to\infty} t^{99} \times 2^{-t}$

Exercise 14.5.5 Evaluate the limits using Equation 14.23. In b. F_0 and G represent initial food production and annual increase in food production, respectively. P_0 and R represent initial population level and population growth rate, respectively.

a. $\lim_{t\to\infty} \frac{t}{R^t}$ where $R > 1$

b. $\lim_{t\to\infty} \frac{F_0 + G\times t}{P_0 \times R^t}$ where $F_0 > 0$, $G > 0$, $P_0 > 0$, $R > 1$

Exercise 14.5.6 If Malthus's two premises are correct, Part b. of the last problem suggests that indeed the amount of food per person will go to zero. Shown in Table 14.5.6 is the record of the world populations since 1900 the world production of cereal grain for 1961 - 2001.

Table for Exercise 14.5.6 Estimates of World Population by the World Health Organization and estimates of world cereal grain production read from a graph in http://www.freedom21.org/alternative/5b-1.jpg

World Population Estimates

Year	10^9 Pop
1900	1.65
1910	1.75
1920	1.86
1930	2.07
1940	2.30
1950	2.52
1960	3.02
1970	3.70
1980	4.45
1990	5.30
2000	6.06

World Cereal Grain Production

Year	Index
1961	0.90
1965	1.10
1969	1.25
1973	1.35
1977	1.45
1981	1.60
1985	1.80
1989	1.88
1993	1.90
1997	2.10
2001	2.12

a. Use your calculator to fit an exponential function to the human population data.

b. Is Malthus's assertion about exponential growth of the human population consistent with the data?

c. Use your calculator to fit both a second degree polynomial and a linear function to the food data. Should the linear function or the second degree polynomial be used?

d. In what sense is the quadratic fit more pessimistic than the linear fit?

e. Would it change Malthus's assertion if food production increases quadratically?

The questions Malthus raised are with us today. There are optimistic reports[4] and criticisms of optimistic reports[5]. The problem is enormously more complex than can be described by two short tables showing population growth and food production. For example, scientists at Goddard Space Flight Center have estimated the total solar energy that is converted to plant material and estimated that humans consume an astounding 14 to 26 percent of that energy as food, clothing, fuel and timber[6].

Exercise 14.5.7 L'Hôspital's Rule can be cascaded. To show that

$$\lim_{t\to\infty} \frac{e^t}{t^2} = \infty, \qquad \text{observe first that} \qquad \lim_{t\to\infty} \frac{e^t}{t} = \infty.$$

Let $F(t) = e^t$ and $G(t) = t^2$. Then $F'(t) = e^t$ and $G'(t) = 2t$ and $G'(t) > 0$ for $t > 1$. Furthermore,

$$\lim_{t\to\infty} \frac{F'(t)}{G'(t)} = \lim_{t\to\infty} \frac{e^t}{2t} = \infty.$$

$$\text{Therefore,} \qquad \lim_{t\to\infty} \frac{F(t)}{G(t)} = \lim_{t\to\infty} \frac{e^t}{t^2} = \infty.$$

---

[4] World Bank Report, November 1993, World Food Outlook

[5] David Pimentel, "Exposition on Skepticism", *Bioscience*, March 2002, 52, 295-298

[6] Roger Doyle, "the Lion's Share", Scientific American, April, 2005, p 30.

Show that

$$\text{a.}\quad \lim_{t\to\infty} \frac{e^t}{t^3} = \infty. \qquad \text{b.}\quad \lim_{t\to\infty} \frac{e^t}{t^4} = \infty. \qquad \text{c.}\quad \lim_{t\to\infty} \frac{e^t}{t^n} = \infty \quad n = 5, 6, 7, \cdots.$$

Exercise 14.5.8 Show that

$$\lim_{n\to\infty} 2^n = \infty$$

using the following procedure.

Argue using mathematical induction that all of the statements in the sequence $\{S_1, S_2, S_3, \cdots\}$ of statements are true where

S_n is the statement that $n < 2^n$, for $n = 1, 2, 3, \cdots$.

Your argument should have two parts.

Part 1. Show that S_1 is true.

Part 2. If one of the statements in $\{S_1, S_2, S_3, \cdots\}$ is not true, there must be a first one that is not true. Let m be the subscript of the first statement in $\{S_1, S_2, S_3, \cdots\}$ that is not true. Show that

1. $1 < m$.
2. S_{m-1} is true.
3. Because S_{m-1} is true, S_m is also true.
4. The assumption that some statement in $\{S_1, S_2, S_3, \cdots\}$ is not true is false. Therefore all of the statements are true.

Conclude from this that as $n \to \infty$, $2^n \to \infty$, and $e^n \to \infty$.

Conclude also that as $n \to \infty$, $2^{-n} \to 0$.

Exercise 14.5.9 Recall that the graph of $y = \ln x$ is the reflection of the graph of $y = e^x$ about the line $y = x$. What is $\lim_{x\to\infty} \ln x$?

Exercise 14.5.10 Review the argument that $\lim_{t\to\infty} \frac{e^t}{t} = \infty$, Equation14.20. Use similar steps to show that

$$\lim_{t\to\infty} \frac{\ln t}{t} = 0 \tag{14.28}$$

You may find the following algebra helpful.

$$\frac{\ln t}{t} = \frac{2\ln t - \ln t}{t} = \frac{\ln t^2 - \ln t}{t} < \frac{\ln t^2 - \ln t}{t^2 - t} \qquad \text{for} \qquad 2 < t$$

Exercise 14.5.11 Evaluate

$$\text{a.}\quad \lim_{t\to\infty} \frac{\ln\sqrt{t}}{\sqrt{t}} \qquad\qquad \text{b.}\quad \lim_{t\to\infty} \frac{\ln t}{\sqrt{t}}$$

Exercise 14.5.12 Use L'Hôspital's Rule, where appropriate, to evaluate the limits

a. $\displaystyle\lim_{t\to\infty}\frac{e^{2t}}{t}$

b. $\displaystyle\lim_{t\to\infty}\frac{e^{t}}{\sqrt{t}}$

c. $\displaystyle\lim_{t\to\infty}\frac{2t^2+1}{5t^2+2}$

d. $\displaystyle\lim_{t\to\infty}\frac{\ln t}{e^{t}}$

e. $\displaystyle\lim_{t\to0^+}\frac{\ln t}{1/t}$

f. $\displaystyle\lim_{t\to0^+}\frac{\sin t}{t}$

g. $\displaystyle\lim_{t\to\infty}\frac{\ln\sqrt{t}}{\sqrt{t}}$

h. $\displaystyle\lim_{t\to0^+}\frac{\sin 2t}{\sin 3t}$

i. $\displaystyle\lim_{t\to\infty}\frac{\ln t}{\sqrt{t}}$

j. $\displaystyle\lim_{t\to\infty}\frac{3^t-1}{2^t-1}$

k. $\displaystyle\lim_{t\to0^+}\frac{t}{\ln(1+t)}$

l. $\displaystyle\lim_{t\to0+}\frac{\tan t}{\sqrt{t}}$

Exercise 14.5.13 Prove Theorem 14.5.2, the Extended Mean Value Theorem. In doing so, you will find it useful to consider the function,

$$D(x) = (\,F(b)-F(a)\,)\times(\,G(x)-G(a)\,) - (\,G(b)-G(a)\,)\times(\,F(x)-F(a)\,)$$

You may then find it useful to show

a. $D(a) = 0$.

b. $D(b) = 0$.

c. There is a number c such that $D'(c) = 0$.

d. $(\,F(b)-F(a)\,)\times G'(c) = (\,G(b)-G(a)\,)\times F'(c)$

e. $G'(c) \neq 0$.

f. $G(b)-G(a) \neq 0$.

14.6 Environmental carrying capacity.

We examine a difference equation used to describe population growth in limited environments.

$$P_{t+1} - P_t \;=\; R\times P_t\times\left(1-\frac{P_t}{M}\right) \qquad \text{Logistic Growth}$$

An alternate equation is examined in Section 14.8

$$P_{t+1} \;=\; \frac{(1+R)\,P_t}{1+R\dfrac{P_t}{M}} \qquad \text{Alternate Logistic Growth}$$

In 1838, Verhulst, in response to Malthus, argued that environments have a carrying capacity and as populations near their environmental limits, the growth rates decrease. Something like this is happening today in the human population. In less developed regions of the world total fertility has fallen from 6 to 3 births per woman in the past fifty years[7].

Verhulst's model of population growth was

> **Mathematical Model of population growth in a limited environment.** The annual increase in human population is proportional to the size of the population at the beginning of the year and is proportional to the fraction of the carrying capacity unused by the population.

If the carrying capacity is measured as M, the number of humans the environment will support, and P_t is the population size in year t then

$$\frac{P_t}{M} \qquad \text{is the fraction of the carrying capacity that is used, and}$$

$$1-\frac{P_t}{M} \qquad \text{is the unused fraction of the carrying capacity.}$$

Because of the double proportionality, the annual increase is proportional to the *product* of the population size and the unused carrying capacity:

$$P_{t+1} - P_t = R\,P_t \times \left(1 - \frac{P_t}{M}\right) \tag{14.29}$$

R is called the *low density growth rate*. When P_t is 'small' compared to M, the factor, $\left(1-\frac{P_t}{M}\right)$ is close to 1 and

$$P_{t+1} - P_t \doteq R\,P_t \qquad \text{for} \qquad \frac{P_t}{M} \qquad \text{'small'}$$

As $P_t \to M$, $\left(1-\frac{P_t}{M}\right)$ approaches 0, so that the population change,

$$P_{t+1} - P_t \doteq R \times P_t \times 0 = 0 \qquad \text{for} \qquad \frac{P_t}{M} \doteq 1$$

For reasons that are unknown to us, the word *logistic* is used to describe Equation 14.29. Equation 14.29 is the discrete logistic equation, and we will study the continuous logistic equation in Chapter 17.

To be concrete, consider a low density growth rate, $R = 0.4$, and a carrying capacity, $M = 1000$ in Equation 14.29. Then

$$P_{t+1} - P_t = 0.4 \times P_t \times \left(1 - \frac{P_t}{1000}\right) \tag{14.30}$$

When $P_t = 10$ (small compared to 1000), then $1 - \frac{P_t}{1000} = 0.99 \doteq 1$ and

$$P_{t+1} - P_t \;\doteq\; 0.4 \times P_t \times 1 = 0.4 \times P_t$$

The population increases approximately 40% each time period, and there is exponential growth. This growth continues until the population size compared to 1000 is significant.

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[7] www.undp.org/popin

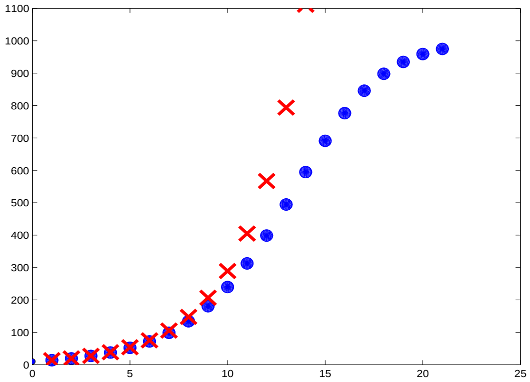


Figure 14.15: The graph of a logistic curve $P_{t+1} = P_t + 0.4P_t\left(1 - \frac{P_t}{1000}\right)$ (circles) and the exponential curve $W_{t+1} = W_t + 0.4W_t$ (X's).

When $P_t = 500$, $1 - \frac{P_t}{1000} = 0.5$ and

$$P_{t+1} - P_t \doteq 0.4 \times P_t \times 0.5 = 0.2 \times P_t$$

The population is still growing, but only at 20% per time period, one-half of that at low density.

When the populations size, P_t reaches 990 (high density, almost to carrying capacity), $1 - \frac{P_t}{1000} = 0.01$ and

$$P_{t+1} - P_t \doteq 0.4 \times P_t \times 0.01 = 0.004 \times P_t$$

and the growth rate has fallen to 0.4% per time period.

In Figure 14.15 are the graphs of

$$\begin{array}{lcll} P_0 = 10 \quad P_{t+1} & = & P_t + 0.4P_t\left(1 - \dfrac{P_t}{1000}\right) & \text{Circles} \\ & \text{and} & & \\ W_0 = 10 \quad W_{t+1} & = & W_t + 0.4W_t & \text{X's} \end{array}$$

It can be seen that W_t follows P_t until $P_t \doteq 200$ and $1 - \frac{P_t}{1000} \doteq 0.8$.

Normalization. The equation

$$P_{t+1} - P_t = R \times P_t \times \left(1 - \frac{P_t}{M}\right)$$

is normalized by dividing each term by M, as in

$$\frac{P_{t+1}}{M} - \frac{P_t}{M} = R \times \frac{P_t}{M} \times \left(1 - \frac{P_t}{M}\right).$$

Then one may define

$$Q_t = \frac{P_t}{M}$$

and write

$$Q_{t+1} - Q_t = R \times Q_t \times (1 - Q_t)$$

This is a simpler equation – it has only one parameter, R – but reflects the dynamics of the first. In practice, P_t is used ambiguously for both the original P_t and what we have written as Q_t.

Explore 14.6.1 Start with an initial population of $P_0 = 100$ and use

$$P_0 = 100, \qquad P_{t+1} - P_t = 0.2 \times P_t \times \left(1 - \frac{P_t}{1000}\right) \tag{14.31}$$

to compute P_1, P_2, $\cdots$, P_{40}. Sketch a graph of P_t *vs* t.

This may be a bit boring, so on your calculator, type

```
100    ENTER
Ans+0.2*Ans*(1-Ans/1000)    ENTER
```

Continue pressing ENTER and the numbers will appear.

Alternatively, you could run the calculator program shown in Explore Figure 14.6.1. If you do so, time will be recorded in TT and population will be recorded in PP. Un-select all functions in Graph, y(x)=, and Select Plot1 and set the Window to fit.

Or you could run the MATLAB program.

Explore Figure 14.6.1 Calculator and MATLAB programs to numerically solve Equation 14.31.

```
PROGRAM:PMAX
:41->dimL TT
:41->dimL PP
:0->TT(1)
:100->PP(1)
:For(K,1,40)
:K->TT(K+1)
:PP(K)+.2*PP(K)*
(1-PP(K)/1000)->PP(K+1)
:End
:Scatter TT,PP
```

```
MATLAB:
close all;clc;clear
P(1)=100;
for k = 1:40
  P(k+1)=P(k)+0.2*P(k)*(1-P(k)/1000);
end
plot(P,'x','linewidth',2)
```

14.6.1 Fitting Logistic Equations to Data.

Shown in the table and graph in Figure 14.16 are data from growing *V. natriegens* in a flask over a two hour period. *V. natriegens* density is measured by Absorbance on a spectrophotometer. The first 80 minutes of this data was shown in the discussion of exponential growth on page **??** where it was noted that the growth rate during the last 16 minutes was a bit below that of 0 to 64 minutes. The population is beginning to exhaust its environment.

The graph of *V. natriegens* density *vs Time* bears a reasonable resemblance to that of the logistic curve shown in previous graphs, and we look for a logistic equation that approximates the *V. natriegens* data. The equation should be of the form

$$P_{t+1} - P_t = R \times P_t \times \left(1 - \frac{P_t}{M}\right) \tag{14.32}$$

pH 6.25		
Time (min)	Index t	Population Density
0	0	0.022
16	1	0.036
32	2	0.060
48	3	0.101
64	4	0.169
80	5	0.266
96	6	0.360
112	7	0.510
128	8	0.704
144	9	0.827
160	10	0.928

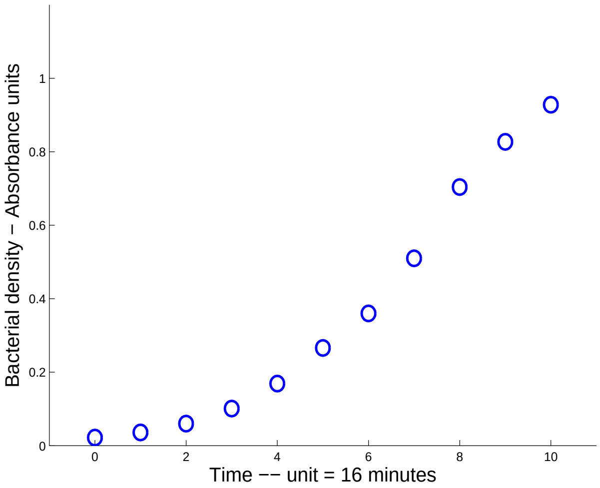


Figure 14.16: Light absorbance data and graph from *V. natriegens* grown by Deb Christensen.

In order to compute with it, we must have an initial value, P_0, and we need to find R, M.

We can divide both sides of Equation 14.32 by P_t and get

$$\frac{P_{t+1} - P_t}{P_t} = R \times \left(1 - \frac{P_t}{M}\right) \tag{14.33}$$

If we think

$$y = \frac{P_{t+1} - P_t}{P_t} \qquad \text{and} \qquad x = P_t$$

we get a linear equation

$$y = R - \frac{R}{M}x$$

Shown in Figure 14.17 are data and a graph of x and y together with a line $y = 0.68 - 0.69x$ fit to the data (by least squares). The graph is a bit scattered, particularly to the left, reflecting the fact that

1. y is the difference of two small experimental values, and
2. That error is magnified by division by a small number.

Now, if we compare

$$y = R - \frac{R}{M}x \qquad \text{with } y = 0.68 - 0.69x$$

we would conclude that

$$R = 0.68 \qquad \frac{R}{M} = 0.69 \qquad \Longrightarrow M = 0.99$$

Shown to in Figure 14.18 is the graph of the original *V. natriegens* data and the graph of

$$P_{t+1} - P_t = 0.68 \times P_t \times \left(1 - \frac{P_t}{0.99}\right) \tag{14.34}$$

They match very well, despite the scatter in the data from which R and M were computed.

P_t	$(P_{t+1} - P_t)/P_t$
0.022	0.634
0.036	0.667
0.060	0.684
0.101	0.673
0.169	0.574
0.266	0.353
0.360	0.417
0.510	0.380
0.704	0.175
0.827	0.122

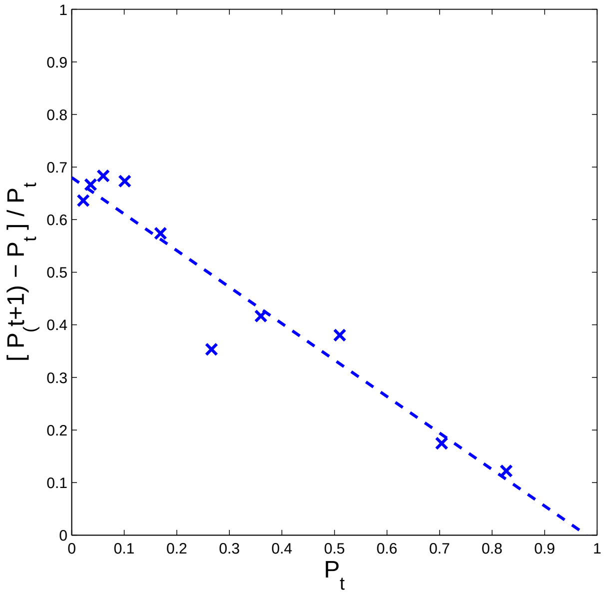


Figure 14.17: Data and graph for $\frac{P_{t+1}-P_t}{P_t}$ versus P_t for P_t data in Figure 14.16.

The number R in Equation 14.29 is the low density growth rate. In Chapter **??** we obtained Equation **??** on page **??**:

$$B_{t+1} - B_t = \frac{2}{3} B_t$$

using just the six early data points at which the bacterial density was low. The low density growth rate $R = 0.68$ above compares well with $\frac{2}{3} = 0.67$.

Exercises for Section 14.6, Environmental carrying capacity.

Exercise 14.6.1 Contrast the growths per time period described by

$$P_{t+1} - P_t = 0.2 \times P_t \times \left(1 - \frac{P_t}{1000}\right)$$

(a) when $P_t = 900$ and (b) when $P_t = 1000$ and (c) when $P_t = 1100$.

Exercise 14.6.2 Do Explore 14.6.1 on page 174.

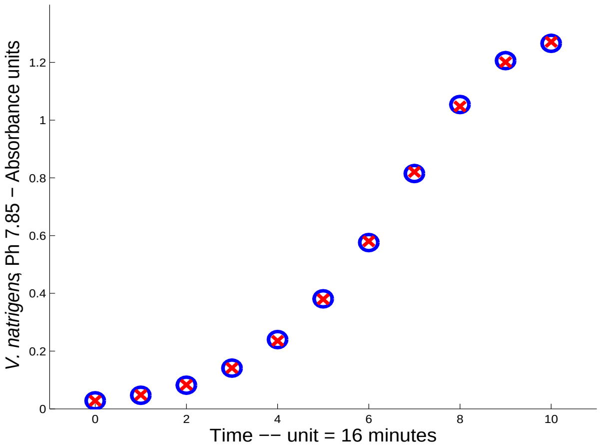


Figure 14.18: Comparison of the *V. natriegens* data of Figure 14.16 with the logistic curve, Equation 14.34.

Exercise 14.6.3 Plot the graphs of P_t and Q_t *vs* t for the pairs shown.

$$\begin{array}{llrcl}
\text{a.} & P_0 = 20 & P_{t+1} & = & P_t + 0.2P_t\left(1 - \frac{P_t}{400}\right) \\
 & Q_0 = 20 & Q_{t+1} & = & Q_t + 0.2Q_t
\end{array}$$

$$\begin{array}{llrcl}
\text{b.} & P_0 = 20 & P_{t+1} & = & P_t + 0.1P_t\left(1 - \frac{P_t}{400}\right) \\
 & Q_0 = 20 & Q_{t+1} & = & Q_t + 0.1Q_t
\end{array}$$

$$\begin{array}{llrcl}
\text{c.} & P_0 = 20 & P_{t+1} & = & P_t + 0.2P_t\left(1 - \frac{P_t}{200}\right) \\
 & Q_0 = 20 & Q_{t+1} & = & Q_t + 0.2Q_t
\end{array}$$

$$\begin{array}{llrcl}
\text{d.} & P_0 = 20 & P_{t+1} & = & P_t + 0.2P_t\left(1 - \frac{P_t}{100}\right) \\
 & Q_0 = 20 & Q_{t+1} & = & Q_t + 0.2Q_t
\end{array}$$

$$\begin{array}{llrcl}
\text{e.} & P_0 = 20 & P_{t+1} & = & P_t + 0.1P_t\left(1 - \frac{P_t}{100}\right) \\
 & Q_0 = 20 & Q_{t+1} & = & Q_t + 0.1Q_t
\end{array}$$

$$\begin{array}{llrcl}
\text{f.} & P_0 = 200 & P_{t+1} & = & P_t + 0.1P_t\left(1 - \frac{P_t}{100}\right) \\
 & Q_0 = 200 & Q_{t+1} & = & Q_t + 0.1Q_t
\end{array}$$

Exercise 14.6.4 Shown in Table 14.6.4A is a data set of *V. natriegens* grown at pH 7.85. This is a continuation of the the data set shown in Exercise **??** on page **??**. Compute R and M for a logistic curve that approximates this data. Compare the computations from the logistic approximation with the data.

Exercise 14.6.5 Many things grow in a logistic fashion. In some instances, the growth of an individual organism is logistic. A crow embryo grows rapidly at first and has an upper bound inside the egg. This is *not* a mathematical model, however. Shown in Table 14.6.4B is the 'wet mass of a crow embryo' measured at days 3 - 17 of the incubation period. Determine whether a logistic curve describes that growth.

Table for Exercise 14.6.4 A. **Exercise 14.6.4.** Light absorbance data from growth of *V. natriegens* in a solution with pH 7.85. B. **Exercise 14.6.5** Wet mass of a crow embryo at days 3 - 17 of the incubation period.

A

pH 7.85		
Time (min)	Index t	Population Density
0	0	0.028
16	1	0.047
32	2	0.082
48	3	0.141
64	4	0.240
80	5	0.381
96	6	0.576
112	7	0.815
128	8	1.054
144	9	1.206
160	10	1.266

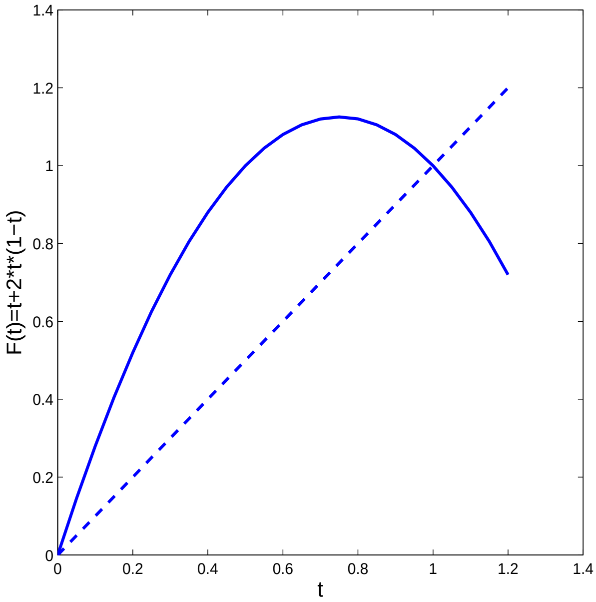
B

Crow embryo mass		
Time (day)	Index t	Embryo mass (gm)
3	0	0.220
4	1	0.435
5	2	0.650
6	3	0.800
7	4	1.200
8	5	1.630
9	6	2.150
10	7	2.800
11	8	3.700
12	9	5.100
13	10	7.500
14	11	10.200
15	12	12.600
16	13	14.850
17	14	16.00

Exercise 14.6.6 A value of $R = 2$ in the logistic equation, $Q_{t+1} = Q_t + R\,Q_t \times (1 - Q_t)$, yields some interesting results. Shown in Table 14.6.6 are computations and graphs for $R = 2$ and $Q_0 = 0.2$. The odd-indexed iterates increase and the even-indexed iterates beyond index 2 decrease. The pattern continues after 100,000 iterations. Do the two sequences converge to 1? 'Cobweb' the graph of $F(x) = x + 2\,x\,(1 - x)$ at the equilibrium point $x = 1$ to formulate an answer. Why does the Cobweb Theorem 14.4.1 not apply in this case?

Table for Exercise 14.6.6 Data and graph for the iteration $Q_{t+1} = Q_t + RQ_t \times (1 - Q_t)$ with $R = 2$ and $Q_0 = 0.2$.

Time	*Pop/M*
0	0.200
1	0.520
2	1.0192
3	0.98006
4	1.01914
5	0.98012
6	1.01909
7	0.98019
8	1.01903
9	0.98025
10	1.01897
⋮	⋮
999,997	0.998882554
999,998	1.001114949
999,999	0.998882565
100,000	1.001114938



Exercise 14.6.7 The logistic iteration $Q_{t+1} = Q_t + R\,Q_t\,(1 - Q_t)$ is but one example of the iterations $Q_{t+1} = F(Q_t)$ that were considered in Section 14.3, Equilibrium Points, Stable and Nonstable.

a. Find the two equilibrium points of this logistic iteration and use Theorem 14.4.1 to determine the conditions on $R \neq 0$ for which at least one of them will be locally stable.

b. Discuss the case $R = 0$.

Exercise 14.6.8 Danger: This problem may scramble and fry your brain. Consider the sequence defined by,

$$Q_0 = 0.2, \qquad Q_{t+1} = Q_t + 2\,Q_t\,(1 - Q_t) \qquad t = 1, 2, \cdots.$$

Show that the subsequence Q_2, Q_4, Q_6, $\cdots$ converges to 1.

It certainly appears so from the data in Table 14.6.6, but even $Q_{100{,}000} = 1.001114938$ is 0.001 above 1. Assume without proof that all of the numbers Q_{2t} satisfy $1 < Q_{2t} < 1.2$.

First you will show that

$$Q_{2t+2} - 1 < Q_{2t} - 1, \qquad t = 1, 2, \cdots.$$

That shows that the sequence decreases toward 1, but does not prove that its limit is 1. (The sequence $\{2 + 1/1,\ 2 + 1/2,\ 2 + 1/3,\ \cdots\ 2 + 1/n,\ \cdots\}$ decreases toward 1, but does not converge to 1.)

You might think that

$$|Q_{t+1} - 1| < |Q_t - 1|, \qquad t = 1, 2, \cdots$$

but that is unfortunately false.

$$|Q_9 - 1| = 0.01975 \not< 0.01903 = |Q_8 - 1|.$$

We therefore have to compose the iteration function $F(x) = x + 2\,x\,(1-x) = 3x - 2x^2$ with itself to get

$$G(x) = F(F(x)) = -8\,x^4 + 24\,x^3 - 24\,x^2 + 9x \tag{14.35}$$

Then $Q_{2t+2} = G(Q_{2t})$, $t = 1, 2, 3, \cdots$.

a. Show that Equation 14.35 is correct.

b. Now $Q_{2t+2} = G(Q_{2t})$, Show that

$$\begin{aligned} Q_{2t+2} - 1 &= G(Q_{2t}) - 1 \\ &= -8\,Q_{2t}^4 + 24\,Q_{2t}^3 - 24\,Q_{2t}^2 + 9\,Q_{2t} - 1 \\ &= (Q_{2t} - 1)\,(-8\,Q_{2t}^3 + 16\,Q_{2t}^2 - 8\,Q_{2t} + 1) \end{aligned} \tag{14.36}$$

c. Draw the graph of $H(x) = -8\,x^3 + 16\,x^2 - 8\,x + 1$ and show that (1,1) is the high point for H on the interval [0.5, 1.2] and that H is positive on this interval.

d. Show that

$$Q_{2t+2} - 1 < Q_{2t} - 1$$

(because the second factor in Equation 14.36 is less than 1).

e. We now know that $\{Q_2,\ Q_4\,Q_6,\ \cdots\}$ is a decreasing sequence of numbers bigger than 1. By Theorem 5.2.1, that bounded non-decreasing sequences have a limit, the sequence $\{Q_2,\ Q_4\,Q_6,\ \cdots\}$ has a limit, L which is either 1 or bigger than 1, and L is less than or equal to every number in $\{Q_2,\ Q_4\,Q_6,\ \cdots\}$.

f. Show that $H(L) \geq H(Q_{2t})$ for every number t.

g. Show that

$$Q_{2t+2} - 1 = (Q_{2t} - 1)\,H(Q_{2t}) \leq (Q_{2t} - 1)\,H(L)$$

h. If $L = 1$, the assertion is proved. Suppose $L > 1$. Then $0 < H(L) < 1$. By cascading the previous inequality, show that

$$Q_{2t} - 1 \leq (Q_2 - 1)(H(L))^t$$

i. Conclude that

$$0 \leq \lim_{t\to\infty} Q_{2t} - 1 = \lim_{t\to\infty} (Q_2 - 1)(H(L))^t = 0.$$

so that

$$1 \quad \leq \quad \lim_{t\to\infty} Q_{2t} \quad = \quad 1. \qquad\qquad \text{Whew!}$$

14.7 Harvest of Natural Populations.

The population difference equations can be modified to reflect the effects of harvest. The harvest may be measured as an amount harvested, H_t, or a fraction of the present population harvested, $h_t \times P_t$. The logistic equation may be

$$
\begin{aligned}
P_{t+1} - P_t &= R \times P_t \times \left(1 - \frac{P_t}{M}\right) - H_t && (14.37)\\
P_{t+1} - P_t &= R \times P_t \times \left(1 - \frac{P_t}{M}\right) - h_t \times P_t && (14.38)
\end{aligned}
$$

Among the questions that one asks is what harvest strategy will provide the maximum long term yield, what is the maximum allowable harvest in order to retain the population, and what are the locally stable equilibrium sizes of the population?

Example 14.7.1 We begin with the equation

$$
\begin{aligned}
P_0 &= 1000\\
P_{t+1} - P_t &= 0.2 \times P_t \times \left(1 - \frac{P_t}{1000}\right) - h \times P_t
\end{aligned}
$$

of a population with an initial population of 1000 individuals, low density growth rate of 0.2 per time interval, carrying capacity 1000 individuals, and ask what constant fraction of individuals present, h, may be harvested and still retain the population?

The equilibrium population is important and we solve for P_e in

$$P_e - P_e = 0.2 \times P_e \times \left(1 - \frac{P_e}{1000}\right) - h \times P_e$$

$$
\begin{aligned}
0 &= 0.2 \times P_e \times \left(1 - \frac{P_e}{1000}\right) - h \times P_e\\
0 &= P_e \times \left(0.2 \times \left(1 - \frac{P_e}{1000}\right) - h\right)
\end{aligned}
$$

The choices are

$$P_e = 0 \qquad \text{and} \qquad 0.2 \times \left(1 - \frac{P_e}{1000}\right) - h = 0$$

for which

$$P_e = 0 \qquad \text{and} \qquad P_e = 1000 \times (1 - 5h)$$

Observe that if $1 - 5h$ is negative, the only realistic equilibrium is 0. Therefore if we harvest more than 20% ($h > 0.2$) of the population present at each time, we will loose the population. This makes sense.

The low density growth rate (births minus natural deaths) is 20% and if harvest exceeds that we will loose the population. In fact, if harvest $h = 0.2$ then $P_e = 1000 \times (1 - 5h) = 0$, and we loose the population.

We next normalize the initial equation by dividing by the carrying capacity.

$$\frac{P_t}{1000} - \frac{P_t}{1000} = 0.2 \times \frac{P_t}{1000} \times \left(1 - \frac{P_t}{1000}\right) - h \times \frac{P_t}{1000}$$

and write (using P_t ambiguously)

$$\begin{aligned} P_0 &= 1 \\ P_{t+1} - P_t &= 0.2 \times P_t \times (1 - P_t) - h \times P_t \end{aligned}$$

Now if the fractional harvest is set at a number, $h < 0.2$, the equilibrium is either 0 or $(1 - 5h) \times 1000$. Shown in Figure 14.19 is the graph of P_{t+1} *vs* P_t for $h = 0.05$ and it is apparent that $P_t = 0$ is a nonstable equilibrium and $P_t = 1 - 5h = 0.75$ is locally stable.

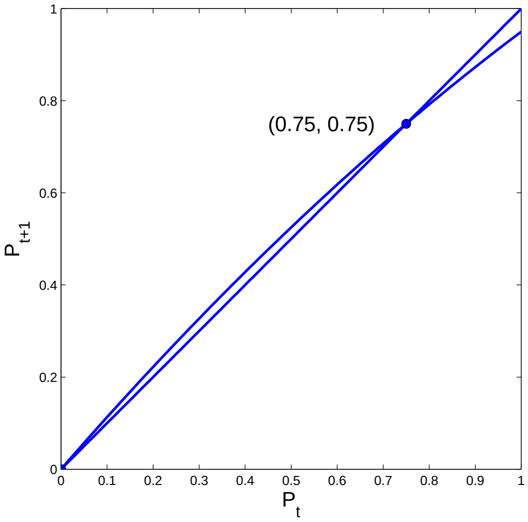


Figure 14.19: Graphs of the iteration equation, $P_{t+1} = P_t + 0.2P_t(1 - P_t) - 0.05P_t$ and $y = x$ showing the equilibrium point at (0.75,0.75).

The objective often will be to maximize harvest. At the equilibrium of $P_e = 1 - 5h$, the harvest amount is $h \times P_e = h \times (1 - 5h)$. The maximum harvest at equilibrium will be obtained at a harvest level that maximizes $h \times (1 - 5h)$.

For what value of h is $\quad h \times (1 - 5h) \quad$ the largest?

Let $G(h) = h \times (1 - 5h)$. Then $G'(h) = 1 - 10h$ and $G'(h) = 0$ when $h = 0.1$ and $G''(h) = -10 < 0$. Therefore $h = 0.1$ yields the maximum value of $h \times (1 - 5h)$. Recall that $R = 0.2$ in this example. Then a 10% harvest strategy is one-half the low density growth rate and will yield a long term maximum harvest value. Furthermore, for $R = 0.2$ and $h = 0.1$, $P_t + 1 = P_t + 0.2P_t \times (1 - P_t) - 0.1P_t$ and the equilibrium value is $P_e = 0.5$ so *the maximum harvest occurs with the population at one-half the maximum supportable population.* This is a general property of logistic models.

Exercises for Section 14.7, Harvest of Natural Populations.

In the next four exercises you will analyze the general form of Equation 17.17 with constant $h_t = h$ and $0 < R < 2$.

$$P_{t+1} - P_t = R \times P_t \times \left(1 - \frac{P_t}{M}\right) - h \times P_t \tag{14.39}$$

Exercise 14.7.1 Divide each term of Equation 14.39 by M to obtain

$$p_{t+1} - p_t = R \times p_t \times (1 - p_t) - h \times p_t$$

Exercise 14.7.2 a. Show that the equilibrium numbers of the iteration

$$p_{t+1} = F(p_t) = p_t + R \times p_t \times (1 - p_t) - hp_t$$

are $p_{*1} = 0$ and $p_{*2} = 1 - h/R$.

b. Show that in order for there to be a positive equilibrium, the fractional harvest rate, h, must be less than the low density growth rate, R.

Exercise 14.7.3 Assume that $h < R < 2$. Compute $F'(p)$ and evaluate $F'(0)$ and $F'(1 - h/R)$. Conclude that 0 is a nonstable equilibrium and $1 - h/R$ is a locally stable equilibrium.

Exercise 14.7.4 Assume that $h < R < 2$ so that $1 - h/R$ is a positive, locally stable equilibrium. The harvest at that equilibrium will be $h \times (1 - h/R)$. Find the value of h for which $h \times (1 - h/R)$ is the largest. Conclude that in order to maximize harvest, the harvest fraction h should be set at one-half the low density growth rate, R. Find the positive equilibrium p_e for this value of h. In terms of M, what is the equilibrium value at the harvest rate that will maximize harvest?

In the next four exercises you will analyze the Equation 14.40,

$$P_{t+1} - P_t = R \times P_t \times \left(1 - \frac{P_t}{M}\right) - H \tag{14.40}$$

where $R < 2$ and a fixed amount,H, is harvested each time period.

Exercise 14.7.5 Divide the terms of Equation 14.40 by M to obtain

$$p_{t+1} - p_t = R \times p_t \times (1 - p_t) - K \tag{14.41}$$

What is the interpretation of K in this equation?

Exercise 14.7.6 a. Show that the equilibrium numbers of the iteration

$$p_{t+1} = F(p_t) = p_t + R \times p_t \times (1 - p_t) - K$$

are

$$p_{*1} = \frac{R - \sqrt{R^2 - 4KR}}{2R} \qquad \text{and} \qquad p_{*2} = \frac{R + \sqrt{R^2 - 4KR}}{2R} \tag{14.42}$$

b. Show that for there to be any positive equilibrium, the harvest K must be less than or equal to one-fourth the low density growth rate R.

Exercise 14.7.7 Assume that $K < 4R$ and $R < 2$. Compute $F'(p)$ for $F(p) = p + r \times p \times (1 - p) - K$. Show that $p_{*,1}$ and $p_{*,2}$ from Equation 14.42 are, respectively, not stable and locally stable.

Exercise 14.7.8 The maximum harvest for $p_{t+1} - p_t = R \times p_t \times (1 - p_t) - K$ is interesting. In order for there to be a positive equilibrium, we must have $K \leq R/4$. We might suppose $K = R/4$.

a. Show that under the assumption that $K = R/4$, $p_{*1} = p_{*2} = 1/2$ and that $F'(p_{*2}) = 1$. We are then uncertain whether 1/2 is a locally stable equilibrium. Note that $p_{*2} = 1/2$ corresponds to one-half the maximum supportable population.

b. Examine the special case

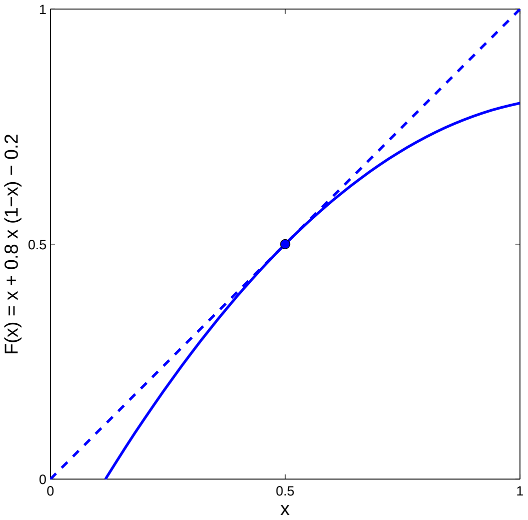
$$p_{t+1} - p_t = 0.8 \times p_t \times (1 - p_t) - 0.2$$

The graph of

$$F(x) = x + 0.8 \times x \times (1 - x) - 0.2$$

and the graph of $y = x$ are shown in Figure 14.7.8. Cobweb the graph starting with $p_0 > 1/2$ and again with the $p_0 < 1/2$. Is $p_{*2} = 1/2$ locally stable? Would you set the constant harvest at one-fourth the low density growth rate and maintain the population at one-half the maximum supportable population?

Figure for Exercise 14.7.8 Graphs of the iteration function, $F(x) = x + 0.8 \times x \times (1 - x) - 0.2$ and $y = x$.



14.8 An alternate logistic equation.

It appears that there is no formula for the solution to Verhulst's logistic equation

$$P_{t+1} - P_t = R\, P_t \left(1 - \frac{P_t}{M}\right)$$

and an alternate form called the Beverton-Holt equation[8] is sometimes used:

$$P_{t+1} = \frac{(1+R)\times P_t}{1+R\frac{P_t}{M}} \tag{14.43}$$

Peculiar as it may seem, this equation involving division does have a formula for its solution.

$$P_t = \frac{MP_0}{P_0 + (M-P_0)(1+R)^{-t}} \tag{14.44}$$

is a solution to Equation 14.43.

Explore 14.8.1 Verify the last statement by showing that

$$\frac{(1+R)\dfrac{MP_0}{P_0+(M-P_0)(1+R)^{-t}}}{1+R\dfrac{\frac{MP_0}{P_0+(M-P_0)(1+R)^{-t}}}{M}} \quad \text{simplifies to} \quad \frac{MP_0}{P_0+(M-P_0)(1+R)^{-(t+1)}}$$

■

Formula 14.44 is more than a Bolt Out of the Blue. A logical derivation from the initial equation appears in Exercise 14.8.5.

Explore 14.8.2 Show that for the alternate logistic equation, R is the low density growth rate. That is, show that in

$$P_{t+1} = \frac{(1+R)\times P_t}{1+R\frac{P_t}{M}}, \qquad \text{for} \qquad P_t \ll M, \qquad P_{t+1} - P_t \doteq R\,P_t$$

■

The alternate logistic Equation 14.43 can also be fit to logistic data, although our personal success with this form has not been so great as with the original Equation 14.32. To fit Equation 14.43 to data, we 'turn it up side down'. Begin with

$$P_{t+1} \quad = \quad \frac{(1+R)\times P_t}{1+R\frac{P_t}{M}}$$

and put the reciprocals of the two sides of the equation equal:

$$\frac{1}{P_{t+1}} \quad = \quad \frac{1}{\frac{(1+R)\times P_t}{1+R\frac{P_t}{M}}} \quad = \quad \frac{1+R\frac{P_t}{M}}{(1+R)\times P_t} \quad = \quad \frac{1}{1+R}\times\frac{1}{P_t}+\frac{1}{M}\frac{R}{1+R}$$

This means that $\frac{1}{P_{t+1}}$ should be linearly related to $\frac{1}{P_t}$.

Data and the graph of $\frac{1}{P_{t+1}}$ *vs* $\frac{1}{P_t}$ for the *V. natriegens* data described in Figure 14.16 is shown in Figure 14.20. Observe the regularity of the graph as compared to the graph of $\frac{P_{t+1}-P_t}{P_t}$ *vs* P_t in Figure e 14.17. Reciprocals are not subject to the roundoff errors that blur differences of small numbers.

---

[8] Introduced in Beverton, R. J. H.; Holt, S. J. (1957), On the Dynamics of Exploited Fish Populations, Fishery Investigations Series II Volume XIX, Ministry of Agriculture, Fisheries and Food

P_t	$1/P_t$	$1/P_{t+1}$
0.022	45.45	27.22
0.036	27.77	16.66
0.060	16.66	9.90
0.101	9.90	5.92
0.169	5.92	3.76
0.266	3.76	2.78
0.360	2.78	1.96
0.510	1.96	1.42
0.704	1.42	1.21
0.827	1.21	

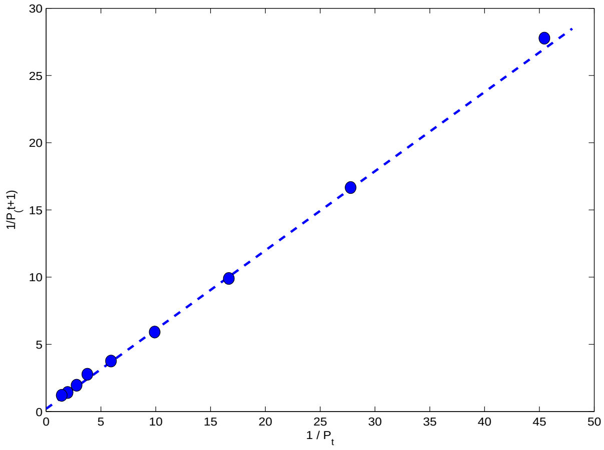


Figure 14.20: Table of reciprocals $1/P_t$ and $1/P_{t+1}$ and graph for the *V. natriegens* data of Figure 14.16 and a line, $y = 0.58914x + 0.20813$, fit to the data.

The equation

$$y = 0.58914x + 0.20813$$

is the least squares fit of a line to the data. By comparison with

$$\frac{1}{P_{t+1}} = \frac{1}{1+R} \times \frac{1}{P_t} + \frac{1}{M}\frac{R}{1+R}$$

We conclude that

$$\frac{1}{1+R} = 0.58914 \qquad \text{and} \quad \frac{1}{M}\frac{R}{1+R} = 0.20813$$

from which it follows that

$$R = \frac{1}{0.58914} - 1 = 0.697 \qquad \text{and} \qquad M = \frac{R}{(1+R)\cdot 0.20813} = 1.974$$

With $R = 0.697$, $M = 1.974$, and $P_{init} = 0.022$, we have from Equation 14.44 that

$$P_t = \frac{0.043}{0.022 + 1.95\,(1.697)^{-t}}$$

A graph of

$$P_t = \frac{0.043}{0.022 + 1.95\,(1.697)^{-t}}$$

is shown in Figure 14.21 together with the original data. The computed data is close to the original data for early times, but a substantial departure for later times, indicating that the value of R is close to correct, but the value of M is off. R is computed from the slope of the reciprocal data graph, which is pretty reliable. However, M is computed from the y-intercept of that graph, which is small and a small change makes a large percentage change. Perhaps another method of fitting the equation to real data should be explored.

Exercises for Section 14.8, An alternate logistic equation.

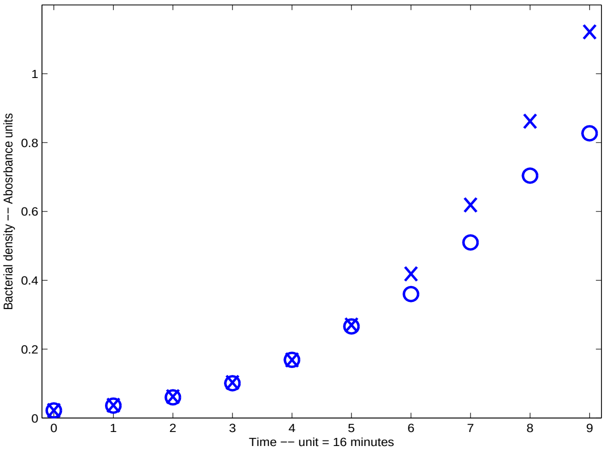


Figure 14.21: Graphs of the *V. natriegens* data (circles) from Figure 14.16 and the predictions from the altered form of the logistic equation, $P_t = \frac{1.974}{1+88.73/1.697^t}$ (x's).

Exercise 14.8.1 Do Explore 14.8.1.

Exercise 14.8.2 Shown in Figure 14.8.2 are comparisons of the two forms of the logistic equation for $R = 0.2$, $M = 10$, and $P_0 = 1$.

$$P_{t+1} - P_t = R \times P_t \times \left(1 - \frac{P_t}{M}\right) \qquad (14.45)$$

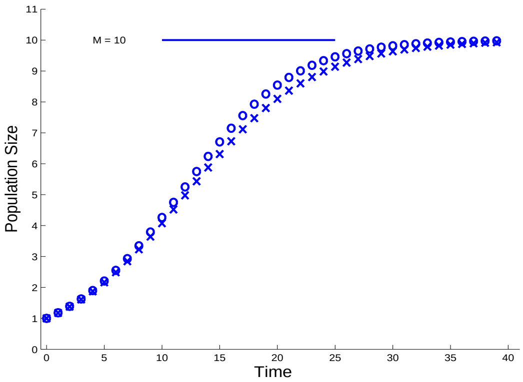
$$P_{t+1} = \frac{(1+R)\,P_t}{1 + R\frac{P_t}{M}} \qquad (14.46)$$

$$P_t = \frac{M\,P_{init}}{P_{init} + (M - P_{init})(1+R)^{-t}} \qquad (14.47)$$

Complete the missing entries in the table using these equations.

Figure for Exercise 14.8.2 Partial data and graphs comparing the two forms of the logistic equation.

t	Eq 14.45	Eq 14.46	Eq 14.47
0	1	1	1
1	_____	_____	_____
2	_____	_____	_____
3	_____	_____	_____
4	_____	_____	_____
5	2.2075	2.1660	2.1660
10	4.2622	4.0757	4.0757
15	6.7069	6.3125	6.3125
20	8.5426	8.0988	8.0988
25	9.4578	9.1379	9.1379
30	9.8140	9.6347	9.6347
35	9.9381	9.8499	9.8499
40	9.9796	9.9391	_____



Exercise 14.8.3 For each of the following systems, note the y-coordinate, M, of the horizontal asymptote, compute P_{10}/M and P_{20}/M and sketch (do not plot) graphs of P_t *vs* t.

$$\begin{array}{llll} \text{a.} & P_0 = 20 & P_{t+1} & = \dfrac{1.4P_t}{1+0.4\dfrac{P_t}{400}} \\[2ex] \text{b.} & P_0 = 20 & P_{t+1} & = \dfrac{1.2P_t}{1+0.2\dfrac{P_t}{400}} \\[2ex] \text{c.} & P_0 = 20 & P_{t+1} & = \dfrac{1.1P_t}{1+0.1\dfrac{P_t}{400}} \\[2ex] \text{d.} & P_0 = 20 & P_{t+1} & = \dfrac{1.2P_t}{1+0.2\dfrac{P_t}{200}} \\[2ex] \text{e.} & P_0 = 20 & P_{t+1} & = \dfrac{1.2P_t}{1+0.2\dfrac{P_t}{100}} \\[2ex] \text{f.} & P_0 = 20 & P_{t+1} & = \dfrac{1.1P_t}{1+0.1\dfrac{P_t}{100}} \end{array}$$

Exercise 14.8.4 Fit the alternate logistic equation 14.43 to each of the data sets for growth of *V. natriegens* in a solution with pH 7.85, and wet mass of a crow embryo at days 3 - 17 of the incubation period shown in Exercise Table 14.6.4 on page 178.

Exercise 14.8.5 Some work is required to see that Equation 14.43

$$P_{t+1} = \frac{(1+R)\times P_t}{1+R\frac{P_t}{M}}$$

has in fact a rather simple solution. This exercise shows the steps.

a. Turn the equation upside down. That is, write

$$\frac{1}{P_{t+1}} = \frac{1+R\frac{P_t}{M}}{(1+R)\times P_t}$$

and obtain

$$\frac{1}{P_{t+1}} = \frac{1}{1+R}\frac{1}{P_t} + \frac{R}{M(1+R)}.$$

b. Use Equation 14.2, $W_{t+1} - W_t = r\,W_t + b$, (notation has been changed to avoid conflict with that of the present exercise) and its solution, Equation 14.3, $W_t = -\frac{b}{r} + \left(W_0 + \frac{b}{r}\right)(1+r)^t$ to show that

$$\frac{1}{P_t} = \frac{1}{M} + \left(\frac{1}{P_0} - \frac{1}{M}\right)\left(\frac{1}{1+R}\right)^t.$$

c. Conclude that

$$P_t = \frac{MP_0}{P_0 + (M-P_0)(1+R)^{-t}} \qquad \text{Equation 14.44}$$

14.9 Chapter 14 Suggested solutions.

Exercise 14.1.1.

	Q_0	Q_1	Q_2	Q_3	Q_4	Q_5
$Q_{t+1} = Q_t + 0.05\sqrt{Q_t}$	4	4.1	4.2012	4.3037		

Exercise 14.1.2.

		Q_0	Q_1	Q_2	Q_3	Q_4	Q_5
a.	$Q_{t+2} = Q_{t+1} + 0.06 \times (1 - Q_t)$	0.1	0.1	0.154	0.208	0.2588	0.3063
c.	$Q_{t+2} = Q_{t+1} + Q_t$	0	1	2	3	5	8

Exercise 14.1.3.

$$\text{a } w_t + 1 = w_t - k \times E \times W_t$$

Exercise 14.1.4.

$$Q_{100} \doteq 49.999999940$$

Exercise 14.1.5.

a.	$Q_1 = 5.375$	$Q_{20} = 14.2028$	c.	$Q_1 = 1.120$	$Q_{20} = 60.317$
e.	$Q_1 = 0.9600$	$Q_{20} = 1.0000$	g.	$Q_1 = 1.5000$	$Q_{20} = 3.0000$
i.	$Q_1 = 0.4493$	$Q_{20} = 0.5671$	k.	$Q_1 = 1.6500$	$Q_{20} = 1.4142$

Exercise 14.1.6.

	Q_0	Q_1	Q_2	Q_3	Q_4	Q_5		Q_0	Q_1	Q_2	Q_3	Q_4	Q_5
a.	1	0	1	1	2	3	c.	1	1	2	3	5	8
e.	1	1	−1	−1	1	1	g.	1	1	1	1	1	1

Exercise 14.1.7. a. $Q_4 = 11$. b. $Q_4 = 1.216$.

Exercise 14.1.8. The results are the values

w_0	w_1	w_2	w_3	w_4	w_5	w_6	w_7
0.5	0.6	0.502	0.507	0.512	0.517	0.522	0.527

where

$$w_0 = 0.5, \quad w_1 = 0.6, \qquad w_{t+1} = w_t + 0.06w_{t-1}(1 - w_{t-1}) - 0.02w_{t-1}.$$

Exercise 14.1.9.

```
a.
Prgm, Edit WH3
:0.3->XM1
:0.4->XM0
:For(I,1,6)
:XN=XM0+0.6*XM1*(1-XM1)-0.02*XM1
:XM0->XM1
:XN->XM0
:Pause Disp XM0
:End
```

```
c.
Prgm, Edit WH3
:0.5->XM1
:0.6->XM0
:For(I,1,6)
::XN=2*XM0-0.96XM1
:XM0->XM1
:XN->XM0
:Pause Disp XM0
:End
```

	P_0	P_1	P_2	P_3	P_4	P_5	P_6	P_7
a.	0.3	0.4	0.520	0.656	0.795	0.918	0.658	0.999
c.	0.5	0.6	0.720	0.864	1.037	1.244	1.493	1.792
e.	1.0	2.0	2.060	1.878	1.689	1.546	1.442	1.3629

Exercise 14.1.10.

	P_0	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}
a.	$\pi/4$	0	$---$								
c.	0.76	0.736	0.785	0.679	0.888	0.398	$1,062$	$---$			

Exercise 14.1.11.

$$S_1 \quad ID_3 \quad DE_5 \quad P_{50} = 202, \qquad S_3 \quad ID_6 \quad DE_4 \quad P_{50} = 3.59E24$$

$$\begin{aligned} DE_5 \quad S_1: \quad P_{t+1} - P_t &= (2+4(t+1)) - (2+4t) = 2+4t+4-2-4t &= 4 \\ DE_4 \quad S_3: \quad P_{t+1} - P_t &= (5\times 3^{t+1}) - 5\times 3^t = 5\times 3^t(3-1) = 2(5\times 3^t) &= 2P_t \end{aligned}$$

Exercise 14.1.12.

a. $$\begin{aligned} P_{t+1} - 0.9P_t &= 2 \\ E - 0.9E &= 2 \\ E &= 20 \end{aligned}$$

c. $$\begin{aligned} P_{t+1} - 0.5P_t &= 3 \\ E - 0.5E &= 3 \\ E &= -2.632 \end{aligned}$$

Exercise 14.1.13. a. $b = 10$, $c = -100$. c. $b = 1.053$ $c = -0.277$. e. $p = 20$ $q = -155$

Exercise 14.1.14. a. $R = 2$ $C = 2.73$, c. $R = 2$ $C = 10$. e. $R = 0.8$ $C = -50$.

Exercise 14.1.15.

$$P_{t+1} - P_t = rP_t + b + c\,t \qquad E_t = p\,t + q$$

$$q - r\,p = b \qquad -r\,q = c; \qquad q = \frac{-c}{r}, \qquad \frac{-c}{r} - r\,p = b, \qquad p = \frac{-c}{r^2} + \frac{-b}{r}.$$

Therefore

$$E_t = p + qt = \frac{-c}{r^2} + \frac{-b}{r} + \frac{-c}{r}t = -\left(\frac{c}{r^2} + \frac{b}{r} + \frac{c}{r}t\right)$$

$$P_t = E_t + (P_0 - E_0)(1+r)^t$$

Substitute

$$E_t = -\left(\frac{c}{r^2} + \frac{b}{r} + \frac{c}{r}t\right) \quad \text{and} \quad E_0 = -\left(\frac{c}{r^2} + \frac{b}{r}\right)$$

to get

$$P_t = -\left(\frac{b}{r} + \frac{c}{r} + \frac{c}{r}t\right) + \left(P_0 + \frac{b}{r} + \frac{c}{r^2}\right)(1+r)^t.$$

In the special case, $c = 0$, this becomes

$$P_t = -\frac{b}{r} + \left(P_0 + \frac{b}{r}\right)(1+r)^t$$

which is Equation 14.3, the solution to the first order difference equation, Equation 14.2, $P_{t+1} - P_t = r\,P_t \,+\, b$.

Exercise 14.1.17.

$$M_t = 49 - 49\,0.98^t \qquad M_{1000} = 48.999999918\,\text{kg}$$

Exercise 14.1.18.

a. $P_0 = 2 \quad r = -0.2 \quad b = 0 \qquad P_t = 2 \times 0.8^t$

c. $P_0 = 2 \quad r = 0.2 \quad b = 0 \qquad P_t = 2 \times 1.2^t$

e. $P_0 = 2 \quad r = -0.2 \quad b = 3 \quad c = 2 \qquad P_t = -35 + 10t + 37 \times 0.8^t$

g. $P_0 = 2 \quad r = 0.2 \quad b = -1 \quad c = 4 \qquad P_t = -95 - 20t + 97 \times 1.2^t$

Exercise 14.2.1.

a.

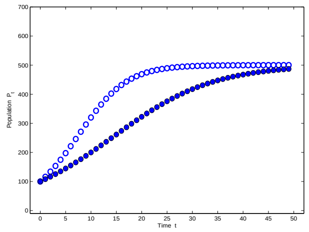


Figure 14.22: **Exercise 14.2.1.**a. Graph of $P_0 = 100$, $P_{t+1} - P_t = 0.2P_t \times \left(1 - \frac{P_t}{500}\right)$ (open circles) and $P_0 = 100$, $P_{t+1} - P_t = 0.1P_t \times \left(1 - \frac{P_t}{500}\right)$ (solid circles). .

Exercise 14.2.3. See Figure 14.23

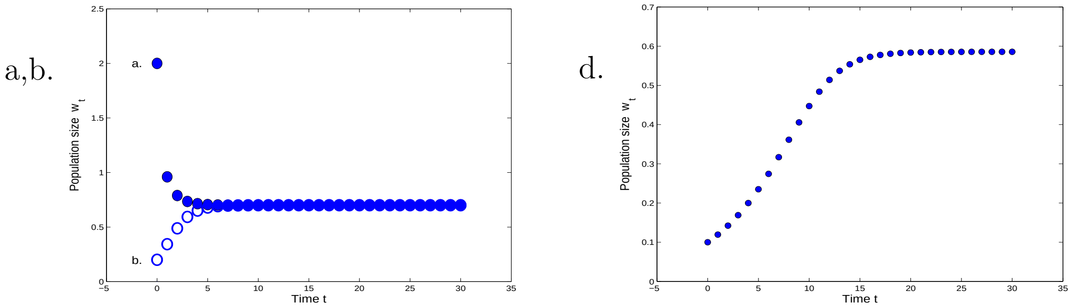


Figure 14.23: **Exercise 14.2.3.** Iteration graphs of a,b. $w_0 = 2$, $w_{t+1} = 1.2w_t/(1 + w_t)$ (filled circles) and $w_0 = 0.2$, $w_{t+1} = 1.2w_t/(1 + w_t)$, (open circles); and d. $w_0 = 0.1$, $w_{t+1} = 1.2w_t \times \cos(w_t)$.

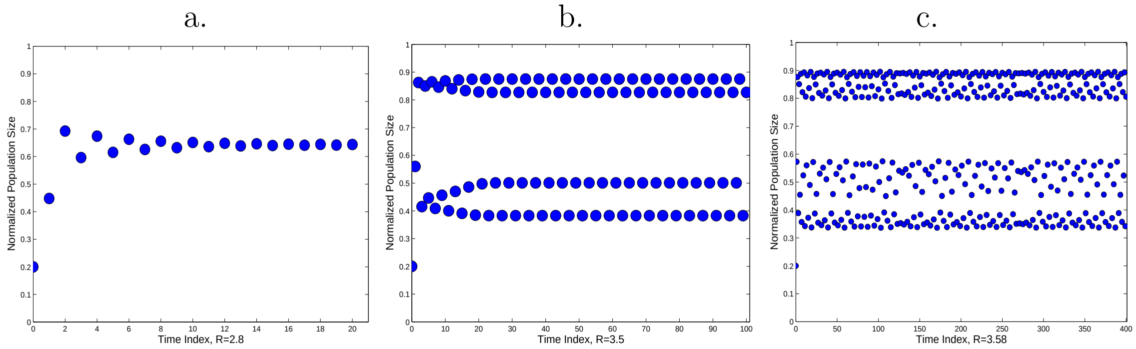


Figure 14.24: Iteration graphs of $w_{t+1} = Rw_t(1 - w_t)$ for $w_0 = 0.2$ and a. $R = 2.8$, c. $R = 3.5$ and e. $R = 3.58$.

Exercise 14.2.4. See Figure 14.24.

Exercise 14.2.5. A MATLAB program to solve this problem is:

```
close all;clc;clear
r_min = 2.5; r_max = 4; %% the range of parameter values we study
n = 150; %% the number of parameter values we consider in this range

t_max = 1000; %% how many iterations to simulate per parameter value
p_max = 50; %% the last p_max iterations are plotted

x0 = 0.2; %% we use the same initial value x0 for all parameters.

R = linspace(r_min, r_max, n+1);

for k = 1:n+1
  x(1) = x0;
  for kk = 1:t_max
    x(kk + 1) = R(k)*x(kk)*(1 - x(kk));
```

```
  end
  pop(:, k) = x(t_max-p_max+1:t_max);
end


plot(0,0)
axis([r_min r_max 0 1])
hold
   plot(R, pop, 'k.');
```

Exercise 14.3.3. See figure 14.25.

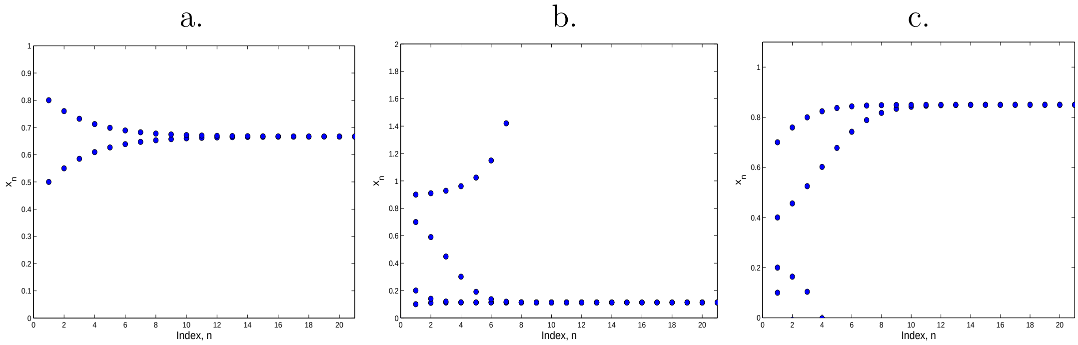


Figure 14.25: Iteration graph of: a $x_{n+1} = 0.7x_n + 0.2$ for $x_0 = 0.5$ and $x_0 = 0.8$. Both sequences move towards 2/3. c. $x_{n+1} = x_n^2 + 0.1$ for $x_0 = 0.1$, $x_0 = 0.2$, $x_0 = 0.7$ and $x_0 = 0.9$ The first three sequences have $x \doteq 0.1127$ as a horizontal asymptote. The fourth sequence grows without bound. e. $x_{n+1} = -0.9x_n^2 + 2x - 0.2$ for $x_0 = 0.1$ (x_1 is negative) and $x_0 = .2$ (x_5 is negative), $x_0 = .4$, $x_0 = .7$ (both sequences have $x \doteq 0.8495$ as a horizontal asymptote).

Exercise 14.3.4. See figures 14.26 and 14.27.

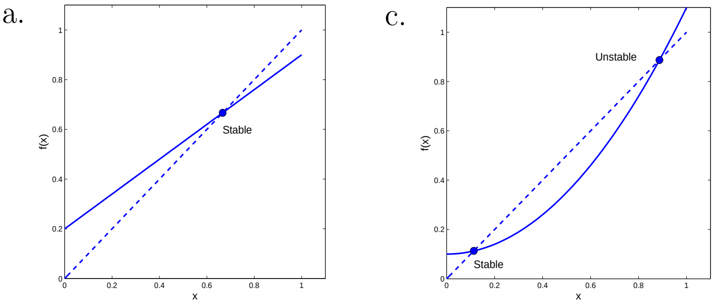


Figure 14.26: **Exercise 14.3.4.** Graphs of a. $F(x) = 0.7x + 0.2$ and c. $F(x) = x^2 + 0.1$ together with the graphs of the diagonal $y = x$ (dashed line) and the equilibrium points of F.

Exercise 14.3.5. b. $F \cdot \alpha > 1$.

Exercise 14.4.1. A MATLAB program to solve part a. follows. The output is shown in Figure 14.28.

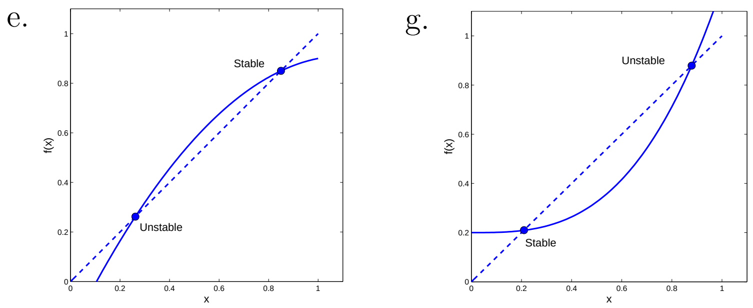


Figure 14.27: **Exercise 14.3.4.** e. $F(x) = -0.9x^2 + 2x - 0.2$ and g. $F(x) = x^3 + 0.2$, h. $F(x) = 8x^3 - 12x^2 + 6x - 1/2$ together with the graphs of the diagonal $y = x$ (dashed line) and the equilibrium points of F.

```
close all;clc;clear
x=[0:0.01:1];  R=1.0;  f=R*x.*(1-x);  y=x;
plot(x,f,'linewidth',2)
hold
axis([-0.01 1.01 -0.01 1.01]);  axis('square')

plot(x,y,'--','linewidth',2)
w(1)=0.5;
for k = 1:80
    w(k+1)=R*w(k)*(1-w(k));
end
for k = 1:30
    plot([w(k) w(k)],[w(k) w(k+1)],'linewidth',2)
    plot([w(k) w(k+1)],[w(k+1) w(k+1)],'linewidth',2)
end
m=(w(1)+2*w(2))/3
direction=(w(2)-w(1))/abs(w(2)-w(1))
plot([w(1) w(1)+0.02],[m m-direction*0.03],'linewidth',2)
plot([w(1) w(1)-0.02],[m m-direction*0.03],'linewidth',2)
th=2*pi*[0:0.05:1];  r=0.01;  xx=r*cos(th);  yy=r*sin(th);
fill(1-1/R+xx,1-1/R+yy,'b')
```

Exercise 14.4.2. The procedure is a fast way of computing square roots. In a. the equilibrium point is $x = \sqrt{2}$ and for $f(x) = (x + x/2)/2$, $f'(\sqrt{2}) = 0$. The sequence sequence converges to the square root of 2.

Exercise 14.4.3.

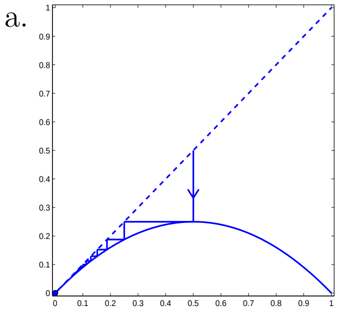


Figure 14.28: **Exercise 14.4.1.** Graphs of $F(x) = r * x * (1 - x)$ and the iterates, $x_0 = 0.5$, $x_{t+1} = F(x_t)$ for a. $r = 1.0$ The diagonal $y = x$ (dashed line) and the stable equilibrium point of F and the path of the iterates are shown.

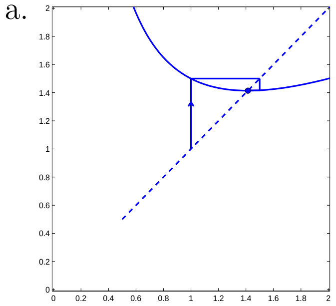


Figure 14.29: **Exercise 14.4.2.** a. Graph of $F(x) = (x + 2/x)/2$ and the iterates, $x_0 = 1.0$, $x_{t+1} = F(x_t)$. The iterates quickly converge to the square root of 2.

Exercise 14.4.7.

$a = 0.5$	$a = 0.625$	$a = 0.75$	$a = 0.875$
1.3750000000			
1.8046875000			
1.9809265136			
1.9998181010			
1.9999999834			
2.0000000000			
$1/a = 2.0$.			

Exercise 14.5.2.

a. $$\lim_{t\to\infty} \frac{e^{0.5t}}{t} = \lim_{t\to\infty} 0.5\frac{e^{0.5t}}{0.5t} = \lim_{\tau\to\infty} 0.5\frac{e^{\tau}}{\tau} = \infty.$$

c. $$\lim_{t\to\infty} \frac{e^{t}}{5t} = \infty$$

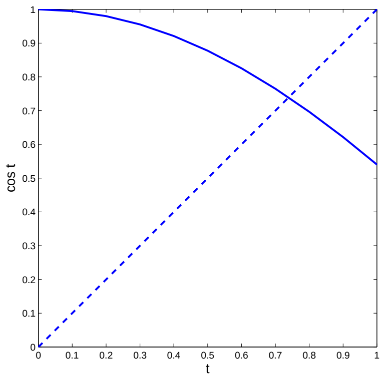
Exercise 14.5.4.

a. $$\lim_{t\to\infty} \frac{e^{t}}{t^2+15} > \lim_{t\to\infty} \frac{e^{t}}{2t^2} > \frac{1}{2}\lim_{t\to\infty} \frac{e^{t}}{t^2} = \infty$$

c. $$\lim_{t\to\infty} \frac{e^{\sqrt{t}}}{\sqrt{t}} = \infty$$

e. $$\lim_{t\to\infty} \frac{t}{2^t} = 0$$

a.

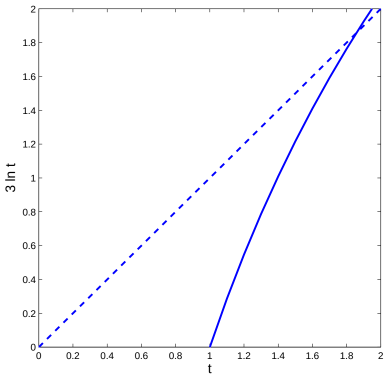


0.7500
0.7317
0.7440
0.7357
0.7413
0.7376
0.7401
0.7384
0.7395
0.7388
0.7393

Stable.

Figure 14.30: **Exercise 14.4.3** a. Graph of $y = \cos t$ and the diagonal, $y = t$ and the equilibrium near $t = 0.75$.

c.



1.8400
1.8293
1.8118
1.7830
1.7348
1.6527
1.5072
1.2308
0.6231
-1.4194
1.0506 + 9.4248i

Unstable.

Figure 14.31: **Exercise 14.4.3** c. Graph of $y = 3\ln t$ and the diagonal, $y = t$ and the equilibrium near $t = 1.84$.

Exercise 14.5.6. Consider and discuss $y = a\,b^t$ with $a = 1.445$ and $b = 1.014$ (correlation with population data 0.985); and $y = 0.306x + 0.994$ and $y = -0.0002x^2 + 0.0406x + 0.87684$. (correlations with the food data 0.9918 and 0.9956, respectively). Graphs appear in Figure 14.33.

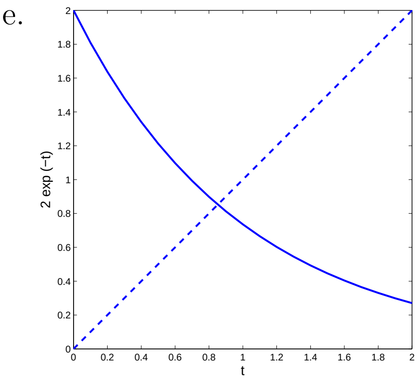
Exercise 14.5.10. Note that, and explain why

$$\frac{\ln t^2 - \ln t}{t^2 - t} = \frac{1}{c_t} \qquad \text{where} \quad t < c_t < t^2 \quad \text{for} \quad 2 < t$$

Exercise 14.5.11.

$$\text{a.} \quad \lim_{t\to\infty} \frac{\ln\sqrt{t}}{\sqrt{t}} \;=\; \lim_{\tau\to\infty} \frac{\ln\tau}{\tau} \;= 0$$

Exercise 14.5.12.



0.8500
0.8548
0.8507
0.8542
0.8512
0.8538
0.8516
0.8535
0.8519
0.8532
0.8521

Stable.

Figure 14.32: **Exercise 14.4.3** e. Graph of $y = 2\exp(-t)$ and the diagonal, $y = t$ and the equilibrium near $t = 0.85$.

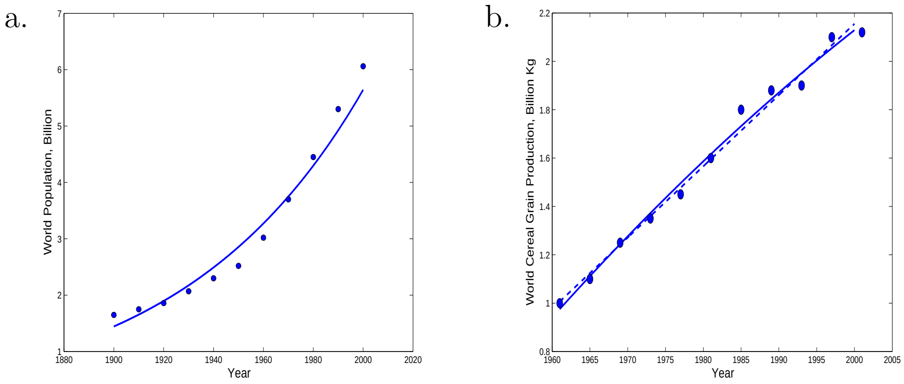


Figure 14.33: **Exercise 14.5.6.** a. World population during the 20th century. b. Cereal grain production, 1961-2001.

a. $$\lim_{t\to\infty}\frac{e^{2t}}{t} = \lim_{t\to\infty}\frac{2e^{2t}}{1} = \infty$$

c. $$\lim_{t\to\infty}\frac{2t^2+1}{5t^2+2} = \lim_{t\to\infty}\frac{4t}{10t} = \lim_{t\to\infty}\frac{4}{10} = 0.4$$

e. $$\lim_{t\to 0}\frac{\ln t}{1/t} = \lim_{t\to 0} t\ln t = \infty$$

g. $$\lim_{t\to\infty}\frac{\ln\sqrt{t}}{\sqrt{t}} = \lim_{t\to\infty}\frac{0.5\ln t}{\sqrt{t}} = \lim_{t\to\infty}\frac{0.5/t}{1/(2\sqrt{t})} = \lim_{t\to\infty}\frac{1}{\sqrt{t}} = 0$$

Exercise 14.6.1. For $P_{t+1} - P_t = 0.2P_t(1 - P_t/1000)$,

When $P_t = 900$, $P_{t+1} - P_t \doteq 0.02P_t$, or 2% growth.

Exercise 14.6.3. See Figure 14.34.

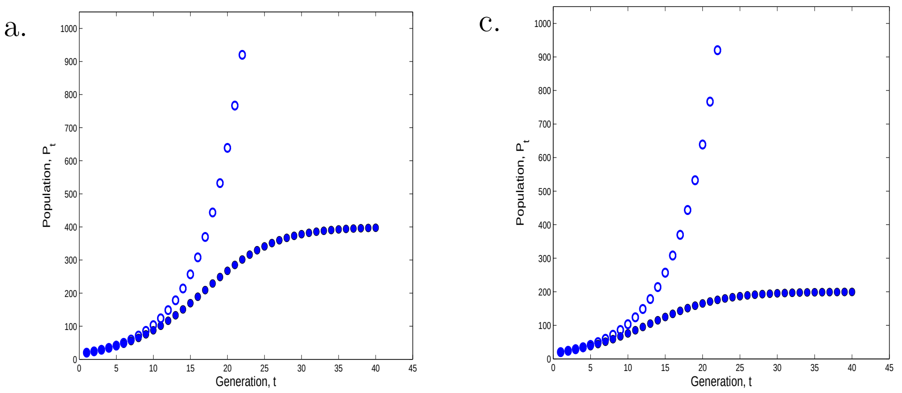


Figure 14.34: **Exercise 14.6.3.** a. Graph of the iterates of $P_0 = 20$, $P_{t+1} = P_t + 0.2P_t(1 - P_t/400)$ and $Q_0 = 20$, $Q_{t+1} = Q_t + 0.2Q_t$. c. Graph of the iterates of $P_0 = 20$, $P_{t+1} = P_t + 0.2P_t(1 - P_t/200)$ and $Q_0 = 20$, $Q_{t+1} = Q_t + 0.2Q_t$.

Exercise 14.6.4. Consider $R = 0.7464$ and $M = 1.3031$.

Exercise 14.6.5. Consider $R = 0.4863$ and $M = 19.9$.

Exercise 14.6.6. The iterates converge to an equilibrium point, 1, of the function $F(x) = x + 2\,x\,(1 - x)$.

Exercise 14.6.7. 0 is a stable equilibrium if $-2 < R < 0$. 1 is also an equilibrium and is stable if ??? If $R = 0$, every number a_e is an equilibrium. Is a_e stable?

Exercise 14.7.3.

$$\begin{aligned} F'(x) &= 1 + R - 2Rx - h \\ F'(0) &= 1 + R - h > 1, \qquad \text{0 is a nonstable equilibrium.} \\ F'(1 - h/R) &= 1 + R - 2R(1 - h/R) - h = 1 - R + h \end{aligned}$$

$$|F'(1 - h/R)| < 1 \Leftrightarrow |1 - R + h| < 1 \Leftrightarrow 0 < R - h < 2$$

Because $h < R$, $0 < R - h$. Because $R < 2$, $R - h < 2$. $1 - h/R$ is a stable equilibrium.

Exercise 14.7.4. The population size at the equilibrium, $p_e = 1 - h/R = 1/2$ is $(1/2)M$, or one-half of the maximum supportable population.

Exercise 14.7.6.

a.

$$F(x) = x + R\,x\,(1 - x) - K, \qquad F(x) = x \Leftrightarrow x\,(1 - x) = K/R \Leftrightarrow x^2 - x + K/R = 0.$$

$$p_{*1} = \frac{1 + \sqrt{1 - 4K/R}}{2} \quad \text{and} \quad p_{*2} = \frac{1 - \sqrt{1 - 4K/R}}{2}$$

b. If $4K/R > 1$, or $K > R/4$, both roots are imaginary and there is no equilibrium. That is, if the harvest is more than one-fourth of the reproduction rate, the population will be lost. See Figure 14.35. For $K = R/4$, $p_{*1} = p_{*2} = 1/2$ and this equilibrium is not stable "from the left".

Exercise 14.7.7. Assume that $K < 4R$ and $R < 2$

$$\begin{aligned} F'(x) &= 1 + R - 2Rx \\ F'\left(\frac{1+\sqrt{1-4K/R}}{2}\right) &= 1 + R - 2R\frac{1+\sqrt{1-4K/R}}{2} \\ &= 1 - R\sqrt{1-4K/R} \quad < 1, \quad \text{and because } R < 2 \quad > -1. \\ F'\left(\frac{1-\sqrt{1-4K/R}}{2}\right) &= \text{Finish} \end{aligned}$$

Thus $\dfrac{1+\sqrt{1-4K/R}}{2}$ is stable and $\dfrac{1-\sqrt{1-4K/R}}{2}$ is not stable.

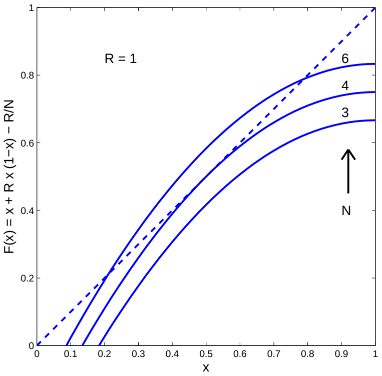
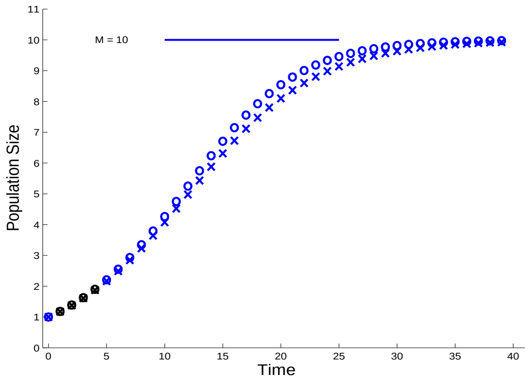


Figure 14.35: **Exercise 14.7.6.** $F(x) = x + R\,x\,(1-x) - K$ for $R = 1$ and $K = R/3$, $R/4$, and $R/6$.

Exercise 14.8.2.

The missing data are in bold face and are plotted in black.

t	Eq 14.45	Eq 14.46	Eq 14.47
0	1	1	1
1	**1.1800**	**1.7765**	**1.7765**
2	**1.3882**	**1.3793**	**1.3793**
3	**1.6272**	**1.6107**	**1.6107**
4	**1.8997**	**1.8726**	**1.8726**
5	2.2075	2.1660	2.1660
⋮	⋮	⋮	⋮



Exercise 14.8.3. See Figure 14.36.

a. $M = 400$, $P_{10}/M = 0.60$, and $P_{20}/M = 0.98$.

c. $M = 400$, $P_{10}/M = 0.12$, and $P_{20}/M = 0.26$.

e. $M = 100$, $P_{10}/M = 0.61$, and $P_{20}/M = 0.91$.

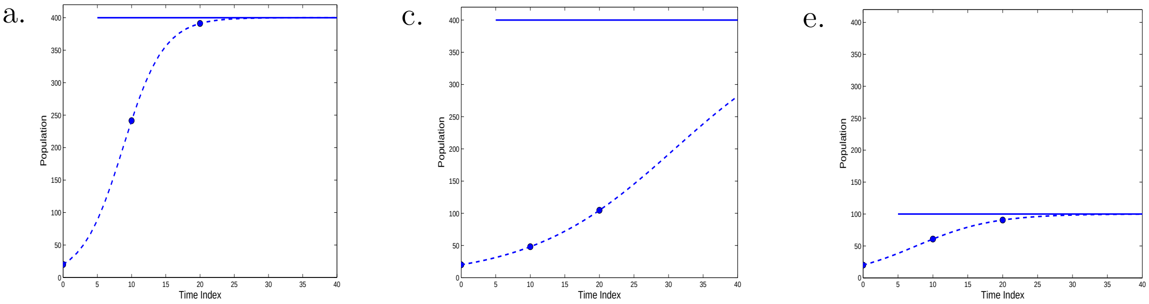


Figure 14.36: **Exercise 14.8.3.** a, c, and e. $(0, P_0)$, $(10, P_{10})$ and $(20, P_{20})$ are plotted and the actual graph is shown in dashed curve.

Exercise 14.8.4. The reciprocals of the data are added to the tables and a plots of reciprocals appear in Figures 14.37A and 14.38B.

$$\frac{1}{P_{t+1}} \quad = \quad \frac{1}{1+R} \times \frac{1}{P_t} + \frac{1}{M}\frac{R}{1+R}$$

A. An equation of a line fit to the data is $y = 0.1731 + 0.5826x$. Thus $1/(1+R) = 0.5826$, $R = 0.72$, $\frac{1}{M}\frac{R}{1+R} = 0.1731$, $M = (R/(1+R))/0.1713) = 2.42$. The data are plotted along with the values of P_t computed from

$$P_0 = 0.028, \qquad P_{t+1} = \frac{1.72P_t}{1 + 0.72P_t/2.42}, \qquad t = 1, \cdots, 10.$$

The value of M is clearly too large.

B. An equation of a line fit to the data is $y = 0.1134 + 0.5305x$. Then $R = 0.88$ and $M = 4.14$. Neither R nor M is suitable. Some improvement is observed by deleting the first data point. Then $y = 0.0196 + 0.6967x$, $R = 0.44$ and $M = 15.5$.

pH 7.85				Crow embryo mass			
Time (min)	Index t	Population Density	Reciprocals	Time (day)	Index t	Embryo mass (gm)	Reciprocals
0	0	0.028	35.7	3	0	0.220	4.55
16	1	0.047	21.3	4	1	0.435	2.30
32	2	0.082	12.2	5	2	0.650	1.54
48	3	0.141	7.09	6	3	0.800	1.25
64	4	0.240	4.17	7	4	1.200	0.83
80	5	0.381	2.62	8	5	1.630	0.61
96	6	0.576	1.74	9	6	2.150	0.47
112	7	0.815	1.23	10	7	2.800	0.36
128	8	1.054	0.95	11	8	3.700	0.27
144	9	1.206	0.83	12	9	5.100	0.20
160	10	1.266	0.79	13	10	7.500	0.13
				14	11	10.200	0.098
				15	12	12.600	0.079
				16	13	14.850	0.067
				17	14	16.00	0.063

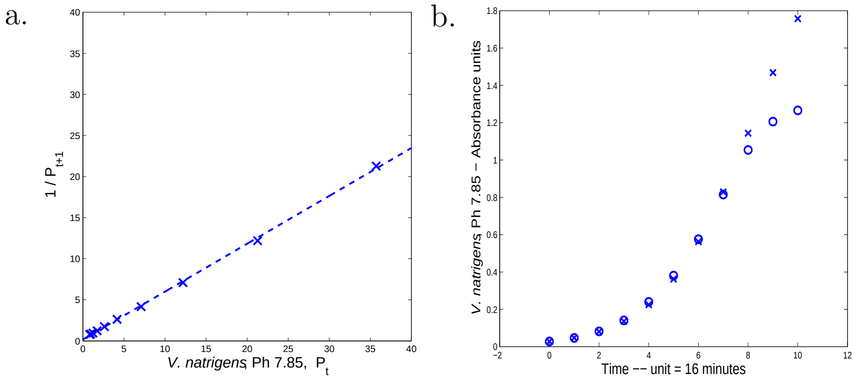


Figure 14.37: **Exercise 14.8.4.** A. Graph of the reciprocals of the population densities for *V. natriegens* grown at pH=7.85. B. Comparison of the original data (open circles) and the values computed from Equation 14.43 with $R = 0.72$ and $M = 2.42$.

Exercise 14.8.5.

$$\begin{aligned}
P_{t+1} &- \frac{(1+R)\times P_t}{1+R\frac{P_t}{M}} \\
\frac{1}{P_{t+1}} &= \frac{1+R\frac{P_t}{M}}{(1+R)\;P_t} \\
\frac{1}{P_{t+1}} &= \frac{1}{1+R}\,\frac{1}{P_t} + \frac{R}{M(1+R)} \\
Q_{t+1} &= \rho\, Q_t + \beta, \qquad Q_t = \frac{1}{P_t}, \quad \rho = \frac{1}{1+R}, \quad \beta = \frac{R}{M(1+R)}
\end{aligned}$$

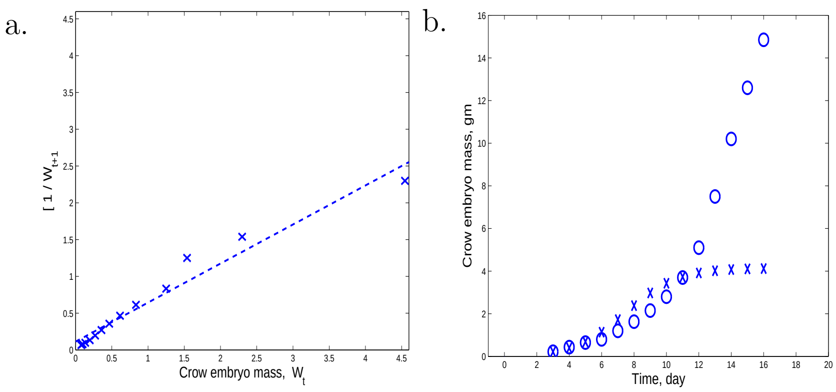


Figure 14.38: **Exercise 14.8.4.** A. Graph of the reciprocals of the crow embryo masses. B. Comparison of the original data (open circles) and the values computed from Equation 14.43 with $R = 0.88$ and $M = 4.14$.

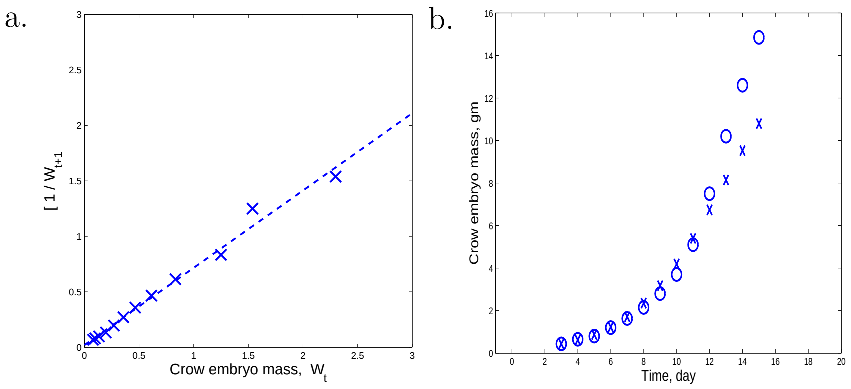


Figure 14.39: **Exercise 14.8.4.** The same as Figure 14.38 with the first point deleted. A. Graph of the reciprocals of the crow embryo masses. B. Comparison of the original data (open circles) and the values computed from Equation 14.43 with $R = 0.44$ and $M = 15.5$.

$$\begin{aligned}
Q_t &= \frac{\beta}{1-\rho} + \rho^t\left(Q_0 - \frac{\beta}{1-\rho}\right) \\
\frac{1}{P_t} &= \frac{1}{M} + (1+R)^{-t}\,(1/P_0 - 1/M) \\
P_t &= \frac{1}{1/M + (1+R)^{-t}(1/P_0 - 1/M)} \\
&= \frac{M\ P_0}{P_0 + (1+R)^{-t}(M - P_0)}
\end{aligned}$$

Chapter 15

Discrete Dynamical Systems

Where are we going?

The first order difference equation models of bacterial growth and penicillin clearance introduced in Chapter **??** are refined to account for additional factors (multiple population categories and absorbance of penicillin into tissue) that allow more accurate matches of models to data. These and other models motivate systems of difference equations that are referred to as discrete dynamical systems. The important biological and mathematical concepts of equilibrium point and stable and unstable equilibria are extended to equilibria of discrete dynamical systems.

15.1 Infectious diseases: The SIR model.

In this section you will develop a model of an infectious disease that is spread directly from human to human (influenza, polio, measles, for examples) and that confers immunity to a person who has had the disease. It does not include diseases carried by mosquitoes or mice or other 'vectors'. The model also assumes a reasonably 'closed' human population in that it does not account for disease being brought into the population from people not included in it, except for the initial source of the disease.

Suppose there are 25,000 students and faculty in a 'closed' university population, and that at the beginning of a semester 50 of the people return to school infected with a certain influenza. Assume that infected people can transmit the disease (are infectious) during a period of 5 days beginning with the day they are first infected.

At any time university people can be classified as

Susceptible. Those people who have not had this influenza and are susceptible to infection.

Infectious. Those people who are infected and are contagious.

Recovered. Those people who have had the flu, are recovered (no longer contagious), and are immune to the disease.

To understand the categories, examine Table 15.1, where it is assumed that the 50 infected people are equally distributed between being infected 4, 3, 2, 1 and 0 days before arriving on campus and 20 new cases are recorded each morning at, say, 6am.

Initial days of an influenza outbreak.
Note that Susceptible, Infectious, and Recovered total to 25,000.

Day	Susceptible	Newly Infected	Infectious	Newly Recovered	Recovered
-4		10	10		
-3		10	20		
-2		10	30		
-1		10	40		
0	24950	10	50		
1	24930	20	60	10	10
2	24910	20	70	10	20
3	24890	20	80	10	30
4	24870	20	____	____	____
5	24850	20	____	____	____
6	24830	20	____	____	____
7	24810	20	100	20	90

The assumption of a fixed number (20) of new cases per day is too restrictive. As the number of infectious people increases, the disease will spread more rapidly. After a large number of people have been infected and recovered, the susceptible population is small and the spread of the disease will slow down. An elementary model of the spread of disease is:

Mathematical Model 15.1.1 Influenza transmission: Influenza is transmitted by an 'infectious contact' between an infectious individual and a susceptible individual. The frequency of such contacts is proportional to the number of infectious people and to the number of susceptible people in the population.

The number of newly infected is proportional to the infectious contact frequency.

The duration of the disease for all individuals is a fixed time period and all people recover.

As usual, for the frequency of contact to be proportional to two quantities we write that the frequency is a constant times the product of the two things:

$$\text{Infectious Contact Frequency} = \beta_0 \times \text{Number of Susceptible} \times \text{Number of Infectious}$$

The number of newly infected is proportional to the infectious contact frequency and we write

$$\text{Number of Newly Infected} = \beta \times \text{Number of Susceptible} \times \text{Number of Infectious} \tag{15.1}$$

The number β called the *transmissibility* incorporates both the biological property of how easily the disease is transferred from one person to another and the sociological property of the frequency and nature of such contacts. Measles, for example, is highly contagious (a child with measles will expose his or her entire class in about 30 minutes in a class room). Measles-β is 'high' and is determined primarily by biological properties. HIV transmission requires an exchange of body fluids; HIV-β is 'low' and is determined largely by sociological properties.

The following notation is useful. Suppose time is measured in periods of hours, days, or weeks and t is the index of time periods. Let

S_t = Number of susceptible people at the beginning of time period t.
I_t = Number of infected people at the beginning of time period t.
R_t =Number of people who have recovered at the beginning of time period t.

R_t includes all people who have been infected and have recovered during any previous time period.

Suppose furthermore that the duration of the disease is d time periods for all people. You are asked to write and analyze equations that model influenza dynamics in the following exercises.

Exercises to complete the analysis of influenza dynamics.

Exercise 15.1.1 Fill in the blanks in Table 15.1 and see that they are consistent with the last line.

Exercise 15.1.2 Write an alternate version of Table 15.1 in which the initial 50 infectious people are all assumed to be newly infected (infected on day zero).

Exercise 15.1.3 a. Write equations for Mathematical Model 15.1.1

$$\begin{aligned} S_{t+1} - S_t &= \\ I_{t+1} - I_t &= \\ R_{t+1} - R_t &= \end{aligned}$$

in terms of S_t, I_t, R_t, and β, a constant of proportionality and d the number of time periods an infected person remains infected. It may help you to think of categories, NI_t of 'newly infected' and NR_t of 'newly recovered' at time t.

b. Convert your equations to iteration form with terms involving S_t, I_t, and R_t on the RHS. They should be equivalent to

$$\begin{aligned} S_{t+1} &= S_t - \beta\, S_t\, I_t \\ I_{t+1} &= I_t + \beta\, S_t\, I_t - \beta\, S_{t-d}\, I_{t-d} \\ R_{t+1} &= R_t + \beta\, S_{t-d}\, I_{t-d} \end{aligned} \tag{15.2}$$

Exercise 15.1.4

a. Explain the term $\beta\, S_{t-d}\, I_{t-d}$ in the Equations 15.2.

b. Explain why

$$\beta\, S_{t-d+1}\, I_{t-d+1} + \cdots + \beta\, S_{t-1}\, I_{t-1} + \beta\, S_t\, I_t = I_t.$$

Use this to rationalize

$$\beta\, S_{t-d}\, I_{t-d} \doteq \frac{1}{d} I_t \tag{15.3}$$

as an acceptable approximation.

We use the approximation 15.3 in Equations 15.2 and write with $\gamma = 1/d$

$$\begin{array}{rcl} S_{t+1} & = & S_t - \beta\, S_t\, I_t \\ I_{t+1} & = & I_t + \beta\, S_t\, I_t - \gamma I_t \\ R_{t+1} & = & R_t + \gamma I_t \end{array} \tag{15.4}$$

A computer or programmable calculator is helpful for the remaining problems.

Exercise 15.1.5 Suppose that the proportionality constant, $\beta = 0.00003$, and that there is a reasonably closed university population of 25,000 people and that 50 return to university at the beginning of a semester infected with an influenza. Use Equation 15.4 to compute S_t, I_t, and R_t for $t = 1, 2, 3$. Assume that infected people remain contagious for 5 days, so that $\gamma = 1/5 = 0.2$ of the infected people recover each day.

Exercise 15.1.6 Repeat the calculations of Exercise 15.1.5, except assume that only one person returns to campus infected with influenza. Use $\beta = 0.00003$ and $\gamma = 0.2$.

Exercise 15.1.7 Repeat the calculations of Exercise 15.1.6 (only one infected person returns to campus and $\gamma = 0.2$) except assume the proportionality constant, $\beta = 0.000005$. Compute S_t, I_t, and R_t for $t = 1, 2, 3$.

Exercise 15.1.8 There is an important qualitative difference in the results of Exercises 15.1.6 and 15.1.7. In Exercise 15.1.6 with $\beta = 0.00003$, $I_1 = 1.55 > I_0 = 1$, the number of infected *increases* and there will be a flu 'epidemic'. In Exercise 15.1.7 with $\beta = 0.000005$, $I_1 = 0.925 < I_0 = 1$, the number of infected *decreases* on the first step and the influenza will quickly die out. Repeat the calculations of Exercise 15.1.7 with various values of β to find the largest number, β_0, such that if $\beta = \beta_0$ in the previous model, $I_1 \leq I_0$.

Exercise 15.1.9 a. Solve for β in

$$\beta \times S_0 \times I_0 - \gamma I_0 = 0.$$

b. Why?

c. Show that if $\dfrac{S_0\beta}{\gamma} > 1$, then $I_1 > I_0$ and if $\dfrac{S_0\beta}{\gamma} < 1$, then $I_1 < I_0$.

d. Solve for β in $\dfrac{S_0\beta}{\gamma} = 1$ with $S_0 = 25,000$ and $\gamma = 0.2$ and compare with value for β_0 in Exercise 15.1.8.

The number $\dfrac{S_0\beta}{\gamma}$ is a commonly used measure of whether there will be an epidemic.

Exercise 15.1.10 Suppose at the beginning of the study there are 24,950 susceptible people, 50 infected people, and no recovered/immune people, and let $\beta = 0.00002$ and $\gamma = 0.2$. Compute the values of S_t, I_t, R_t for 70 days.

a. When is the epidemic at its height?

b. Do all of the people get sick?

c. Repeat the computation for $\beta = 0.00006$ and only 12 days. You will find that all of the people will get sick. What is the least value of β for which all people get the flu?

The event that everyone gets sick is a property of our discrete model, and perhaps a peculiar property. You will show in Exercise 18.5.11 that in the continuous model analogous to Equations 15.4 it never happens that everyone gets sick.

Exercise 15.1.11 a. What are the dimensions of the constants β and γ in Equations 15.4?

b. Convert Equations 15.4 to variables, x, y, and z, that are fractions of the whole population? The constants β and γ have to change, to, say, β^* and γ^*. What are β^*? and γ^*

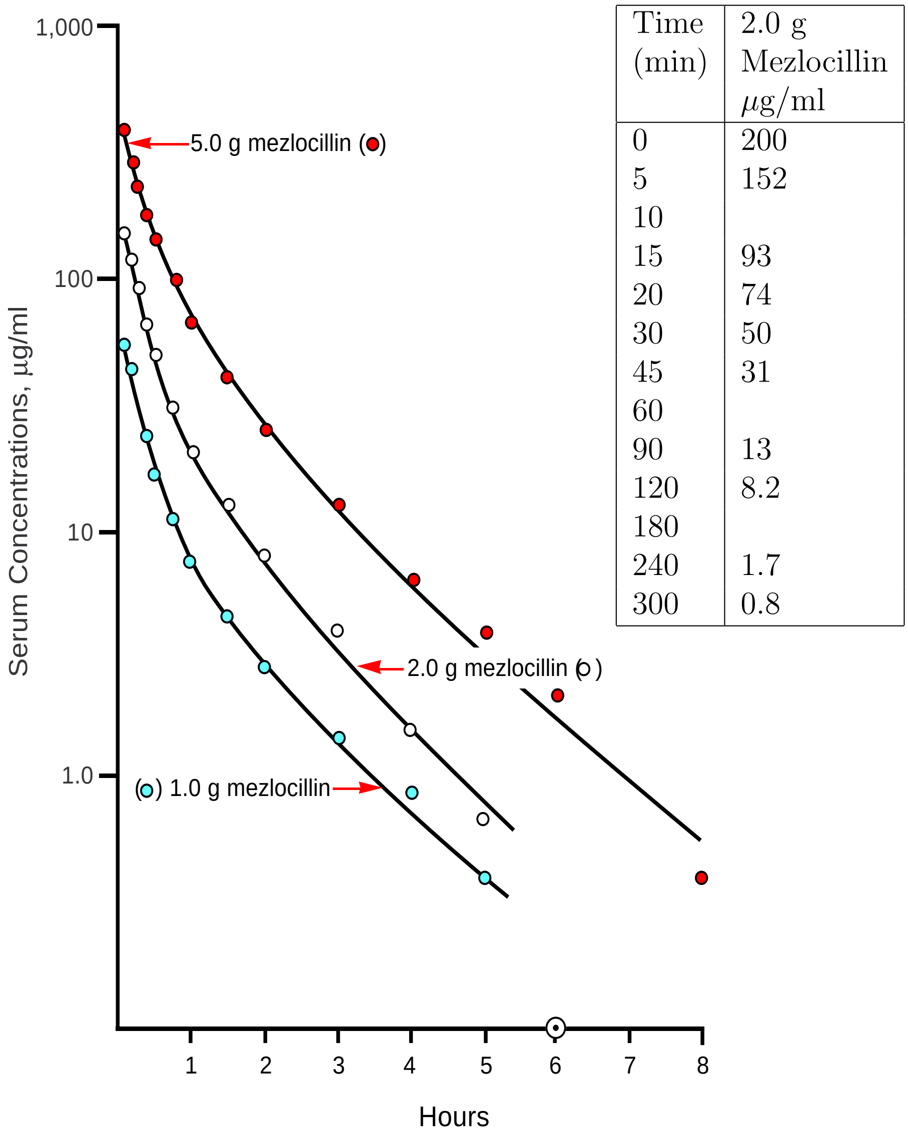
15.2 Pharmacokinetics of Penicillin

The movement of penicillin in the body is considerably more complex than has been suggested earlier by the simple elimination of penicillin by the kidney. We present here some of the data from T. Bergans, Penicillins, in *Antibiotics and Chemotherapy, Vol. 25*, H. Schøonfeld, Ed., S. Karger, Basel, New York, 1978, and suggest some of the improvements in the model of penicillin kinetics that are suggested by the data.

Figure 15.1 shows simple clearance of mezlocillin from serum after intravenous administration of a 1, 2, or 5 g dose during a 5 minute period. This is referred to as a 'bolus' dose, and is imagined as an instantaneous administration of the drug. We will focus attention on the 2 g data. The data appears to have been taken 5, 10, 15, 20, 30, 45, 60, 120, 180, 240 and 300 minutes after administration was completed. The table to the right of the graphs contains numbers read from the 2 g dose graph.

Explore 15.2.1 Three data points, corresponding to times $t = 10$, $t = 60$, and $t = 180$ minutes are omitted from the table. Read approximate values for these data from the graph of 2 g data and complete the table. ■

Observe that the vertical scale in Figure 15.1 is measured in logarithms to the base 10 of μg/ml. It appears that at time $t = 0$ the data would be about 200 μg/ml. Assume a serum volume of 2.75 liters (Rhoades and Tanner, page 210, $2.75 = 0.55 \times 5.0$). Administration of 2 gm yields an initial value of 727 μg/ml ($= 2 \times 10^6/(2.75 \times 10^3)$). However, the dose was administered over a 5 minute interval; if we assume that 4% is lost each minute, one has 644 μg/ml at the end of the administration period. This is quite high compared to the $t = 0$ data point which we estimated to be about 200 μg/ml. It appears that



Time (min)	2.0 g Mezlocillin μg/ml
0	200
5	152
10	
15	93
20	74
30	50
45	31
60	
90	13
120	8.2
180	
240	1.7
300	0.8

Figure 15.1: Clearance of penicillin from serum, Figure 29, page 92 of Tom Bergans, Penicillins, in H. Schönfeld Ed. Antibiotics and Chemotherapy, V. 25, Pharmacokinetics, S Karger, Basel, 1978.

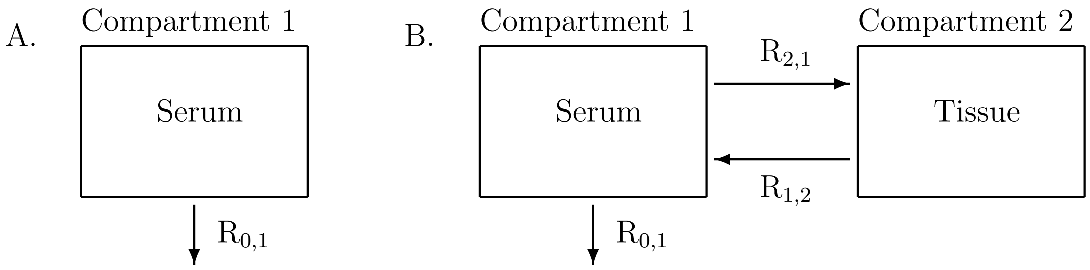


Figure 15.2: Schematic diagrams of one- and two-compartment physiological models.

the penicillin equilibrates quickly in a pool of larger volume, possibly including the serum, the blood cells, and part of the interstitial fluid.

We wish to write a mathematical model descriptive of penicillin kinetics. In Section **??** we suggested that penicillin clearance would be 4% per minute, or approximately 20% per five minutes. If so, by the methods of that section we would conclude that $P_t = 200 \times 0.8^t$ and we would conclude that the graph of P_t versus t would be a straight line on semilog paper. See Exercise 15.2.2.

The graphs of the observed data in Figure 15.1 are not straight lines. The departure from straight lines is due to the attachment of mezlocillin to proteins in the vascular pool and migration of mezlocillin into tissue, followed by return of the penicillin to the vascular pool. Attachment to proteins and migration to tissue both remove penicillin from the vascular pool and have similar effects on the model, although the flow rates will be different. To simplify the model the two are combined into only migration into tissue.

Schematic diagrams that enable discussion of the kinetics are commonly used to depict pharmacokinetic models. Shown in Figure 15.2 are models showing (A) simple elimination from the serum and (B) migration to and from tissue and elimination from the serum pool. The symbols, $\mathrm{R}_{i,j}$, represent flow rates between compartments of the substance of interest (penicillin, for example). By well established convention, $\mathrm{R}_{i,j}$ represents flow to compartment i from compartment j, and compartment 0 is “outside” the system.

The difference between one and two compartment models can be explored using some very simple concepts.

Explore 15.2.2 We do not have data on the mesocilin concentration in the tissue compartment. What is the tissue concentration at time $t = 0$? Sketch a possible graph of that concentration. Do you expect that concentration to exceed the serum mesocilin concentration at any time?

Kinetics of a one-pool model A flask (Serum) has 5 liters of pure water and 200 grams of salt. At the end of each five minute interval, 1 liter (20%) of the liquid is removed from the flask, after which one liter of pure water is added and the flask is well stirred.

Let P_t denote the amount of salt in the flask at time t (at the end of the $t\underline{th}$ five minute time interval and at the beginning of the $t + 1\underline{st}$ time interval. We can write

$$\begin{array}{rcll} P_0 & = & 200 & \text{200 grams initially in the flask.} \\ P_{t+1} - P_t & = & -0.2 * P_t & \text{Remove 20\% each time period.} \\ P_{t+1} & = & 0.8 \times P_t & \text{Iteration equation.} \\ P_t & = & 200 \times 0.8^t & \text{Solution to the iteration equation.} \end{array}$$

Kinetics of a two-pool model. An example of the two-pool model shown in Figure 15.2 is analyzed as follows. Suppose there are two flasks, flask 1 (Serum) contains 1 liter of water and 200 milligrams (mg) of salt and flask 2 (Tissue) contains 0.68 liters of water. (The numbers have been selected to give an answer similar to the data of Figure 15.1 and expressed in rational numbers). Note that the concentration of salt in flask 1 (Serum) is 200 mg/liter = 200 μg/ml.) At the end of each five minute interval, 68 ml of liquid is removed from flask 2 (10%) and temporarily set aside, 68 ml of liquid is removed from flask 1 (6.8%) and added to flask 2 and 162 ml of liquid is removed from flask 1 (16.2%) and discarded. Then the 68 ml removed from flask 2 is added to flask 1, and 162 ml of pure water is added to flask 1.

Let A_t and B_t denote the amount of salt measured in milligrams in flask 1 and flask 2, respectively, at the end of the $t\underline{th}$ 5 minute time interval and at the beginning of the $t + 1\underline{st}$ time interval. The value of $R_{1,2}$, for example, in the two-pool model diagram is either 68 ml per five minutes or the fractional value 0.1 (10% of the pool) per five minutes. Both forms are used in the pharmacokinetics literature; we use the fractional value.

The value of A_0 is 200 and the value of B_0 is 0. The changes in the amount of salt in the two flasks may be accounted for by:

$$\begin{array}{rcccc} \text{Change} & & \text{Increase} & \text{Decrease} & \text{Units} \\ A_{t+1} - A_t & = & 0.1 B_t & (-0.162 - 0.068) A_t & \text{mg} \\ B_{t+1} - B_t & = & 0.068 A_t & -0.1 B_t & \text{mg} \end{array} \qquad (15.5)$$

The initial conditions and difference equations lead to the iteration equations

$$\begin{array}{llcl} A_0 = 200 & A_{t+1} & = & 0.77 \times A_t + 0.10 \times B_t \\ B_0 = 0 & B_{t+1} & = & 0.068 \times A_t + 0.90 \times B_t \end{array} \qquad (15.6)$$

Note: A_t and B_t represent the amounts of salt in their respective flasks. The concentrations of salt in their respective flasks would be $A_t/1$ and $B_t/0.68$ grams per liter.

Reader Beware: Incoming Twin Bolts of Lightning. The equations

$$\begin{array}{rcl} A_t & = & \frac{200}{2.1} \times (0.4 \times 0.94^t + 1.7 \times 0.73^t) \\ B_t & = & \frac{200}{2.1} \times (0.68 \times 0.94^t - 0.68 \times 0.73^t) \end{array} \qquad (15.7)$$

exactly solve the Equations 15.6. Equations 15.7 are easier to use and give us a picture of the solution more quickly than the iteration Equations 15.6. You will see how Equations 15.7 are obtained in Section 15.4.

We show that the first iteration equation is satisfied by the proposed solution equations and leave for you to show in Exercise 15.2.8 that the second iteration equation is satisfied. First observe that

$$A_0 = \frac{200}{2.1} \times \left(0.4 \times 0.94^0 + 1.7 \times 0.73^0\right) = \frac{200}{2.1} \times (0.4 \times 1 + 1.7 \times 1) = \frac{200}{2.1} \times 2.1 = 200$$

so that **the initial value of $\mathbf{A}_t$ is satisfied**. Next we substitute the two expressions for A_t and B_t into the right hand side of the equation for A_{t+1}.

$$\begin{aligned}
A_{t+1} &= 0.77 \times A_t + 0.10 B_t \\
&= 0.77 \times \left[\frac{200}{2.1} \times \left(0.4 \times 0.94^t + 1.7 \times 0.73^t\right)\right] \\
&\qquad\qquad +0.10 \times \left[\frac{200}{2.1} \times \left(0.68 \times 0.94^t - 0.68 \times 0.73^t\right)\right] \\
&= \frac{200}{2.1}\left[(0.77 \times 0.4 + 0.1 \times 0.68)0.94^t + (0.77 \times 1.7 - 0.1 \times 0.68)0.73^t\right] \\
&= \frac{200}{2.1}\left[\frac{0.77 \times 0.4 + 0.1 \times 0.68}{0.94}0.94^{t+1} + \frac{0.77 \times 1.7 - 0.1 \times 0.68}{0.73}0.73^{t+1}\right] \\
&= \frac{200}{2.1}\left[0.4 \times 0.94^{t+1} + 1.7 \times 0.73^{t+1}\right] \qquad = \quad A_{t+1}
\end{aligned}$$

Whew!! The first iteration equation is satisfied.

Exercises for Section 15.2, Pharmacokinetics of Penicillin

Exercise 15.2.1 Assuming that immediately after a 2 gm injection of penicillin into the vascular pool, the penicillin concentration throughout the pool is 200 μg/ml, what is the estimated volume of the vascular pool?

How does this estimate compare with the blood volume of an adult of about 5 liters and a serum volume of 2.75 liters?

Exercise 15.2.2 a. Suppose 2 gm of mezlocillin are injected (bolus injection) into the serum of a patient with the result that at time $t = 0$ the serum concentration of mezlocillin is 200 μg/ml. Assume for this problem that the mezlocillin stays in the serum until removed by the kidneys and that the kidneys remove 20% of the mezlocillin from the serum every five minutes. Let P_t denote the concentration of mezlocillin in the serum at the end of the t^{th} 5 minute time interval. Write equations for

$$\begin{aligned} P_{t+1} - P_t &= \underline{\hspace{5cm}} & \\ P_{t+1} &= \underline{\hspace{5cm}} & \text{in terms of } P_t \\ P_t &= \underline{\hspace{5cm}} & \text{in terms of } t. \end{aligned}$$

b. Compute the mezlocillin levels predicted by $P_t = 0.8^t \times 200$ where t marks 5 minute intervals for the times, $t = 0$, $t = 1$, $\cdots$ $t = 24$ and plot the data along with the original data in Figure 15.1 on a semilog graph.

c. Explain why the graph of P_t versus t is a straight line on the semilog graph.

Exercise 15.2.3 Compute A_1 and B_1, A_2 and B_2, A_3 and B_3 Using the Iteration Equations 15.6.

Exercise 15.2.4 Compute A_1, A_2 and A_3 using the Solution Equations 15.7. You should get the same answers as for A_1, A_2 and A_3 in Exercise 15.2.3.

Exercise 15.2.5 Compute A_{12} and B_{12} using either the Difference Equations 15.5 or the Iteration Equations 15.6 or the Solution Equations 15.7 (your choice). Compare A_{12} with the observed value of penicillin concentration at 60 minutes in Figure 15.1.

Exercise 15.2.6 Compute A_{24} and B_{24} using either the Difference Equations 15.5 or the Iteration Equations 15.6 or the Solution Equations 15.7 (your choice). Compare A_{24} with the observed value of penicillin concentration at 120 minutes in Figure 15.1.

Exercise 15.2.7 Explain the reasons for the terms in Equations 15.5 that account for Change, Increase, and Decrease of A_t and of B_t.

Exercise 15.2.8 Show that the proposed solutions, Equations 15.7, also satisfy $B_0 = 0$ and the second of Equations 15.6. Do exact arithmetic.

Exercise 15.2.9 .

Let AC_t and BC_t denote salt concentrations in the Serum and Tissue flasks, respectively. Then (note: milligrams/liter = μg/ml).

$$AC_t = A_t/1.0 \quad \mu\text{g/ml, and } BC_t = B_t/0.68 \quad \mu\text{g/ml.}$$

a. Compute A_t and B_t from Equations 15.7 for the times shown in Figure 15.1 and plot the data from Figure 15.1 and AC_t and BC_t on a semilog graph.

b. How do the values of AC_t compare with with the observed concentration of penicillin?

c. Except that they are discrete points, the two graphs for AC_t and BC_t cross. One might guess that the tissue concentration maximum occurs where the tissue concentration is the same as the serum concentration. Examine the graphs and determine whether this appears to be true.

d. Use $B_{t+1} - B_t = 0.068A_t - 0.1B_t$ to determine whether the maximum tissue concentration occurs when $AC_t = BC_t$.

Exercise 15.2.10 Residence time is a measure of the average amount of time a chemical or biological agent spends in a well defined volume. You may read of the residence time of, for example, greenhouse gases in the atmosphere, salt in an estuary, water in the atmosphere or in a lake or in ground water, or even vegetables in the market. In the one compartment example where 200 gm of salt are dissolved (into Na^+ and Cl^- atoms) in 5 liters of water and 20% of the solution is replaced every five minutes with pure water, what is the average time that the Na^+ atoms spend in the flask? This requires a bit of warmup.

a. Compute the average age of your class if there are 22 students who are 18 years old, 16 students who are 19 years old, and 6 students who are 20 years old.

b. The 'age' of the Na^+ atoms removed at the first replacement is 5 minutes and 20% of the total, T, are removed and 80% of the total remain in the flask. Argue that the average age, $\overline{A}_3$, of the Na^+ removed during the first three replacements is

$$\overline{A}_3 = \frac{0.2T \cdot 5 + 0.2 \cdot 0.8T \cdot 10 + 0.2 \cdot 0.8^2T \cdot 15}{0.2T + 0.2 \cdot 0.8T + 0.2 \cdot 0.8^2T}$$

$$\overline{A}_3 = 5\frac{1 + 2 \cdot 0.8 + 3 \cdot 0.8^2}{1 + 0.8 + 0.8^2}$$

c. Write an expression for the average age, $\overline{A}_{20}$, of the Na^+ removed during the first twenty replacements.

d. $\overline{A}_{20}$ and even $\overline{A}_{2000}$ is easily evaluated using a computer or calculator program (or even a lot of patience), but the old guys developed some clever methods.

From Equation 3.1 of Chapter 3, we can write

$$1 - x^{n+1} = (1-x)(1 + x + x^2 + \cdots x^{n-1} + x^n),$$

$$\text{and define} \qquad P_n(x) = 1 + x + x^2 + \cdots x^{n-1} + x^n = \frac{1 - x^{n+1}}{1-x}.$$

You can immediately see that the denominator of $\overline{A}_{20}$ is

$$P_{19}(0.8) = \frac{1 - 0.8^{20}}{1 - 0.8} \doteq 4.92424$$

and is age easily computed. It gets better. Show that the numerator of $\overline{A}_{20}$ is $P'_{20}(x)|_{x=0.8}$.

e. Think out of the box a minute. Imagine that

$$P_\infty(x) = 1 + x + x^2 + \cdots = \lim_{n\to\infty} \frac{1-x^n}{1-x} = \frac{1}{1-x} \quad \text{and} \quad P_\infty'(x) = \frac{1}{(1-x)^2},$$

and that

$$\overline{A}_\infty = 5\frac{P_\infty'(x)}{P_\infty(x)}\bigg|_{x=0.8} = 5\frac{\dfrac{1}{(1-x)^2}}{\dfrac{1}{1-x}}\Bigg|_{x=0.8} = 25.$$

Thus the average residence time of all Na^+ atoms is $25 \times 5 = 125$ minutes.

The **box** follows.

f. Compute

$$P_{n+1}'(x) = \left[\frac{1-x^{n+2}}{1-x}\right]'$$

g. Use L'Hospital's Rule, Theorem 14.5.1, to show that if $0 < a < 1$ then

$$\lim_{t\to\infty} t\,a^t = \lim_{t\to\infty} \frac{t}{(1/a)^t} = 0.$$

h. Suppose $0 < x < 1$ and show that

$$\lim_{n\to\infty} 5\frac{\dfrac{1-(n+2)x^{n+1}+(n+1)x^{n+2}}{(1-x)^2}}{\dfrac{1-x^{n+1}}{1-x}} = \frac{5}{1-x}.$$

15.3 Continuous infusion and oral administrations of penicillin.

Shown in Figures 15.3 and 15.4 are two more graphs from T. Bergans, *ibid.* Figure 15.3 reports the serum concentration during constant infusion of carbenicillin into the serum pool; a diagram that can be used to model the process is included. The diagram has only a serum pool, a source from outside (infusion) leading into the serum pool and an exit from the pool (kidneys). A tissue compartment (dashed box) could be added to the diagram. As in the bolus injection model of the previous section, continuous infusion can be modeled in terms of salt in flasks, a task left for you in Exercise 15.3.3.

Analysis of oral ingestion of ampicillin. We will analyze the data for oral ingestion of ampicillin shown in Figure 15.4. We try to model the dark line in Figure 15.4. Estimates of serum concentrations of ampicillin read from the graph are shown in Table 15.1.

It appears that there are at most six data points represented in the graph of Figure 15.4, and the two compartment model may be as 'rich' as can be justified. We suggest, however, a three compartment model for you to try in Exercise 15.3.5.

Two compartment model. Again we write in terms of water and salt in flasks.

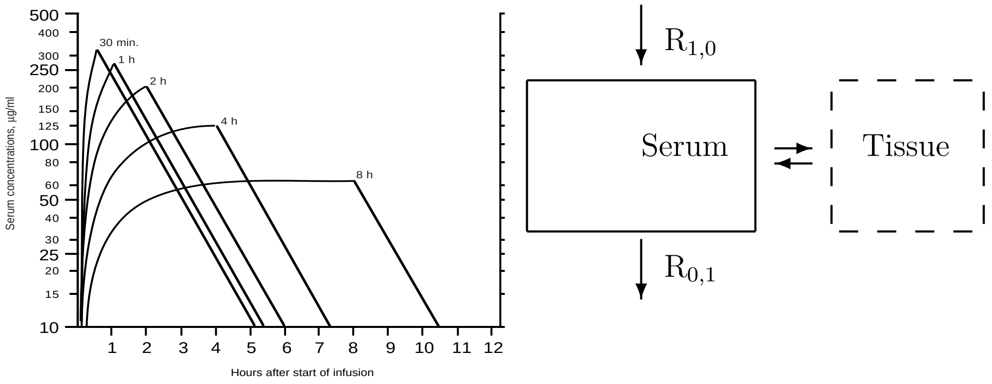


Figure 15.3: Graph of continuous infusion administration of carbenicillin, from Tom Bergans, *ibid.*, page 73, Figure 16.

The original data was in Palmer and Höffler, Carbenicillin, Basisdaten zur Therapie mit einem Antibiotikum (Urban & Schwarzenberg, München, 1977), except for the 30 minute curve which was from T. Bergan and B. Ødvin, Cross-over study of penicillin pharmacokinetics after intravenous infusions. Chemotherapy, 20: 263-279 (1974).

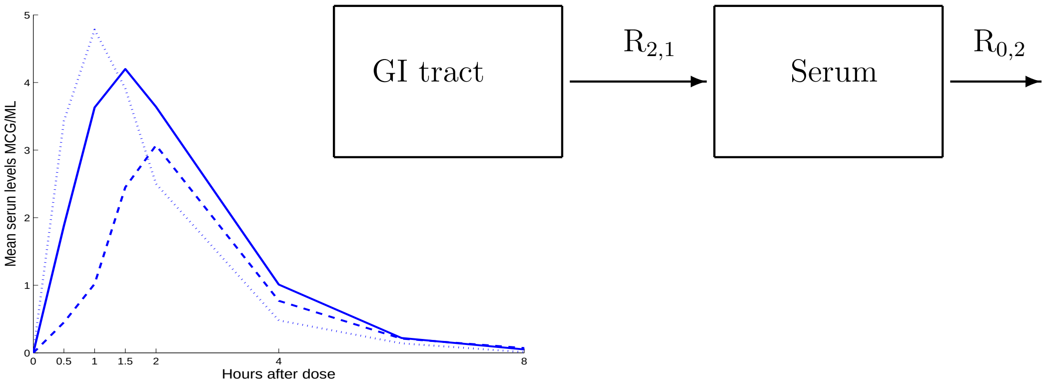
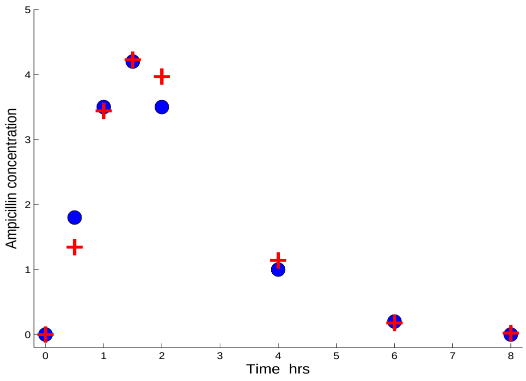


Figure 15.4: Data for oral administration of 500 mg of pivamphicillin and ampicillin to healthy volunteers (medical students) and a schematic diagram used to model this experiment. Note: MCG/ML = μg/ml. Graph reproduced from Fernandez, C.A. et al: *J. Int. Med. Res.*, **1**, 530-533 by permission from Field House Publishing LLP, all rights reserved. — Amphicillin, Fasting; ··· Pivamphicillin, Fasting; - - - Pivamphicillin with food. Pivamphicillin was first being tested, and without food patients experienced nausea.

Table 15.1: Estimates of ampicillin concentrations (Conc.) read from the dark line graph in Figure 15.4. Graph of that data (dots) and of computed data (+).

Time (hr)	Conc. μg/ml	Computed Conc. μg/ml
0	0.0	0.0
0.5	1.8	3.6
1.0	3.5	4.5
1.5	4.2	4.2
2.0	3.5	3.5
4.0	1.0	1.1
6.0	0.2	0.3
8.0	0.0	0.1



Flask 1 (GI tract) has 1 liter of water and at time $t = 0$, 18 milligrams of salt are added to it and it dissolves immediately.

Flask 2 (Serum) has 2 liters of pure water in it.

Each 30 minutes 35% of the liquid in flask 2 (700 ml) is removed and discarded, after which 40% of the liquid in flask 1 (400 ml) plus 300 ml of pure water is added to flask B and 400 ml of pure water is added to flask A.

Notation. Let A_t denote the amount of salt in flask 1 at time t and B_t denote the amount of salt in flask 2 at time t, **where t is measured in 30 minute intervals.**

The initial conditions are

$$A_0 = 18.0 \qquad \text{and} \qquad B_0 = 0.0$$

The changes in A_t and B_t can be accounted for as follows.

$$\begin{array}{rcrr} \text{Change} & & \text{Increase} & \text{Decrease} \\ A_{t+1} - A_t & = & & -0.40A_t \\ B_{t+1} - B_t & = & 0.40A_t & -0.35B_t \end{array} \tag{15.8}$$

These can be converted to iteration equations with initial conditions:

$$\begin{array}{rclcrcl} A_0 & = & 18 & \qquad & A_{t+1} & = & 0.6A_t \\ B_0 & = & 0 & & B_{t+1} & = & 0.4A_t + 0.65B_t \end{array} \tag{15.9}$$

You can readily write a formula for A_t and are asked to do so in Exercise 15.3.2. In Section 15.4 you will find how to write the formula for B_t:

$$B_t = 144 \times (0.65^t - 0.6^t) \quad \text{milligrams.} \qquad \text{Let} \quad BC_t = \frac{B_t}{2} \quad \mu\text{g/ml.} \tag{15.10}$$

Using the equation for BC_t, the numbers in the 'Computed Conc.' column of Table 15.1 are readily computed. It is apparent from the graph in Table 15.1 that the overall shape of the computed graph is similar to that of the original data and the concentrations for the later times are similar in both. The computed data 'peaks early,' however. It would peak even earlier were it not for the large discrete time interval of 30 minutes. An alternate model with 10 minute intervals is suggested in Exercise 15.3.5 that more closely fits the data.

We have given three examples of data for penicillin administration and difference equation models that could be descriptive of the kinetics of the penicillin. We have also stated solutions to the difference equations, without giving procedures to find the solutions. There are some general procedures for finding the solutions, and the next two sections are directed to that end.

Exercises for Section 15.3, Continuous infusion and oral administrations of penicillin.

Exercise 15.3.1 Explain the reasons for the terms in Equations 15.8 that account for Increase and Decrease of A_t and of B_t.

Exercise 15.3.2 Find the solution to

$$A_0 = 18.0 \qquad A_{t+1} = 0.6A_t,$$

which is the first equation in Equations 15.9.

Exercise 15.3.3 Analysis of continuous infusion of carbenicillin. Figure 15.3 contains data of concentrations of carbenicillin in serum during and after continuous infusion into healthy volunteers (medical students). You are to analyze a model of that system.

A flask (Serum) has 10 liters of pure water. Each 10 minutes, one liter (10%) of the liquid is removed from the flask, after which one liter of water containing salt with a concentration of 66 milligrams per liter is added to the flask and the flask is well stirred.

Let A_t denote the amount of salt in the flask at time t in 10 minute intervals.

a. Read and record the carbenicillin concentrations of the 8h curve in Figure 15.3 for the hours 1 to 8. Assume that concentration at hour 0 was 0. Interpret that curve to show that infusion continued at a constant rate for 8 hours and was terminated at the end of 8 hours.

b. What is A_0?

c. Write an equation that accounts for the Change, Increase, and Decrease of A_t each ten minute interval, similar in form to that of Equation 15.8.

d. Write the initial condition and an iteration equation that will enable the computation of A_t for any 10 minute interval.

e. Your iteration equation should be of the form $A_{t+1} = q \times A_t + p$. You solved equations of this form in Section **??**. Write a solution to your iteration equation.

f. Compare carbenicillin concentrations that you read from the graph with the salt *concentrations* for **hours** 0 to 8 (0, $6t$, $12t$, $\cdots$).

g. After infusion is terminated at 8 hours, the concentration decreases linearly on the semilog graph. Compare this with the nonlinear decrease following bolus injection shown in Figure 15.1. What difference in the states of the two patients may account for the difference in response?

h. Discuss the advisability of including a tissue compartment (dashed box in Figure 15.3) in the model of the data of the figure.

Exercise 15.3.4 Repeat the analysis of Exercise 15.3.3 for the 4 hour curve. Use the model of Exercise 15.3.3 and adjust the concentration of the salt in the water added to the flask at each ten minute interval.

Exercise 15.3.5 As discussed in the analysis of the oral ingestion of ampicillin, the concentration predicted by our solution 'peaked' too early to match the data. One possible interpretation is that following ingestion of the pill into the stomach, there was a time of transition to the intestine where it was then absorbed into serum. This would suggest a model described by the diagram in Exercise Figure 15.3.5

Imagine three flasks A, B, and C, containing 1.5, 2.0 and 1.0 liters, respectively, of pure water. At time $t = 0$, 21 milligrams of salt is placed in flask A and immediately dissolves. Every 10 minutes 300 ml (30%) of solution is removed from flask C and discarded, 300 ml (15%) of solution is transferred from flask B to flask C, and 300 (20%) ml of water is transferred from flask A to flask B. Finally, 300 ml of pure water is added to flask A. Let A_t, B_t, and C_t denote the amount of salt in flasks A, B, and C at time interval t.

a. What are A_0, B_0 and C_0?

b. Write equations that accounts for the Change, Increase, and Decrease in each of A_t, B_t, and C_t each ten minute interval. similar in form to that of Equation 15.8.

c. Write the initial conditions and an iteration equations that will enable the computation of A_t, B_t, and C_t for any 10 minute interval.

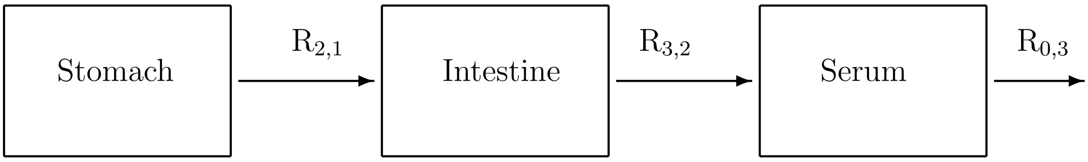
d. Show by substitution that

$$\begin{aligned} A_t &= 21 \times 0.8^t \\ B_t &= 84 \times (0.85^t - 0.8^t) \\ C_t &= 84 \times 0.85^t - 126 \times 0.8^t + 42 \times 0.7^t \end{aligned}$$

solve your iteration equations.

e. Use the solution equation for C_t from part d. to compute ampicillin *concentrations* for times 0, 0.5, 1.0, 1.5, 2, 4, 6, and 8 hours and compare those values with the values shown in Table 15.1.

Figure for Exercise 15.3.5 Diagram of 3 compartment model of oral ingestion of ampicillin for Exercises 15.3.5 and 15.3.6



Exercise 15.3.6 The three compartment model for oral ingestion of 500 mg of ampicillin diagrammed in Exercise Figure 15.3.5 can be described more simply than was done in Exercise 15.3.5. The solutions of the iteration equations are unusual, but match the data well.

Imagine three flasks A, B, and C, each containing 1.0 liter of pure water. At time $t = 0$, 14 milligrams of salt are placed in flask A and immediately dissolve. Every 10 minutes 200 ml (20%) of solution is removed from flask C and discarded, 200 ml (20%) of solution is transferred from flask B to flask C, and 200 ml (20%) of water is transferred from flask A to flask B. Finally, 200 ml of pure water is added to flask A. Let A_t, B_t, and C_t denote the amount of salt in flasks A, B, and C at time interval t.

a. What are A_0, B_0 and C_0?

b. Write equations that accounts for the Change, Increase, and Decrease in each of A_t, B_t, and C_t each ten minute interval. similar in form to that of Equation 15.8.

c. Write the initial conditions and an iteration equations that will enable the computation of A_t, B_t, and C_t for any 10 minute interval.

d. Show by substitution that

$$\begin{aligned} A_t &= 14 \times 0.8^t \\ B_t &= 3.5 \times t \times 0.8^t \\ C_t &= 0.4375 \times t \times (t-1) \times 0.8^t \end{aligned}$$

solve your iteration equations.

e. Use the solution equation for C_t from part d. to compute ampicillin concentrations for times 0, 0.5, 1.0, 1.5, 2, 4, 6, and 8 hours and compare those values with the values shown in Table 15.1.

15.4 Solutions to pairs of difference equations.

The solution Equations 15.7

$$\begin{aligned} A_t &= \frac{200}{2.1} \times \left(0.4 \times 0.94^t + 1.7 \times 0.73^t\right) \\ B_t &= \frac{200}{2.1} \times \left(0.68 \times 0.94^t - 0.68 \times 0.73^t\right) \end{aligned}$$

to

$$\begin{array}{ll} A_0 = 200 & A_{t+1} = 0.77 \times A_t + 0.10 \times B_t \\ B_0 = 0 & B_{t+1} = 0.068 \times A_t + 0.90 \times B_t \end{array}$$

are presented as **Twin Lightning Bolts Out of the Blue** without indication of their source. We present here a method for finding solutions to all dynamical systems of the form

$$\begin{array}{llrcl} A_0 & \text{Given} & A_{t+1} & = & a_{1,1} \times A_t + a_{1,2} \times B_t \\ B_0 & \text{Given} & B_{t+1} & = & a_{2,1} \times A_t + a_{2,2} \times B_t \end{array} \tag{15.11}$$

There are four stages to finding the solution.

Stage 1. The Stage 1 goal is to obtain an equation that only involves the sequence A_0, A_1, A_2, $\cdots$. We eliminate B_t from the Equations 15.11 by

$$\begin{aligned} a_{2,2} \times A_{t+1} &= a_{2,2} \times a_{1,1} \times A_t + a_{2,2} \times a_{1,2} \times B_t \\ a_{1,2} \times B_{t+1} &= a_{1,2} \times a_{2,1} \times A_t + a_{1,2} \times a_{2,2} \times B_t \\ a_{2,2} \times A_{t+1} - a_{1,2} \times B_{t+1} &= a_{2,2} \times a_{1,1} \times A_t - a_{1,2} \times a_{2,1} \times A_t \end{aligned} \tag{15.12}$$

Increase the index, t, in $A_{t+1} = a_{1,1} \times A_t + a_{1,2} \times B_t$ to get

$$A_{t+2} = a_{1,1} \times A_{t+1} + a_{1,2} \times B_{t+1} \tag{15.13}$$

Eliminate $a_{1,2} \times B_{t+1}$ between Equations 15.12 and 15.13

$$\begin{aligned} a_{2,2} \times A_{t+1} - a_{1,2} \times B_{t+1} &= a_{2,2} \times a_{1,1} \times A_t - a_{1,2} \times a_{2,1} \times A_t \\ a_{1,1} \times A_{t+1} + a_{1,2} \times B_{t+1} &= A_{t+2} \\ a_{1,1} \times A_{t+1} + a_{2,2} \times A_{t+1} &= (a_{2,2} \times a_{1,1} - a_{1,2} \times a_{2,1}) \times A_t + A_{t+2} \end{aligned}$$

The last equation may be written

$$A_{t+2} - (a_{1,1} + a_{2,2}) \times A_{t+1} + (a_{1,1} \times a_{2,2} - a_{1,2} \times a_{2,1}) \times A_t = 0 \tag{15.14}$$

Equation 15.14 is a *second order linear* difference equation that we write as

$$A_{t+2} - p \times A_{t+1} + q \times A_t = 0 \qquad p = a_{1,1} + a_{2,2} \qquad q = a_{1,1} \times a_{2,2} - a_{1,2} \times a_{2,1} \tag{15.15}$$

If the first two initial values, A_0 and A_1, are known, then all subsequent values of A_t can be computed from

$$A_{t+2} - p \times A_{t+1} + q \times A_t = 0$$

Explore 15.4.1 Do this.

a. Show that for the penicillin equations

$$\begin{array}{lrcl} A_0 = 200 & A_{t+1} & = & 0.77 \times A_t + 0.10 \times B_t \\ B_0 = 0 & B_{t+1} & = & 0.068 \times A_t + 0.90 \times B_t \end{array}$$

$p = 1.67$ and $q = 0.6862$, so that

$$A_{t+2} - 1.67A_{t+1} + 0.6862A_t = 0$$

b. Show that $A_0 = 200$, $B_0 = 0$, and $A_1 = 0.77 \times A_0 + 0.10 \times B_0$ yield $A_0 = 200$ and $A_1 = 154$.

c. Compute A_2 and A_3 from

$$A_0 = 200 \qquad A_1 = 154 \qquad A_{t+2} - 1.67 A_{t+1} + 0.6862 A_t = 0$$

■

For Equation 15.11

$$\begin{array}{llrcl} A_0 & \text{Given} & A_{t+1} & = & a_{1,1} \times A_t + a_{1,2} \times B_t \\ B_0 & \text{Given} & B_{t+1} & = & a_{2,1} \times A_t + a_{2,2} \times B_t \end{array}$$

$$A_1 = a_{1,1} \times A_0 + a_{1,2} \times B_0$$

so that A_0 and A_1 are known for $A_{t+2} - p \times A_{t+1} + q \times A_t = 0$, and A_2, A_3, $\cdots$ are determined.

Stage 2. The Stage 2 goal is to find a solution to

$$A_{t+2} - p \times A_{t+1} + q \times A_t = 0$$

Observe that $A_t = 0$ for all time t is a solution, but is not very helpful in describing penicillin pharmacokinetics, (with $A_0 = 200$ and $A_1 = 154$) and we need a more interesting solution. Thinking optimistically, we note that the first order iteration equation

$$P_{t+1} = R \times P_t \qquad \text{has a solution:} \qquad P_t = C \times R^t$$

and we try for a similar solution

$$A_t = C \times r^t \qquad \text{to} \qquad A_{t+2} - p \times A_{t+1} + q \times A_t = 0.$$

If $A_t = C \times r^t$, then $A_{t+1} = C \times r^{t+1}$ and $A_{t+2} = C \times r^{t+2}$ and we try

$$\begin{aligned} C \times r^{t+2} - p \times C \times r^{t+1} + q \times C \times r^t &= 0 \\ C \times \left(r^t \times r^2 - p \times r^t \times r^1 + q \times r^t\right) &= 0 \\ C \times r^t \times \left(r^2 - p \times r + q\right) &= 0 \end{aligned}$$

One of the factors in the previous product must be zero. The choices $C = 0$ and $r^t = 0$ yield $A_t \equiv 0$ – not helpful. Therefore we try

$$r^2 - p\,r + q = 0 \tag{15.16}$$

which is a quadratic equation and has (with luck) two nonzero real roots

$$r_1 = \frac{p + \sqrt{p^2 - 4q}}{2} \qquad \text{and} \qquad r_2 = \frac{p - \sqrt{p^2 - 4q}}{2}$$

For this section we assume that $\mathbf{r}_1$ and $\mathbf{r}_2$ are distinct and real.

We have reason to hope that we have found two solutions $A_t = C \times r_1^t$ and $A_t = C \times r_2^t$. It gets even better than that! We will show in Stage 3 that *for any two numbers*, C_1 *and* C_2,

$$A_t = C_1 \times r_1^t + C_2 \times r_2^t \qquad \textit{solves} \qquad A_{t+2} - pA_{t+1} + qA_t = 0. \tag{15.17}$$

Equation 15.16, $r^2 - pr + q = 0$, is called the *characteristic equation* and

$$r_1 = \frac{p + \sqrt{p^2 - 4q}}{2} \qquad \text{and} \qquad r_2 = \frac{p - \sqrt{p^2 - 4q}}{2}$$

are called the *characteristic roots* of both the difference equation

$$A_{t+2} - p \times A_{t+1} + q \times A_t = 0$$

and (when $p = a_{1,1} + a_{2,2}$ and $q = a_{1,1} \times a_{2,2} - a_{1,2} \times a_{2,1}$) the iteration equations

$$\begin{aligned} A_{t+1} &= a_{1,1} \times A_t + a_{1,2} \times B_t \\ B_{t+1} &= a_{2,1} \times A_t + a_{2,2} \times B_t. \end{aligned}$$

Explore 15.4.2 Do this. Find r_1 and r_2 for

$$\begin{aligned} A_0 = 200 \qquad A_{t+1} &= 0.77 \times A_t + 0.10 \times B_t \\ B_0 = 0 \qquad B_{t+1} &= 0.068 \times A_t + 0.90 \times B_t \end{aligned}$$

Set your calculator to display at least five decimal digits.

Compare r_1 and r_2 with terms in the claimed solution

$$\begin{aligned} A_t &= \frac{200}{2.1} \times \left(0.4 \times 0.94^t + 1.7 \times 0.73^t\right) \\ B_t &= \frac{200}{2.1} \times \left(0.68 \times 0.94^t - 0.68 \times 0.73^t\right) \end{aligned}$$ ■

Stage 3. We show that for any two numbers, C_1 and C_2

$$A_t = C_1 \times r_1^t + C_2 \times r_2^t \qquad \text{solves} \qquad A_{t+2} - pA_{t+1} + qA_t = 0.$$

$$\begin{aligned} & A_{t+2} - pA_{t+1} + qA_t \\ = \ & \left(C_1 \times r_1^{t+2} + C_2 \times r_2^{t+2}\right) - p \times \left(C_1 \times r_1^{t+1} + C_2 \times r_2^{t+1}\right) \\ & \qquad + q \times \left(C_1 \times r_1^t + C_2 \times r_2^t\right) \\ = \ & C_1 \times \left(r_1^{t+2} - p \times r_1^{t+1} + q \times r_1^t\right) + C_2 \times \left(r_2^{t+2} - p \times r_2^{t+1} + q \times r_2^t\right) \end{aligned}$$

$$\begin{aligned}
&= C_1 \times r_1^t \times \left(r_1^2 - p \times r_1^1 + q\right) + C_2 \times r_2^t \times \left(r_2^2 - p \times r_2^1 + q\right) \\[2em]
&= C_1 \times r_1^t \times (\quad 0 \quad) + C_2 \times r_2^t \times (\quad 0 \quad) \\[2em]
&= 0
\end{aligned}$$

Whew! The 0's in the last step are because r_1 and r_2 are the roots to $r^2 - pr + q = 0$.

Stage 4. (Hang on, we are about there!) We need to evaluate C_1 and C_2 and will choose them so that the solution

$$A_t = C_1 \times r_1^t + C_2 \times r_2^t$$

to the iteration

$$A_{t+2} - pA_{t+1} + qA_t = 0$$

matches the initial data. A_0 and B_0 are Given. From $A_1 = a_{1,1}A_0 + a_{1,2}B_0$ we can assume that A_1 is also known. For any C_1 and C_2

$$A_t = C_1 \times r_1^t + C_2 \times r_2^t$$

is a solution and we impose

$$\begin{aligned}
A_0 = C_1 \times r_1^0 + C_2 \times r_2^0 &= A_0 \quad \text{Given} \\
A_1 = C_1 \times r_1^1 + C_2 \times r_2^1 &= A_1 \quad \text{Given}
\end{aligned}$$

With r_1 distinct from r_2 the equations

$$\begin{aligned}
C_1 + C_2 &= A_0 \\[1em]
r_1C_1 + r_2C_2 &= A_1
\end{aligned}
\qquad \text{imply that} \qquad
\begin{aligned}
C_1 &= \frac{A_1 - r_2A_0}{r_1 - r_2} \\[1em]
C_2 &= \frac{r_1A_0 - A_1}{r_1 - r_2}
\end{aligned}
\tag{15.18}$$

Summary. The pair of first order iteration equations

$$\begin{aligned}
&A_0 \quad \text{Given} \qquad & A_{t+1} &= a_{1,1} \times A_t + a_{1,2} \times B_t \\[1em]
&B_0 \quad \text{Given} \qquad & B_{t+1} &= a_{2,1} \times A_t + a_{2,2} \times B_t
\end{aligned}$$

uniquely determine A_0, A_1, $\cdots$ and B_0, B_1, $\cdots$.

If the characteristic equation

$$r^2 - pr + q = 0, \quad \text{where} \quad p = a_{1,1} + a_{2,2} \quad \text{and} \quad q = a_{1,1} \times a_{2,2} - a_{1,2} \times a_{2,1},$$

has distinct real roots r_1 and r_2, the sequence A_0, A_1, $\cdots$ is expressed by

$$A_t = \frac{A_1 - r_2A_0}{r_1 - r_1} r_1^t + \frac{r_1A_0 - A_1}{r_1 - r_2} r_2^t. \tag{15.19}$$

You will develop in Exercise 15.4.8 the equation

$$B_t = \frac{B_1 - r_2 B_0}{r_1 - r_2} r_1^t + \frac{r_1 B_0 - B_1}{r_1 - r_2} r_2^t. \tag{15.20}$$

Because the dynamical system, Equations 15.11, uniquely determines A_0, $A_1\, A_2\, \cdots$ and B_0, $B_1\, B_2\, \cdots$ and the sequences in Equations 15.19 and 15.20 satisfy the initial conditions and the Equations 15.11 of the dynamical system, Equations 15.19 and 15.20 are the only solutions to the dynamical system.

The method of **Stage 2** applies to any second order difference equation of the form

$$w_{t+2} - pw_{t+1} + qw_t = 0.$$

The solution is

$$w_t = C_1 r_1^t + C_2 r_2^t \qquad \text{or} \qquad w_t = C_1 r_1^t + C_2 t r_1^t$$

where r_1 and r_2 are the roots to $r^2 - p\,r + q = 0$ and the second solution applies only if $p^2 - 4q = 0$ and there is only one root, r_1.

Stage 3 has an important generalization that applies to solutions of $A_{t+2} - pA_{t+1} + qA_t = 0$ and even higher order linear difference equations.

Theorem 15.4.1 If u_t and v_t are solutions to the linear homogeneous difference equation

$$w_{t+n} + p_{n-1} w_{t+n-1} + \cdots + p_1 w_{t+1} + p_0 w_t = 0 \tag{15.21}$$

where p_{n-1}, $\cdots$ p_1 and p_0 are constants, then for any numbers C_1 and C_2,

$$C_1 u_t + C_2 v_t \quad \text{is a solution to Equation 15.21}$$

.

Proof.

$$\begin{aligned} C_1(u_{t+n} + C_2 v_{t+n}) + \cdots + p_1(C_1 u_{t+1} + C_2 v_{t+1}) + p_0(C_1 u_t + C_2 v_t) &= \\ C_1(u_{t+n} + p_{n-1}u_{t+n-1} + \cdots + p_1 u_{t+1} + p_0 u_t) & \\ + \quad C_2(v_{t+n} + p_{n-1}v_{t+n-1} + \cdots + p_1 v_{t+1} + p_0 v_t) &= \\ C_1 \times (\quad 0 \quad) + C_2 \times (\quad 0 \quad) &= 0. \end{aligned}$$

End of proof.

Example 15.4.1 Find formulas for A_t and B_t if

$$\begin{array}{rclrcl} A_0 & = & 1 & A_{t+1} & = & 0.52A_t + 0.04B_t \\ B_0 & = & 2 & B_{t+1} & = & 0.24A_t + 0.4B_t \end{array}$$

Solution

$$p = 0.52 + 0.4 = 0.92, \quad q = 0.52 \cdot 0.4 - 0.24 \cdot 0.04 = 0.1984, \quad r^2 - 0.92r + 0.1984 = 0$$

$$r_1 = \frac{0.92 + \sqrt{0.92^2 - 4 \cdot 0.1984}}{2} = 0.46 + \sqrt{0.0132}, \qquad r_2 = 0.46 - \sqrt{0.0132}$$

$$A_1 = 0.52 \cdot 1 + 0.04 \cdot 2 = 0.6, \qquad B_1 = 0.24 \cdot 1 + 0.4 \cdot 2 = 1.04.$$

$$C_1 = \frac{A_1 - r_2A_0}{r_1 - r_2} = \frac{0.6 - (0.46 - \sqrt{0.0132}) \cdot 1}{2\sqrt{0.0132}} = (0.07/\sqrt{0.0132}) + 0.5$$

$$C_2 = \frac{r_1A_0 - A_1}{r_1 - r_2}\frac{(0.46 + \sqrt{0.132}) \cdot 1 - 0.6}{2\sqrt{0.0132}} = (-0.07/\sqrt{0.0132}) + 0.5$$

$$\begin{aligned} A_t &= C_1\, r_1^t + C_2\, r_2^t \\ &= ((0.07/\sqrt{0.0132}) + 0.5)\,(0.46 + \sqrt{0.0132})^t + \\ &\qquad ((-0.07/\sqrt{0.0132}) + 0.5)\,(0.46 - \sqrt{0.0132})^t \end{aligned}$$

$$\begin{aligned} D_1 &= \frac{B_1 - r_2B_0}{r_1 - r_2} = (0.06/\sqrt{0.0132}) + 1 \qquad \text{See Exercise 15.4.8} \\ D_2 &= \frac{r_1B_0 - B_1}{r_1 - r_2} = (-0.06/\sqrt{0.0132}) + 1 \\ B_t &= ((0.06/\sqrt{0.0132}) + 1)\,(0.46 + \sqrt{0.0132})^t + \\ &\qquad ((-0.06/\sqrt{0.0132}) + 1)\,(0.46 - \sqrt{0.0132})^t \qquad \blacksquare \end{aligned}$$

Exercises for Section 15.4, Solutions to pairs of difference equations.

Exercise 15.4.1 Do Explore 15.4.1 and Explore 15.4.2.

Exercise 15.4.2 Now put it all together. Show that for

$$\begin{array}{lrcl} A_0 = 200 & A_{t+1} & = & 0.77 \times A_t + 0.10 \times B_t \\ B_0 = 0 & B_{t+1} & = & 0.068 \times A_t + 0.90 \times B_t \end{array}$$

$$A_t = \frac{200}{2.1} \times 0.4 \times 0.94^t + \frac{200}{2.1} \times 1.7 \times 0.73^t$$

Do exact arithmetic.

Exercise 15.4.3 Show by substitution that the proposed solution satisfies the difference equation.

$$\begin{array}{llll} \text{a.} & A_t = C\times 2^t & & A_{t+2} - 2.5\times A_{t+1} + A_t = 0 \\ \text{b.} & A_t = C\times 0.9^t & & A_{t+2} - 1.3\times A_{t+1} + 0.36\times A_t = 0 \\ \text{c.} & A_t = C_1\times 0.5^t + C_2\times 0.7^t & & A_{t+2} - 1.2\times A_{t+1} + 0.35\times A_t = 0 \\ \text{d.} & A_t = C_1 + C_2\times 0.2^t & & A_{t+2} - 3\times A_{t+1} + 2\times A_t = 0 \\ \text{e.} & A_t = C_1\times 0.6^t + C_2\times(-0.5)^t & & A_{t+2} - 0.1\times A_{t+1} - 0.3\times A_t = 0 \end{array}$$

Exercise 15.4.4 Find the roots to the characteristic equation, $r^2 - pr + q = 0$ and write the solutions for A_t in the following systems.

$$\begin{array}{lrclrcl} \text{a.} & A_0 &=& 10 & A_{t+1} &=& 0.8A_t + 0.2B_t \\ & B_0 &=& 0 & B_{t+1} &=& 0.1A_t + 0.7B_t \\[1ex] \text{b.} & A_0 &=& 0 & A_{t+1} &=& 0.6A_t + 0.3B_t \\ & B_0 &=& 5 & B_{t+1} &=& 0.2A_t + 0.7B_t \\[1ex] \text{c.} & A_0 &=& 1 & A_{t+1} &=& 0.26A_t + 0.04B_t \\ & B_0 &=& 1 & B_{t+1} &=& 0.06A_t + 0.24B_t \\[1ex] \text{d.} & A_0 &=& 2 & A_{t+1} &=& 1.04A_t + 0.16B_t \\ & B_0 &=& 3 & B_{t+1} &=& 0.24A_t + 0.96B_t \\[1ex] \text{e.} & A_0 &=& 20 & A_{t+1} &=& 0.86A_t + 0.04B_t \\ & B_0 &=& 10 & B_{t+1} &=& 0.06A_t + 0.84B_t \end{array}$$

Exercise 15.4.5 Find the solutions to

$$\begin{array}{lrclrclrcl} \text{a.} & w_0 &=& 3 & w_1 &=& 1 & w_{t+2} - 5w_{t+1} + 6w_t &=& 0 \\ \text{b.} & w_0 &=& 0 & w_1 &=& 0 & w_{t+2} + 8w_{t+1} + 12w_t &=& 0 \\ \text{c.} & w_0 &=& 2 & w_1 &=& 1 & w_{t+2} - 6w_{t+1} + 8w_t &=& 0 \\ \text{d.} & w_0 &=& 1 & w_1 &=& 1 & w_{t+2} - 5w_{t+1} + 4w_t &=& 0 \end{array}$$

Exercise 15.4.6 Show that if r_1 is zero (or r_2 is zero), then $q = 0$. In this case, Equation 15.15, $A_{t+2} - pA_{t+1} + qA_t = 0$, is $A_{t+2} - pA_{t+1} = 0$ which is first order and easy to solve.
It is curious that when $q = 0$, $A_1 - pA_0$ may not be zero, but $A_{t+1} - pA_t = 0$ for $t \geq 1$.
Write an equation for A_t in terms of A_1 and p.
Solve the systems for A_t, $t = 0, 1, 2, \cdots$.

$$\begin{array}{lrclrcl} \text{a.} & A_0 &=& 2 & A_{t+1} &=& 0.3A_t + 0.6B_t \\ & B_0 &=& 1 & B_{t+1} &=& 0.2A_t + 0.4B_t \\[1ex] \text{b.} & A_0 &=& 0 & A_{t+1} &=& 0.6A_t + 0.3B_t \\ & B_0 &=& 5 & B_{t+1} &=& 0.2A_t + 0.1B_t \end{array}$$

Exercise 15.4.7 Solve for B_t.

For the initial conditions and iteration equations:

$$\begin{array}{rclrcl} A_0 &=& 2 & A_{t+1} &=& 0.3A_t + 0.1B_t \\ B_0 &=& 5 & B_{t+1} &=& 0.1A_t + 0.3B_t \end{array}$$

a. Eliminate A_t by subtraction $(0.1A_{t+1} - 0.3B_{t+1})$.

b. Explain why $B_{t+2} = 0.1A_{t+1} + 0.3B_{t+1}$.

c. Use the two equations from parts a. and b. to write

$$B_{t+2} - 0.6B_{t+1} + 0.08B_t = 0$$

d. Suppose that for some number, r, $B_t = Cr^t$. Show that $C \neq 0$, $r \neq 0$ imply that $r^2 - 0.6r + 0.08 = 0$.

e. Find the roots of $r^2 - 0.6r + 0.08 = 0$.

f. Show that for any two numbers, C_1 and C_2,

$$B_t = C_1 \times 0.2^t + C_2 \times 0.4^t \qquad \text{solves} \qquad B_{t+2} - 0.6B_{t+1} + 0.08B_t = 0.$$

g. Compute B_1.

h. Show that

$$B_t = 1.5 \times 0.2^t + 3.5 \times 0.4^t$$

Exercise 15.4.8 Follow the steps of Exercise 15.4.7 to find an equation for B_t satisfying:

$$\begin{array}{rclrcl} A_0 &=& \text{Given} & A_{t+1} &=& a_{1,1}A_t + a_{1,2}B_t \\ B_0 &=& \text{Given} & B_{t+1} &=& a_{2,1}A_t + a_{2,2}B_t \end{array}$$

Stepping stones will be

$$\begin{array}{rcl} a_{1,1}B_t - a_{2,1}A_t &=& (a_{1,1}\, a2,2 - a_{2,1}\, a_{1,2})B_t \\ B_{t+2} &=& a_{2,1}A_{t+1} + a_{2,2}B_{t+1} \\ B_{t+2} - (a_{1,1} + a_{2,2})B_{t+1} + (a_{1,1}\, a_{2,2} - a_{2,1}\, a_{1,2})B_t &=& 0 \\ B_t &=& D_1 \times r_1^t + D_2 \times r_2^t \end{array}$$

where r_1 and r_2 are roots of $r^2 - (a_{1,1} + a_{2,2})\, r + (a_{1,1}\, a2,2 - a_{2,1}\, a_{1,2}) = 0$ and

$$D_1 = \frac{B_1 - r_2 B_0}{r_1 - r_2} \qquad D_2 = \frac{r_1 B_0 - B_1}{r_1 - r_2}$$

Assume r_1 and r_2 are two real numbers.

Exercise 15.4.9 Show that for

$$\begin{array}{lrcl} A_0 = 200 & A_{t+1} & = & 0.77 \times A_t + 0.10 \times B_t \\ \\ B_0 = 0 & B_{t+1} & = & 0.068 \times A_t + 0.90 \times B_t \end{array}$$

$$B_t = \frac{200}{2.1} \times 0.68 \times 0.94^t - \frac{200}{2.1} \times 0.68 \times 0.73^t$$

Exercise 15.4.10 Find solutions for both A_t and B_t satisfying

$$\begin{array}{llrcl} a. & A_0 = 10 & A_{t+1} & = & 0.50A_t + 0.2B_t \\ & B_0 = 0 & B_{t+1} & = & 0.15B_t + 0.7B_t \end{array}$$

$$\begin{array}{llrcl} b. & A_0 = 5 & A_{t+1} & = & 0.6A_t + 0.1B_t \\ & B_0 = 10 & B_{t+1} & = & 0.2A_t + 0.7B_t \end{array}$$

15.5 Roots equal to zero, multiple roots, and complex roots

We examine special cases of the dynamical system

$$\begin{array}{llrcl} A_0 & \text{Given} & A_{t+1} & = & a_{1,1} \times A_t + a_{1,2} \times B_t \\ \\ B_0 & \text{Given} & B_{t+1} & = & a_{2,1} \times A_t + a_{2,2} \times B_t \end{array}$$

for which the roots are zero, repeated, or complex and consider the $\lim_{t\to\infty} A_t$ and $\lim_{t\to\infty} B_t$. As in the previous section, $A_{t+2} - pA_{t+1} + qA_t = 0$ and $B_{t+2} - pB_{t+1} + qB_t = 0$ for $p = a_{1,1} + a_{2,2}$ and $q = a_{1,1} \times a_{2,2} - a_{1,2} \times a_{2,1}$.

The case of a zero root. It is a generally useful result that:

Theorem 15.5.1 If r_1 and r_2 are the roots to $x^2 - px + q = 0$ then

$$r_1 + r_2 = p \qquad \text{and} \qquad r_1 \times r_2 = q. \tag{15.22}$$

(The result is valid even in the case that $p^2 - 4q < 0$ and $\sqrt{p^2 - 4q}$ is written $\sqrt{-1}\,\sqrt{4q - p^2} = i\,\sqrt{4q - p^2}$.)

Proof.

$$r_1 + r_2 = \frac{p + \sqrt{p^2 + 4q}}{2} + \frac{p - \sqrt{p^2 - 4q}}{2} = \frac{p + \sqrt{p^2 - 4q} + p - \sqrt{p^2 - 4q}}{2} = p$$

$$r_1 \times r_2 = \frac{p + \sqrt{p^2 - 4q}}{2} \times \frac{p - \sqrt{p^2 - 4q}}{2} = \frac{p^2 - (p^2 - 4q)}{4} = q$$

End of proof.

The case of a root equal to zero, $\mathbf{r_2 = 0}$. If the characteristic equation, $r^2 - pr + q = 0$ has a root equal to zero, $r_2 = 0$, then $q = r_1 \times r_2 = 0$ and both

$$\begin{array}{lrcll} A_{t+2} = p \times A_{t+1} & A_t & = & A_1 \times p^{t-1} & \text{for} \quad t \geq 1 \\ B_{t+2} = p \times B_{t+1} & B_t & = & B_1 \times p^{t-1} & \text{for} \quad t \geq 1. \end{array}$$

A_1 and B_1 are computed from $A_1 = a_{1,1}A_0 + a_{1,2}B_0$ and $B_1 = a_{2,1}A_0 + a_{2,2}B_0$, respectively. It may not be that $A_1 = p \times A_0$.

The case of repeated roots, $\mathbf{r_1 = r_2}$. Biological systems are unlikely to be so finely tuned that $r_1 = r_2$. However, there is a simple solution. If $r_1 = r_2$, because $r_1 + r_2 = p$, the repeated root, $r_1 = \frac{p}{2}$.

In Exercise 15.5.1 you are asked to show that when $r_1 = r_2$ is a repeated root,

$$A_t = t \times r_1^t \qquad \text{solves} \qquad A_{t+2} - pA_{t+1} + qA_t = 0,$$

and for any two numbers C_1 and C_2,

$$A_t = C_1 r_1^t + C_2 \times t \times r_1^t \qquad \text{solves} \qquad A_{t+2} - pA_{t+1} + qA_t = 0.$$

We next show that every solution of $A_{t+2} - pA_{t+1} + qA_t = 0$ is of the form

$$A_t = C_1 r_1^t + C_2 \times t \times r_1^t.$$

To see this, we assume that A_0 and B_0 are known and $A_1 = a_{1,1}A_0 + a_{1,2}B_0$ has been computed. Then we solve

$$\begin{array}{rcl} A_0 & = & C_1 r_1^0 + C_2 \times 0 \times r_1^0 \\ A_1 & = & C_1 r_1^1 + C_2 \times 1 \times r_1^1 \end{array}$$

for C_1 and C_2. Assume that $r_1 \neq 0$. Then $C_1 = A_0$ and $C_2 = \dfrac{A_1 - A_0 r_1}{r_1}$ are uniquely determined. In the case that $r_1 = 0$, repeated root of zero, then both $p = 0$ and $q = 0$ and $A_t = 0$ and $B_t = 0$ for $t \geq 2$.

The case $\mathbf{r_1 = a + bi}$, $\mathbf{i = \sqrt{-1}}$. An interesting and important case occurs when the discriminant $p^2 - 4q < 0$ and the roots $r_1 = a + bi$ and $r_2 = a - bi$ are complex numbers. The same formulas

$$A_t = C_1 \times r_1^t + C_2 \times r_2^t \qquad C_1 = \frac{A_1 - r_2 A_0}{r_1 - r_2} \qquad C_2 = \frac{r_1 A_0 - A_1}{r_1 - r_2}$$

are valid, but are painful to work with. Normally, A_0, A_1, p and q are real so that all of A_2, A_3, $\cdots$ are real. Because r_1 and r_2 are complex, it is difficult to see that the previous formulas define real values for A_t (but they do).

Such systems arise, for example in Section 16.4.3 when considering equilibrium points of predator-prey systems. The following system is similar to Equation 16.46 of that section.

$$\begin{array}{rclrcrcr} \xi_0 & = & 0 & \xi_{t+1} & = & 0.98\,\xi_t & + & 0.08\,\eta_t \\ \eta_0 & = & 1 & \eta_{t+1} & = & -0.13\,\xi_t & + & 0.94\,\eta_t \end{array} \tag{15.23}$$

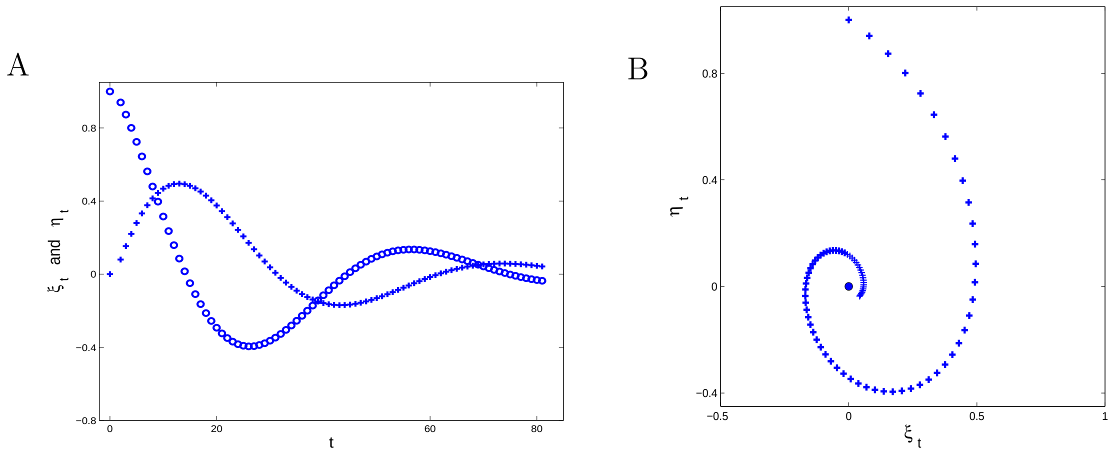


Figure 15.5: Graphs of the dynamical system 15.23. A. Graphs of the excess predator ξ_t (+) and excess prey η_t (o) versus time. B. Graph of η_t versus ξ_t.

ξ_t and η_t are linear approximations to departure from equilibrium of a predator and prey, respectively, and the initial conditions reflect an excess of prey. The roots to the characteristic equation, $p^2 - 1.92p + 0.9316 = 0$ are $r_1 = 0.96 + 0.10\,i$ and $r_1 = 0.96 - 0.10\,i$.

Two graphs of ξ_t and η_t are shown in Figure 15.5. Both graphs illustrate the periodic variation associated with complex roots and with predator-prey systems. With an excess of prey at time 0, the predator population increases causing a decrease in the prey population to a level below the equilibrium population, which is followed by a decrease in the prey population.

Alert: Incoming Bolt Out of the Blue. For the difference equation

$$A_0 \quad A_1 \qquad \text{Given,} \qquad A_{t+2} - pA_{t+1} + qA_t = 0 \qquad p^2 - 4q < 0$$

let

$$a = \frac{p}{2} \qquad b = \frac{\sqrt{4q - p^2}}{2} \qquad \rho = \sqrt{a^2 + b^2} \qquad \theta = \arccos\frac{a}{\rho}$$

Then the solution A_t is given by

$$A_t = C_1\rho^t \cos t\theta + C_2\rho^t \sin t\theta \qquad C_1 = A_0 \qquad C_2 = \frac{A_1 - aA_0}{b}. \tag{15.24}$$

Similarly, the solution B_t in Equations 15.11 is given by

$$B_t = D_1\rho^t \cos t\theta + D_2\rho^t \sin t\theta \qquad D_1 = B_0 \qquad D_2 = \frac{B_1 - aB_0}{b}. \tag{15.25}$$

Explore 15.5.1 Use Theorem 15.5.1 to show that $\rho^2 = q$.

All numbers used in Equation 15.24 and 15.25 are real, so the formulas specify real numbers for A_t and B_t. Readers experienced with complex arithmetic will recognize that the solution is related to De Moivre's formula

$$(a + bi)^n = \rho^n(\cos n\theta + i\sin n\theta)$$

For the system 15.23, the roots are $a \pm b\,i = 0.96 \pm 0.1\,i$, $\xi_0 = 0$ and $\xi_1 = 0.98\xi_0 + 0.08\eta_0 = 0.08$. Then

$$C_1 = \xi_0 = 0 \qquad C_2 = \frac{\xi_1 - a\,\xi_0}{b} = \frac{0.08 - 0.1 \times 0}{0.96} = 1/12$$

$$\rho = \sqrt{0.96^2 + 0.1^2} = \sqrt{0.9316} \doteq 0.965 \qquad \theta = \arccos\frac{a}{\rho} \doteq 0.104 \quad \text{radians.}$$

Then from Equation 15.24

$$\xi_t = C_1\rho^t \cos t\theta + C_2\rho^t \sin t\theta = \frac{1}{12} \times 0.965^t \times \sin(t \times 0.104).$$

We next show that

$\mathbf{A_t} = \rho^{\mathbf{t}} \cos \mathbf{t}\theta$ **solves** $\mathbf{A_{t+2} - pA_{t+1} + qA_t = 0}$ **when** $\mathbf{p^2 - 4q < 0}$

We recall

$$a = \frac{p}{2} \qquad b = \frac{\sqrt{4q - p^2}}{2} \qquad \rho = \sqrt{a^2 + b^2} \qquad \theta = \arccos\frac{a}{\rho}$$

and note that

$$\rho^2 = a^2 + b^2 = \left(\frac{p}{2}\right)^2 + \left(\frac{\sqrt{4q - p^2}}{2}\right)^2 = \frac{p^2}{4} + \frac{4q - p^2}{4} = q \tag{15.26}$$

and that $\cos\theta = \frac{a}{\rho} = \frac{p}{2\rho}$ We also recall a trigonometric identity

$$\cos(x + y) + \cos(x - y) = 2\cos x \cos y$$

Substitute $A_t = \rho^t \cos t\theta$ into $A_{t+2} - pA_{t+1} + qA_t$ and get

$$\begin{aligned}
\rho^{t+2}\cos((t+2)\theta) - p\rho^{t+1}\cos((t+1)\theta) + q\rho^t\cos(t\theta) &= \\
\rho^t \times \Big(\rho^2\cos\left((t+1)\theta + \theta\right) - p\rho\cos((t+1)\theta) + q\cos\left((t+1)\theta - \theta\right)\Big) &= \\
\rho^t \times \Big(\rho^2\cos\left((t+1)\theta + \theta\right) - p\rho\cos((t+1)\theta) + \rho^2\cos\left((t+1)\theta - \theta\right)\Big) &= \\
\rho^t \times \Big(\rho^2\left(\cos\left((t+1)\theta + \theta\right) + \cos\left((t+1)\theta - \theta\right)\right) - p\rho\cos((t+1)\theta)\Big) &= \\
\rho^t \times \Big(\rho^2 2\cos\left((t+1)\theta\right)\cos(\theta) - p\rho\cos((t+1)\theta)\Big) &= \\
\rho^t \times \left(\rho^2 2\cos\left((t+1)\theta\right)\frac{p}{2\rho} - p\rho\cos((t+1)\theta)\right) &= 0 \\
\text{Whew!} &
\end{aligned}$$

We suggest in Exercise 15.5.5 that you show that $A_t = \rho^t \sin t\theta$ also solves $A_{t+2} - pA_{t+1} + qA_t = 0$. Then it will follow from Theorem 15.4.1 that for any two numbers C_1 and C_2 that $A_t = C_1\rho^t \sin t\theta + C_2\rho^t \cos t\theta$ solves $A_{t+2} - pA_{t+1} + qA_t = 0$.

Exercises for Section 15.5, Roots equal to zero, multiple roots, and complex roots.

Exercise 15.5.1

a. Show that if $r_1^2 - pr_1 + q = 0$ and $p^2 - 4q = 0$ then

$$A_t = t \times r_1^t \qquad \text{solves} \qquad A_{t+2} - pA_{t+1} + qA_t = 0$$

You will need to show that

$$(t+2) \times r_1^{t+2} - p \times (t+1) \times r_1^{t+1} + q \times t \times r_1^t = 0 \qquad \text{for all } t$$

Remember that $r_1 = \frac{p}{2}$.

b. As in previous work, $A_t = r_1^t$ solves $A_{t+2} - pA_{t+1} + qA_t = 0$. Use Theorem 15.4.1 to show that for any two numbers, C_1 and C_2

$$A_t = C_1 r_1^t + C_2 \times t \times r_1^t \qquad \text{solves} \qquad A_{t+2} - pA_{t+1} + qA_t = 0 \tag{15.27}$$

c. Show that

$$C_1 = A_0 \qquad \text{and} \qquad C_2 = A_1/r_1 - A_0 \tag{15.28}$$

Exercise 15.5.2 Use formulas 15.24 and 15.25 to show that for the iteration equations

$$\begin{array}{lcl} A_0 = 1 \quad A_{t+1} & = & A_t - B_t \\ B_0 = 0 \quad B_{t+1} & = & A_t + B_t \end{array}$$

$$A_t = \left(\sqrt{2}\right)^t \cos\left(t\frac{\pi}{4}\right) \qquad \text{and} \qquad B_t = \left(\sqrt{2}\right)^t \sin\left(t\frac{\pi}{4}\right)$$

Exercise 15.5.3 Solve the following systems.

a. $\begin{array}{lcllcl} A_0 & = & 1 & A_{t+1} & = & 0.52A_t + 0.04B_t \\ B_0 & = & 2 & B_{t+1} & = & 0.24A_t + 0.4B_t \end{array}$

b. $\begin{array}{lcllcl} A_0 & = & 1 & A_{t+1} & = & 0.3A_t + 0.9B_t \\ B_0 & = & 0 & B_{t+1} & = & 0.2A_t + 0.6B_t \end{array}$

c. $\begin{array}{lcllcl} A_0 & = & 3 & A_{t+1} & = & 0.3A_t - 0.5B_t \\ B_0 & = & 2 & B_{t+1} & = & 0.2A_t + 0.1B_t \end{array}$

d. $\begin{array}{lcllcl} A_0 & = & 2 & A_{t+1} & = & 0.5A_t - 0.1B_t \\ B_0 & = & 3 & B_{t+1} & = & 0.1A_t + 0.3B_t \end{array}$

e. $\begin{array}{lcllcl} A_0 & = & 4 & A_{t+1} & = & 3.0A_t - 0.5B_t \\ B_0 & = & 5 & B_{t+1} & = & 2.0A_t + 1.0B_t \end{array}$

f. $\begin{array}{lcllcl} A_0 & = & 0 & A_{t+1} & = & 0.42A_t + 0.04B_t \\ B_0 & = & 1 & B_{t+1} & = & 0.24A_t + 0.38B_t \end{array}$

g. $\begin{array}{lcllcl} A_0 & = & 0 & A_{t+1} & = & 0.62A_t - 0.08B_t \\ B_0 & = & -1 & B_{t+1} & = & 0.48A_t + 0.18B_t \end{array}$

h. $\begin{array}{lcllcl} A_0 & = & 1 & A_{t+1} & = & 3A_t + 6B_t \\ B_0 & = & 4 & B_{t+1} & = & 2A_t + 4B_t \end{array}$

i. $\begin{array}{lcllcl} A_0 & = & 1 & A_{t+1} & = & 0.5A_t - 1.00B_t \\ B_0 & = & 3 & B_{t+1} & = & 0.1A_t + 0.30B_t \end{array}$

j. $\begin{array}{lcllcl} A_0 & = & 3 & A_{t+1} & = & 0.3A_t + 0.6B_t \\ B_0 & = & 4 & B_{t+1} & = & 0.6A_t + 0.3B_t \end{array}$

Exercise 15.5.4 The following MATLAB code solves Exercise 15.5.3 a. and can be changed to solve any of the other problems. Use the code for a. and write equations for A_t and B_t. Discuss the role of "epsilon." Modify the code to solve each of the other parts of Exercise 15.5.3. The computation of a4 and b4 and `(m^4)`*[a0;b0] is an internal check that will be more apparent after Section 15.6, Matrices.

```
m=[0.5 0.04;0.24 0.4];a0=1;b0=2;
    r1=(p+sqrt(p^2-4*q))/2;
z=m*[a0;b0];a1=z(1);b1=z(2);
p=trace(m);q=det(m); epsilon=norm(m)/10^8;
if p^2-4*q > epsilon
    r2=(p-sqrt(p^2-4*q))/2;
    C1=(a1-r2*a0)/(r1-r2);C2=(r1*a0-a1)/(r1-r2);
    D1=(b1-r2*b0)/(r1-r2);D2=(r1*b0-b1)/(r1-r2);
    a4=C1*r1^4 + C2*r2^4;b4=D1*r1^4 + D2*r2^4;
    [p^2-4*q r1 r2 C1 C2 D1 D2 a4 b4 ]
elseif abs(p^2-4*q) <=epsilon
    r=p/2;
    C1=a0;C2=(a1/r-a0);D1=b0;D2=b1/r-b0;
    a4=C1*r^4+C2*4*r^4;b4=D1*r^4+D2*4*r^4;
    [p^2-4*q r C1 C2 D1 D2 a4 b4]
elseif p^2-4*q < -epsilon
    a=p/2;b=sqrt(4*q-p^2)/2;
    rho=sqrt(a^2+b^2);theta=acos(a/rho);
    C1=a0;C2=(a1-a*a0)/b;D1=b0;D2=(b1-a*b0)/b;
    a4=C1*rho^4*cos(4*theta)+C2*rho^4*sin(4*theta);
    b4=D1*rho^4*cos(4*theta)+D2*rho^4*sin(4*theta);
    [p^2-4*q rho theta C1 C2 D1 D2 a4 b4]
end
(m^4)*[a0;b0]
```

Exercise 15.5.5 For the trigonometrically strong. Show that

$$A_t = \rho^t \sin t\theta \qquad \text{solves} \qquad A_{t+2} - pA_{t+1} + qA_t = 0$$

where

$$p^2 - 4q < 0, \qquad \rho^2 = q \qquad \text{and} \qquad \theta = \arccos\left(\frac{p}{2\rho}\right).$$

You may wish to know that $\sin(x+y) + \sin(x-y) = 2\sin x \cos y$

15.6 Matrices

Matrices greatly simplify study of dynamical systems such as

$$\begin{aligned} A_0 = 200 \quad A_{t+1} &= 0.77 \times A_t + 0.10B_t \\ B_0 = 0 \qquad B_{t+1} &= 0.068 \times A_t + 0.90 \times B_t \end{aligned}$$

A matrix is a rectangular array of numbers. We use only 2 by 2 and 2 by 1 arrays of the form

$$\text{2 by 2} \qquad \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} \qquad \text{and 2 by 1} \qquad \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

For larger matrices, an m by n matrix has m rows and n columns and $a_{i,j}$ is the entry in the ith row and jth column. Always the row dimension or row index is listed first. An example of a 4 by 6 matrix, A, is

$$A = \begin{bmatrix} 85 & 37 & 60 & 61 & -8 & -2 \\ -1 & 70 & -70 & -88 & 37 & 12 \\ 45 & -99 & -70 & 54 & -64 & 93 \\ 67 & 80 & 55 & 34 & -22 & 62 \end{bmatrix} \qquad A_{2,4} = -88$$

Matrices of the same dimension can be added

$$\begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} + \begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{bmatrix} = \begin{bmatrix} a_{1,1}+b_{1,1} & a_{1,2}+b_{1,2} \\ a_{2,1}+b_{2,1} & a_{2,2}+b_{2,2} \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} c_1+d_1 \\ c_2+d_2 \end{bmatrix}.$$

$$\text{But} \qquad \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \qquad \text{is not defined.}$$

A matrix may be multiplied by a number, α.

$$\alpha \times \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} = \begin{bmatrix} \alpha\times a_{1,1} & \alpha\times a_{1,2} \\ \alpha\times a_{2,1} & \alpha\times a_{2,2} \end{bmatrix} \qquad \alpha \times \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \alpha\times c_1 \\ \alpha\times c_2 \end{bmatrix}$$

An n by k matrix A can multiply a k by m matrix, B. The product, $A \times B$ is an n by m matrix. (n by $\mathbf{k}$) $\times$ ($\mathbf{k}$ by m) $= n$ by m.

$$\begin{array}{ccc} \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} \times \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} & = & \begin{bmatrix} a_{1,1}\times c_1 + a_{1,2}\times c_2 \\ a_{2,1}\times c_1 + a_{2,2}\times c_2 \end{bmatrix} \\ (2 \text{ by } \mathbf{2}) \times (\mathbf{2} \text{ by } 1) & = & (2 \text{ by } 1) \end{array}$$

$$\begin{array}{ccc} \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \\ a_{3,1} & a_{3,2} \end{bmatrix} \times \begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{bmatrix} & = & \begin{bmatrix} a_{1,1}\times b_{1,1} + a_{1,2}\times b_{2,1} & a_{1,1}\times b_{1,2} + a_{1,2}\times b_{2,2} \\ a_{2,1}\times b_{1,1} + a_{2,2}\times b_{2,1} & a_{2,1}\times b_{1,2} + a_{2,2}\times b_{2,2} \\ a_{3,1}\times b_{1,1} + a_{3,2}\times b_{2,1} & a_{3,1}\times b_{1,2} + a_{3,2}\times b_{2,2} \end{bmatrix} \\ (3 \text{ by } \mathbf{2}) \times (\mathbf{2} \text{ by } 2) & = & (3 \text{ by } 2) \end{array}$$

Explore 15.6.1 To understand why matrix multiplication is defined the way it is, consider two systems of equations,

$$\begin{aligned} z_1 &= a_{1,1}y_1 + a_{1,2}y_2 \\ z_2 &= a_{2,1}y_1 + a_{2,2}y_2 \end{aligned} \quad \text{and} \quad \begin{aligned} y_1 &= b_{1,1}x_1 + b_{1,2}x_2 \\ y_2 &= b_{2,1}x_1 + b_{2,2}x_2 \end{aligned}$$

and compute the matrix, C, such that

$$\begin{aligned} z_1 &= c_{1,1}x_1 + c_{1,2}x_2 \\ z_2 &= c_{2,1}x_1 + c_{2,2}x_2. \end{aligned}$$ ▬

It is said that 'the rows of the matrix on the left multiply the columns of the matrix on the right.' In order for this to work, the number of columns in the matrix on the left (k) must be the same as the number of rows in the matrix on the right (also k).

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \times \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} \qquad \text{is not defined: } (2 \text{ by } \mathbf{1}) \times (\mathbf{2} \text{ by } 2)$$

The following algebraic rules apply as long as the dimensions of the matrices conform:

$$\begin{aligned} A + (B + C) &= (A + B) + C && \text{Addition is associative.} \\ A + B &= B + A && \text{Addition is commutative.} \\ A \times (B \times C) &= (A \times B) \times C && \text{Multiplication is associative.} \\ A \times (B + C) &= A \times B + A \times C && \text{Mult distributes over addition.} \end{aligned}$$

Missing from the list is $A \times B = B \times A$ and:

In most cases $A \times B$ is not $B \times A$.

For example

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \times \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix} = \begin{bmatrix} 7 & -10 \\ 15 & -22 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix} \times \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} -5 & -6 \\ -9 & -10 \end{bmatrix}$$

The matrix

$$Z = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

is called the **additive identity matrix** or **zero** matrix because for any 2 by 2 matrix, A,

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} = \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix}$$

The matrix

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

is called the **multiplicative identity matrix** or commonly the **identity** matrix because for any 2 by 2 matrix, A,

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} = \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix}$$

The determinant of a 2 by 2 matrix A is denoted by $\det A$ and

$$\det A = \det \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} = a_{1,1}a_{2,2} - a_{1,2}a_{2,1} \tag{15.29}$$

If $\det A \neq 0$ then a matrix denoted by A^{-1} and called **A inverse**, is defined for 2 by 2 matrices by

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} a_{2,2} & -a_{1,2} \\ -a_{2,1} & a_{1,1} \end{bmatrix}. \tag{15.30}$$

An analogous, but more involved formula works for n by n matrices, $n > 2$.

A^{-1} has the property that

$$A^{-1} \times A = A \times A^{-1} = I$$

Explore 15.6.2 Show that

$$\frac{1}{a_{1,1}a_{2,2} - a_{1,2}a_{2,1}} \left[\begin{array}{cc} a_{2,2} & -a_{1,2} \\ -a_{2,1} & a_{1,1} \end{array} \right] \times \left[\begin{array}{cc} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{array} \right] = \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right]$$

■

The **characteristic roots** of a 2 by 2 matrix, $A = [a_{i,j}]$, are the roots to the **characteristic polynomial**

$$r^2 - (a_{1,1} + a_{2,2})r + a_{1,1}a_{2,2} - a_{1,2}a_{2,1} = 0 \tag{15.31}$$

The diagonal of a matrix

$$M = \left[\begin{array}{cc} m_{1,1} & m_{1,2} \\ m_{2,1} & m_{2,2} \end{array} \right]$$

consists of the entries $m_{1,1}$ $m_{2,2}$ and the trace of M is $m_{1,1} + m_{2,2}$. If $m_{2,1} = m_{1,2} = 0$, M is said to be a diagonal matrix. If $m_{2,1} = 0$, M is said to be upper triangular. I suspect you can guess what a lower triangular matrix is.

15.6.1 Iteration equations with matrices.

In the system

$$\begin{array}{llcrcr} A_0 = 200 & A_{t+1} & = & 0.77 \times A_t & + & 0.10 \times B_t \\ & & & & & \\ B_0 = 0 & B_{t+1} & = & 0.068 \times A_t & + & 0.90 \times B_t \end{array}.$$

we let

$$A = \left[\begin{array}{cc} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{array} \right] = \left[\begin{array}{cc} 0.77 & 0.10 \\ 0.068 & 0.90 \end{array} \right] \qquad X_t = \left[\begin{array}{c} A_t \\ B_t \end{array} \right] \qquad t = 0, 1, 2, \cdots$$

Then A is (2 by **2**) and X_t is (**2** by 1) so that $A \times X_t$ is well defined and

$$A \times X_t = \left[\begin{array}{cc} 0.77 & 0.10 \\ 0.068 & 0.90 \end{array} \right] \times \left[\begin{array}{c} A_t \\ B_t \end{array} \right] = \left[\begin{array}{c} 0.77 \times A_t + 0.10 \times B_t \\ 0.068 \times A_t + 0.90 \times B_t \end{array} \right] = \left[\begin{array}{c} A_{t+1} \\ B_{t+1} \end{array} \right] = X_{t+1}$$

The equation

$$X_{t+1} = K \times X_t$$

can be cascaded just like its scalar cousin

$$\begin{aligned} X_1 &= K \times X_0 \\ \\ X_2 &= K \times X_1 = K \times (K \times X_0) \end{aligned}$$

$$= \quad (K \times K) \times X_0 \qquad\qquad \text{Mult is associative.}$$

$$\begin{aligned} X_3 &= K \times X_2 = K \times ((K \times K) \times X_0) \\ &= (K \times K \times K) \times X_0 \qquad\qquad \text{Mult is associative.} \end{aligned}$$

We quite reasonably define for a square matrix, K,

$$K^t = \underbrace{K \times K \times \cdots \times K}_{\text{t factors}}$$

and claim that

$$X_t = K^t \times X_0. \tag{15.32}$$

Exercises for Section 15.6, Matrices.

Exercise 15.6.1 Do Explore 15.6.1.

Exercise 15.6.2 Do Explore 15.6.2.

Exercise 15.6.3 For each matrix, A, compute the characteristic roots of A using Equation 15.31 and compute A^{-1} using Equations 15.29 and 15.30 and compute $A \times A^{-1}$ and $A^{-1} \times A$.

$$\text{a.} \quad \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \qquad \text{b.} \quad \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad \text{c.} \quad \begin{bmatrix} 1 & -3 \\ 2 & 4 \end{bmatrix} \qquad \text{d.} \quad \begin{bmatrix} -1 & -2 \\ -3 & -4 \end{bmatrix}$$

Exercise 15.6.4 Show that the roots of the upper triangular matrix

$$\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$$

are a and b.

Exercise 15.6.5 For

$$A = \begin{bmatrix} 3 & 4 \\ -1 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 6 & 3 \\ 4 & 2 \end{bmatrix} \qquad C = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

use your calculator or a computer or your head to compute the following expressions if the matrices of the expression conform.

a. $A + B$ b. $B + A$ c. $A + C$ d. $C + I$

e. $A + I$ f. $A \times B$ g. $B \times A$ h. $A \times C$

i. $C \times A$ j. $A \times I$ k. $A \times A^{-1}$ l. $B \times B^{-1}$

m. $A^{-1} \times C$ n. $A \times A$ o. A^2 p. $B \times A^{-1}$

q. e^A r. $e^A \times e^{(-A)}$ s. e^I t. $\sin A$

Exercise 15.6.6 Define the matrices

$$A = \begin{bmatrix} 1.2 & -0.6 \\ 0.4 & 0.2 \end{bmatrix} \quad B = \begin{bmatrix} 1.2 & 0.4 \\ 0.8 & 0.4 \end{bmatrix} \quad C = \begin{bmatrix} 2.5 & -2.5 \\ -5.0 & 7.5 \end{bmatrix} \quad L = \begin{bmatrix} 0.8 & 0 \\ 0 & 0.6 \end{bmatrix}$$

a. Compute the characteristic roots of A.

b. Use pencil and paper to show that $B \times C = I$.

c. Use pencil and paper to show that $B \times L \times C = A$.

d. Use pencil and paper to compute L^2.

e. Use pencil and paper to show that $B \times L^2 \times C = A^2$.

f. Show that $B \times L^5 \times C = A^5$.

Exercise 15.6.7 In Problem 15.2.6 for time = 120 minutes, $t = 24$, you found that

$$A_{24} = 8.71 \qquad B_{24} = 14.6$$

Enter

$$D = \begin{bmatrix} 0.77 & 0.10 \\ 0.068 & 0.90 \end{bmatrix} \qquad X = \begin{bmatrix} 200 \\ 0 \end{bmatrix}$$

in your calculator and compute $D^{24} \times X$. You should get $\begin{bmatrix} 8.713 \\ 14.635 \end{bmatrix}$.

Exercise 15.6.8 Equations 15.7 found earlier for A_t and B_t

$$\begin{aligned} A_t &= \tfrac{200}{2.1} \times (0.4 \times 0.94^t + 1.7 \times 0.73^t) \\ B_t &= \tfrac{200}{2.1} \times (0.68 \times 0.94^t - 0.68 \times 0.73^t) \end{aligned}$$

can be written as

$$X_t = \frac{200}{2.1} \begin{bmatrix} 0.4 \\ 0.68 \end{bmatrix} \times 0.94^t + \frac{200}{2.1} \begin{bmatrix} 1.7 \\ -0.68 \end{bmatrix} \times 0.73^t$$

Let

$$D = \begin{bmatrix} 0.77 & 0.10 \\ 0.068 & 0.90 \end{bmatrix}$$

$$E = \begin{bmatrix} 0.40 & 1.70 \\ 0.68 & -0.68 \end{bmatrix} \qquad L = \begin{bmatrix} 0.94 & 0.00 \\ 0.00 & 0.73 \end{bmatrix} \qquad X_0 = \begin{bmatrix} 200 \\ 0 \end{bmatrix}$$

a. Compute the characteristic roots of D (using Equation 15.31).

b. Compute $D\,E - E\,L$.

c. Use algebra to show that $D = E\,L\,E^{-1}$.

d. Use algebra to show that $D^2 = (E\,L\,E^{-1})\,(E\,L\,E^{-1}) = E\,L^2\,E^{-1}$.

e. Show that $D^t = E \times L^t \times E^{-1}$.

f. Use a calculator or computer to compute $D^{24}\, X_0 = X_{24}$ and $E \times L^t \times E^{-1} X_0$. You should get $\begin{bmatrix} 8.713 \\ 14.635 \end{bmatrix}$.

g. Evaluate $\lim_{t\to\infty} D^t = \lim_{t\to\infty} E \times L^t \times E^{-1}$.

Exercise 15.6.9 The MATLAB program shown below left produces the matrices shown in two columns to the right, the the output is not in the correct order. Match the lines of code with the various output entries.

```
close all;clc;clear              154.000            0.682
X=[200; 0];                      13.600
A=[0.77 0.10; 0.068 0.90];
A*X                       8.7135           1.0000   -1.6700  0.6828
A^2*X                            22.7120
A^24*X
eye(2)                             1    0          0.730
A*eye(2)                           0    1          0.940
eye(2)*A
P=poly(A)                        0.770  0.100      1.3116  -0.1457
E=eig(A)                         0.680  0.900     -0.0991   1.1221
det(A)
A^(-1)                          119.940           0.770  0.100
                                 22.712           0.680  0.900
```

Exercise 15.6.10 The two MATLAB programs shown below compute solutions for the iteration

$$\begin{aligned} x_1 &= 2 \qquad x_{t+1} &= a_{1,1}\, x_t + a_{1,2}\, y_t \\ y_1 &= 1 \qquad y_{t+1} &= a_{2,1}\, x_t + a_{2,2}\, y_t \end{aligned}$$

for the matrix

$$A = \begin{bmatrix} 1.1 & -0.4 \\ 0.1 & 0.86 \end{bmatrix}$$

a. The graphs generated by the programs are shown in Figure 15.6. Explain the difference in the two graphs.

b. Compute the characteristic values of A.

c. Modify either PRGM 1 or PRGM 2 to compute the iterations for the following matrices. Compute the characteristic roots of the matrices and interpret the nature of the graphs in terms of the characteristic roots.

$$A = \begin{bmatrix} 1.1 & -0.1 \\ 0.1 & 0.86 \end{bmatrix} \quad A = \begin{bmatrix} 1.1 & -0.4 \\ 0.2 & 0.86 \end{bmatrix} \quad A = \begin{bmatrix} 1.0 & -0.1 \\ 0.2 & 0.86 \end{bmatrix} \quad A = \begin{bmatrix} 1.1 & -0.1 \\ 0.2 & 1.0 \end{bmatrix}$$

`MATLAB PRGM 1`

```
close all;clc;clear
x(1)= 2; y(1) = 1; t(1)=0;
for i = 1:50
  t(i+1) = t(i)+1;
  x(i+1) = 1.1*x(i)-0.4*y(i);
  y(i+1) = 0.1*x(i)+0.86*y(i);
end
plot(t,x,'x','linewidth',2);
hold
plot(t,y,'+','linewidth',2)
```

`MATLAB PRGM 2`

```
close all;clc;clear
x(1)= 2; y(1) = 1; t(1)=0;
for i = 1:50
  t(i+1) = t(i)+1;
  x(i+1) = 1.1*x(i)-0.4*y(i);
  y(i+1) = 0.1*x(i)+0.86*y(i);
end
plot(x,y,'o','linewidth',2)
text(x(1)-0.1,y(1),'S','fontsize',16)
```

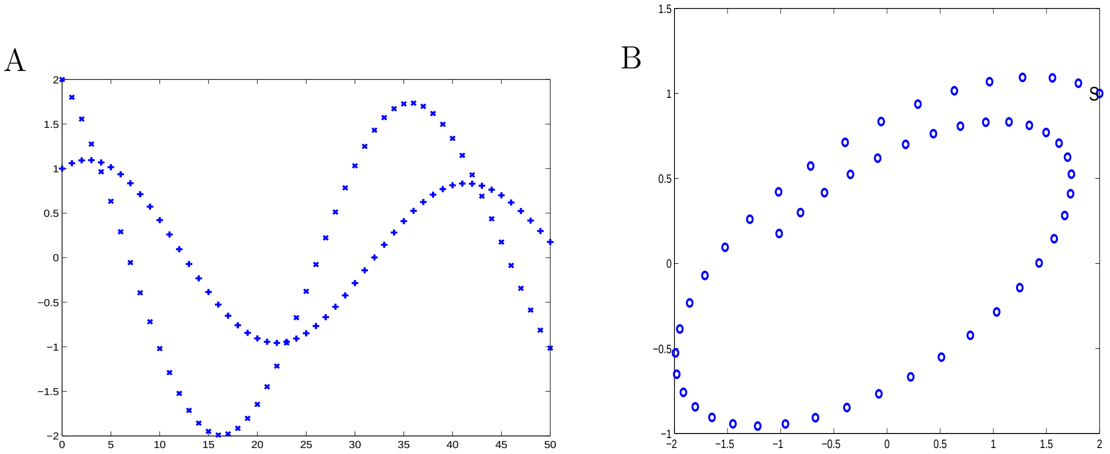


Figure 15.6: Graphs produced by MATLAB PRGM 1 in Exercise 15.6.10. A. Graphs produced by MATLAB PRGM 2 in Exercise 15.6.10.

15.7 Chapter 15 Suggested solutions.

Exercise 15.1.4.

a. The term $\beta \times S_{t-d} \times I_{t-d}$ is the newly infected at time $t-d$, they recover at time t and are the newly recovered at time t.

b. The people infected at time t are those who became infected at previous times and have not recovered. Those people are accounted for by

$$\beta\, S_{t-d+1}\, I_{t-d+1}, \;\; \beta\, S_{t-d+2}\, I_{t-d+2}, \;\; \cdots \;\beta\, S_{t-1}\, I_{t-1}, \quad \text{and} \quad \beta\, S_t\, I_t.$$

The sum of those numbers is I_t.

c. *Assuming* that the number of newly infected is reasonably constant over $t-d$ to t, we might conclude that

$$\beta\, S_{t-d}\, I_{t-d} = \frac{\beta\, S_{t-d+1}\, I_{t-d+1} + \beta\, S_{t-d+2}\, I_{t-d+2} + \cdots + \beta\, S_{t-1}\, I_{t-1} + \beta\, S_t\, I_t}{d} = \frac{1}{d} I_t$$

Exercise 15.1.5.

MATLAB Program To Solve The Model of Epidemics

```
close all;clc;clear
beta=0.00003; gamma=0.2;
S(1)=24950;  %% Susceptible
I(1)=50;     %% Infected
R(1)=0;      %% Recovered
day(1)=0;
for t = 1:70
    S(t+1) = S(t) - beta*S(t)*I(t);
    I(t+1) = I(t) + beta*S(t)*I(t) - gamma*I(t);
    R(t+1) = R(t) + gamma*I(t);
    day(t+1)=t;
end
[S.' I.' R.' day.']
```

TI-86 Program To Solve The Model of Epidemics

```
PROGRAM:INFL
:Fix 03                 Sets window to display 3 digits
:0.00001  -> BETA       The symbol -> is STORE
:{71,4}-> dim C         Table of 71 rows (days) and 4 columns.
:24950 -> C(1,1)        Initial susceptible.  Units are 1000 people.
:50    -> C(1,2)        Initial infected.
:0     -> C(1,3)        Initial recovered.
:0     -> C(I,4)        Day zero.
:For(I,1,70)            Compute for 70 days.
:BETA*C(I,1)*C(I,2) -> NINF     Newly infected on day I.
:0.2*C(I,2) -> NRCR             20% of infected are newly recovered.
:C(I,1)-NINF -> C(I+1,1)        Update susceptible.
:C(I,2)+NINF-NRCR -> C(I+1,2)   Update infected.
```

```
:C(I,3)+NRCR -> C(I+1,3)        Update recovered.
:I-> C(I+1,4):
End
:Pause C                Display data. Use arrows to view data.
:Fix 9                  Reset the display to 9 decimal place.
```

```
    S        I      R     Day
    24950     50     0    0
    24913     77    10    1
    24855    120    25    2
    24765    185    49    3
```

Exercise 15.1.6.

```
   S           I       R       Day
   24999       1       0       0
   24998.25    1.55    0.20    1
   24997.09    2.40    0.51    2
   2.499529    3.72    0.99    3
```

Exercise 15.1.7.

```
   S           I       R        Day
   24999.0       1       0        0
   24998.9     0.925   0.200    1
   24998.8     0.856   0.385    2
   24998.7     0.791   0.556    3
```

Exercise 15.1.8. For $\beta = 0.0000080003$, $I_1 < 1 = I_0$ and for $\beta = 0.0000080004$, $I_1 > 1 = I_0$.

Exercise 15.1.9.

a.

$$\beta_0 = \frac{\gamma}{S_0}.$$

b. In the equation for I_{t+1},

$$I_{t+1} = I_t + \beta\, S_t\, I_t - \gamma I_t \quad = I_t + I_t\,(\beta\, S_t - \gamma)\,.$$

c. $I_1 = I_0$ if and only if $\beta\, S_0 - \gamma = 0$.

d. If $\frac{S_0\beta}{\gamma} > 1$, $S_0\beta > \gamma$ and $S_0\beta - \gamma > 0$ and

$$I_1 = I_0 + \beta\, S_0\, I_0 - \gamma I_0 \quad = I_0 + I_0\,(\beta\, S_0 - \gamma) > I_0.$$

e. $\beta = 0.00008$, the same as in Exercise 15.1.8

Exercise 15.1.10.

a. The height of the epidemic occurs on day 24 when 2347 people are sick.

b. It appears that some of the people do not get sick. On day 87 there are still 2343 susceptible and no infected. However, there is some uncertainty about what the eventual limits will be with high accuracy computing.

c. The number of susceptible turns negative at day 9, and one can assume that all of the people have become sick. For $\beta = 0.00005797$, negatives appear and for $\beta = 0.00005796$ no negatives appear, but zeros appear. For $\beta = 0.00005$ it appears that there are eventually 3 susceptible and no infected.

Exercise 15.1.11.

a. The dimension of β is 1/people. γ is dimensionless.

b. $\beta * = N\beta$ and is dimensionless. $\gamma * = \gamma$ and is dimensionless.

Exercise 15.2.1. The estimated volume is 10 liters.

Exercise 15.2.2. See Figure 15.7.

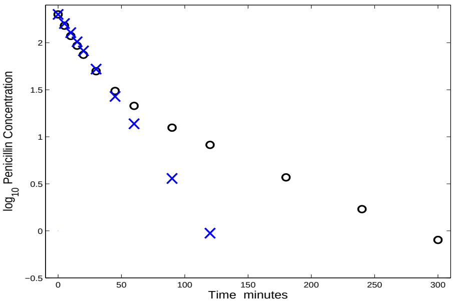


Figure 15.7: **Exercise 15.2.2.** Graph of P_t *vs* t for $P_0 = 200$, $P_{t+1} = 0.8P_t$.

Exercise 15.2.3. $A_3 = 94.6$, $B_3 = 28.6$.

Exercise 15.2.4. $A_3 = 94.6$, $B_3 = 28.6$.

Exercise 15.2.5. $A_{12} = 21.8$ and is close to the observed penicillin value at 60 minutes of 21.3. B_{12} =???.

Exercise 15.2.7. $A_{t+1} - A_t$ is the change in vascular penicillin between times t and $t + 1$. $0.1B_t$ is the flow into the vascular pool from tissue. $(-0.162 - 0.068)A_t$ is flow out of the vascular pool to tissue (-0.162) and removed by the kidneys (-0.068). $B_{t+1} - B_t$ is $\cdots$.

Exercise 15.2.9. Calculator program to solve the iteration equations 15.6.

```
:Fix 1
:.068->P
:.100->Q
:.162->R
:{37,3}->dim C
:200->C(1,1)
:0.0->C(1,2)
:0.0->C(1,3)
:For(I,1,36)
:(1-P-R)*C(I,1)+Q*C(I,2)->C(I+1,1)
:P*C(I,1)+(1-Q)*C(I,2)  ->C(I+1,2)
:I*5->C(I+1,3)
:End
:Pause  C
:Fix 9
```

MATLAB program to solve the iteration equations 15.6.

```
close all;clc;clear
Tplot= [0 5 10 15 20 30 45 60 90 120 180 240 300];
Pplot= [200 151.6 118.0 93.0 74.4 49.8 30.7 21.3 ...
             12.5  8.2  3.7 1.7 0.8];
t=[0:5:300];
a(1)=200;
b(1)=0;
for i = 1:60
    a(i+1)=0.77*a(i)+ 0.1*b(i);
    b(i+1)=0.068*a(i)+0.9*b(i);
end

semilogy(Tplot,Pplot,'o')
hold
semilogy(t,a,'x')
```

```
semilogy(t,b,'+')
```

Exercise 15.2.10.

a. Average Age = $18.63\overline{3}$.

b. At exchange 1, 20 percent of T is removed and the atoms are of age 5 minutes; $0.8 \cdot T$ remains.
At exchange 2, 20 percent of $0.8T$ is removed and the atoms are of age 10 minutes; $0.8 \cdot 0.8T$ remains.
At exchange 3, 20 percent of 0.8^2T is removed and the atoms are of age 15 minutes.
In the three exchanges, $0.2T + 0.2 \cdot 0.8T + 0.2 \cdot 0.8^2T$ atoms are removed.

$$\overline{A}_3 = \frac{0.2T \cdot 5 + 0.2 \cdot 0.8T \cdot 10 + 0.2 \cdot 0.8^2T \cdot 15}{0.2T + 0.2 \cdot 0.8T + 0.2 \cdot 0.8^2T} = 5\,\frac{1 + 0.8 \cdot 2 + 0.8^2 \cdot 3}{1 + 0.8 + 0.8^2}$$

c.

d.

$$P_{20}(x) = 1 + x + x^2 + \cdots + x^{19} + x^{20} \qquad P_{20}'(x) = 1 + 2x + 3x^2 + \cdots + 19x^{18} + 20x^{19}$$

e. Out of the box is more fun.

f.

$$\begin{aligned}\left[\frac{1 - x^{n+2}}{1-x}\right]' &= \frac{(1-x)(-(n+2)x^{n+1}) - (1 - x^{n+2})(-1)}{(1-x)^2} \\ &= \frac{1 - (n+2)x^{n+1} + (n+1)x^{n+2}}{(1-x)^2} \\ P_{20}'(0.8) &= 23.8001, \overline{A}_{20} = 5\frac{P_{20}'(0.8)}{P_{19}(0.8)} = 24.149.\end{aligned}$$

g.

$$\lim_{t\to\infty} t\,a^t = \lim_{t\to\infty} \frac{t}{(1/a)^t} = \lim_{t\to\infty} \frac{t}{[(1/a)^t]'} = \lim_{t\to\infty} \frac{1}{(1/a)^t \ln(1/a)} = 0$$

h.

$$\lim_{n\to\infty} 5\frac{\dfrac{1 - (n+2)x^{n+1} + (n+1)x^{n+2}}{(1-x)^2}}{\dfrac{1 - x^{n+1}}{1-x}} =$$

$$\frac{5}{1-x} \lim_{n\to\infty} \frac{1 - (n+2)x^{n+1} + (n+1)x^{n+2}}{1 - x^{n+1}}$$

$$= \frac{5}{1-x}\,\frac{1 - \lim_{n\to\infty}(n+2)x^{n+1} + \lim_{n\to\infty}(n+1)x^{n+2}}{1 - \lim_{n\to\infty} x^{n+1}}$$

$$= \frac{5}{1-x}\,\frac{1 - 0 + 0}{1 - 0} = \frac{5}{1-x} \quad \text{and} \quad \left.\frac{5}{1-x}\right|_{x=0.8} = 25.$$

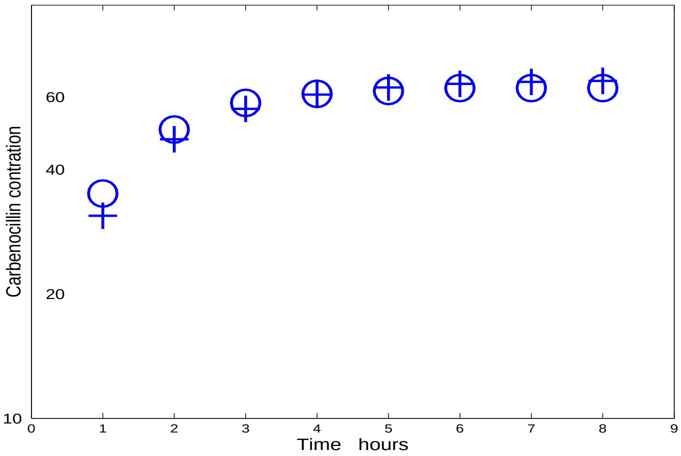


Figure 15.8: Exercise 15.3.3 Observed (o) and computed (+) estimates of Carbenicillin (see Figure 15.3)

Exercise 15.3.2. $A_t = 9 \cdot 0.6^t$

Exercise 15.3.3. a. Data read from the 8-hr curve of Figure 15.3.

Time (hours)	0	1	2	3	4	5	6	7	8	9.5
Carbenicillin	0	35	50	58	61	62	63	63	63	10
Computed	0	31	47	56	61	63	65	65	66	

b. $A_0 = 0$. e. $A_t = 660 - 0.9^t \cdot 660$. h. The linear response suggests a model, $P_{t+1} - P_t = -rP_t$, $P_t = (1-r)^t P_0$. Let $t = 0$ correspond to 0 hours and t index 10 minute intervals. It appears that $P_0 = 63$ and $P_9 = 10$. Then $10 = (1-r)^9 \cdot 63$, $1 - r = \sqrt[9]{10/63}$, $r = 0.185$.

One might assume that the tissue compartment is saturated and slow to release carbenicillin, or that it is saturated and releases it so fast that it immediately equilibrates with the serum.

i. The original linear model encompasses both infusion (the 66) and kidney removal (the -0.1). One could fit the data more accurately with a tissue compartment, but would then be fitting a model with four parameters to 9 data points.

Exercise 15.3.4.

Time (hr)	0	1	2	3	4
Carbenicillin (4 hr curve)	0	65	100	120	125
Computed, 125	0	59	90	106	115
Computed, 135	0	63	97	115	124

We suggest basically the same salt model with higher concentration of salt:

A flask (Serum) has 10 liters of pure water. Each 10 minutes, one liter (10%) of the liquid is removed from the flask, after which one liter of water containing salt with a concentration of 125 grams per liter is added to the flask and the flask is well stirred.

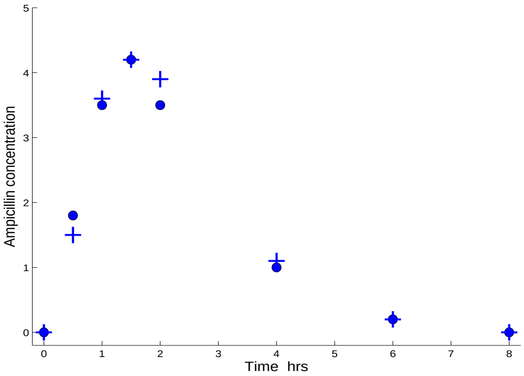
Then $A_t = 1250 - 1250 \cdot 0.9^t$ where t is the index for ten minute intervals. A_0, A_6, A_{12}, A_{18}, and A_{24} are shown in Computed 125.

If you assume that the observed concentration is still increasing at hour 4, you might assume a higher saturation value of, say, 135. Then $A_t = 1350 - 1350 \cdot 0.9^t$ and Computed, 135 is a better fit to the data.

Exercise 15.3.5.

Table 15.2: Exercise 15.3.5 Estimates of ampicillin concentrations read from the dark line graph in Figure 15.4. Graph of that data (dots) and of computed data (+) from the three-compartment model.

Time hr	Conc. g/ml	Comp. g/ml
0	0.0	0.0
0.5	1.8	1.5
1.0	3.5	3.6
1.5	4.2	4.2
2.0	3.5	2.3
4.0	1.0	1.1
6.0	0.2	0.2
8.0	0.0	0.0



c.

$$\begin{array}{llll} A_0 = 21 & A_{t+1} & = & 0.80A_t \\ B_0 = 0 & B_{t+1} & = & 0.20A_t \quad + \quad 0.85B_t \\ C_0 = 0 & C_{t+1} & = & 0.15B_t \quad + \quad 0.70C_t \end{array}$$

e.

$$A_{t+1} = ???? = 0.8A_t$$

$$\begin{aligned} B_{t+1} &= ???? \\ 0.20A_t + 0.85B_t &= ???? \\ C_{t+1} &= 84 \cdot 0.85^{t+1} - 126 \cdot 0.8^{t+1} + 42 \cdot 0.7^{t+1} \\ &= 71.4 \cdot 0.85^t - 100.8 \cdot 0.8^t + 29.4 \cdot 0.7^t \end{aligned}$$

$$\begin{aligned} 0.15B_t + 0.70C_t &= 0.15 \cdot 84(0.85^t - 0.8^t) + \\ &\qquad 0.7 \left(84 \times 0.85^t - 126 \cdot 0.8^t + 42 \cdot 0.7^t\right) \\ &= (0.15 \cdot 84 + 0.7 \cdot 84)0.85^t - (0.15 \cdot 84 + 0.7 \cdot 126)0.8^t \\ &\qquad +0.7 \cdot 42 \cdot 0.7^t \\ &= 71.4 \cdot 0.85^t - 100.8 \cdot 0.8^t + 29.4 \cdot 0.7^t \end{aligned}$$

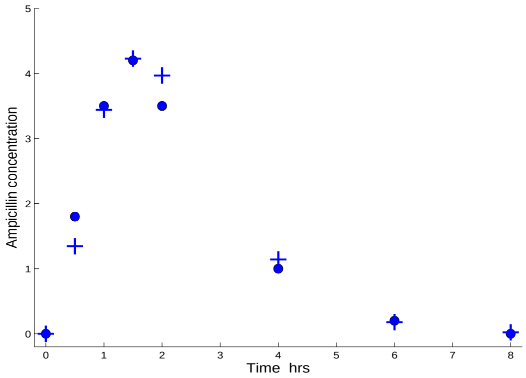
f. See Table 15.2.

Exercise 15.3.6. c.

$$\begin{array}{llll} A_0 = 14 & A_{t+1} & = & 0.80A_t \\ B_0 = 0 & B_{t+1} & = & 0.20A_t \quad + \quad 0.8B_t \\ C_0 = 0 & C_{t+1} & = & 0.2B_t \quad + \quad 0.8C_t \end{array}$$

Table 15.3: Exercise 15.3.6 Estimates of ampicillin concentrations read from the dark line graph in Figure 15.4. Graph of that data (dots) and of computed data (+) from the special three-compartment model.

Time hr	Conc. g/ml	Comp. g/ml
0	0.0	0.0
0.5	1.8	1.3
1.0	3.5	3.4
1.5	4.2	4.2
2.0	3.5	4.0
4.0	1.0	1.1
6.0	0.2	0.2
8.0	0.0	0.0



e.

$$\begin{aligned}
A_{t+1} &= ???? = 0.8A_t \\
B_{t+1} &= ???? = 0.8B_t + 0.2A_t \\
C_{t+1} &= 0.4375(t+1)(t)0.8^{t+1} \\
&= 0.4375t(t-1+2)0.8\cdot 0.8^t \\
&= 0.8\cdot 0.4375t(t-1)0.8^t + 2\cdot 0.4375t0.8\cdot 0.8^t \\
&= 0.8C_t + 0.2\cdot 8\cdot .4375t0.8^t = 0.8C_t + 0.2\cdot 3.5\cdot 0.8^t \\
&= 0.8C_t + 0.2B_t
\end{aligned}$$

Exercise 15.4.2. In Explore 15.4.1 and 15.4.2 we show that 0.94 and 0.73 are the characteristic roots of the system. Then for any numbers, C_1 and C_2, $A_t = C_1 0.94^t + C_2 0.73^t$ solves $A_{t+2} - 1.67A_{t+1} + 0.6862A_t = 0$. We will choose C_1 and C_2 so that $A_0 = 200$ and $A_1 = 154$. Therefore,

$$\begin{aligned}
C_1 0.94^0 + C_2 0.73^0 &= 200 \\
C_1 0.94^1 + C_2 0.73^1 &= 154 \\
0.94C_1 + 0.94C_2 &= 0.94\cdot 200 = 188 \\
C_2(0.94-0.73) &= 188 - 154 = 34 \\
C_2 &= \frac{34}{0.21} = \frac{200\cdot 1.7}{2.1} \\
C_1 &= 200 - C_2 = 200 - \frac{200\cdot 1.7}{2.1} = \frac{200\cdot 0.4}{2.1}
\end{aligned}$$

Exercise 15.4.3. a. $A_t = C\,2^t$, $A_{t+2} - 2.5\,A_{t+1} + A_t = 0$.

$$C\,2^{t+2} - 2.5C\,2^{t+1} + C\,2^t = C\,2^t(2^2 - 2.5 \cdot 2 + 1) = C\,2^t(4 + 5 - 1) = 0$$

c. $A_t = C_1\,0.5^t + C_2\,0.7^t$, $A_{t+2} - 1.2\,A_{t+1} + 0.35\,A_t = 0$.

$$C_1\,0.5^{t+2} + C_2\,0.7^{t+2} - 1.2(C_1\,0.5^{t+1} + C_2\,0.7^{t+1}) + 0.35(C_1\,0.5^t + C_2\,0.7^t) =$$

$$C_1 0.5^t(0.5^2 - 1.2 \cdot 0.5 + 0.35) + C_2 0.7^t(0.7^2 - 1.2 \cdot 0.7 + 0.35) =$$

$$C_1 0.5^t(0.5^2 - 1.2 \cdot 0.5 + 0.35) + C_2 0.7^t(0.7^2 - 1.2 \cdot 0.7 + 0.35) = C_1 \cdot 0 + C_2 \cdot 0 = 0$$

Exercise 15.4.4.

$$\textbf{a}.\quad \begin{array}{rclcrcl} A_0 &=& 10 & & A_{t+1} &=& 0.8A_t + 0.2B_t \\ B_0 &=& 0 & & B_{t+1} &=& 0.1A_t + 0.7B_t \end{array}$$

$$p = 0.8 + 0.7 = 1.5, \quad q = 0.8 \cdot 0.7 - 0.1 \cdot 0.2 = 0.54 \qquad r^2 - 1.5r + 0.54 = 0,$$

$$r_1 = 0.9, \quad r_2 = 0.6 \qquad A_t = C_1\,0.9^t + C_2\,0.6^t$$

$$A_0 = 10, \quad A_1 = 0.8 \cdot 10 + 0.2 \cdot 0 = 8, \qquad C_1 + C_2 = 10, \quad 0.9C_1 + 0.6C_2 = 8,$$

$$C_1 = 20/3, C_2 = 10/3 \qquad A_t = \frac{20}{3}0.9^t + \frac{10}{3}0.6^t$$

$$B_0 = 0, \quad B_1 = 0.1 \cdot 10 + 0.7 \cdot 0 = 1, \qquad B_t = C_1\,0.9^t + C_2\,0.6^t$$

$$C_1 + C_2 = 0, \quad 0.9C_1 + 0.6C_2 = 1, \qquad C_1 = 10/3, C_2 = -10/3 \qquad B_t = \frac{10}{3}0.9^t - \frac{10}{3}0.6^t$$

$$\textbf{c}.\quad \begin{array}{rclcrcl} A_0 &=& 1 & & A_{t+1} &=& 0.26A_t + 0.04B_t \\ B_0 &=& 1 & & B_{t+1} &=& 0.06A_t + 0.24B_t \end{array}$$

$$p = 0.26 + 0.24 = 0.5, \quad q = 0.26 \cdot 0.24 - 0.06 \cdot 0.04 = 0.06 \qquad r^2 - 0.5r + 0.06 = 0,$$

$$r_1 = 0.3, \quad r_2 = 0.2 \qquad A_t = C_1\,0.3^t + C_2\,0.2^t$$

$$A_0 = 1, \quad A_1 = 0.26 \cdot 1 + 0.04 \cdot 1 = 0.3, \qquad C_1 + C_2 = 1, \quad 0.3C_1 + 0.2C_2 = 0.3,$$

$$C_1 = 1, \quad C_2 = 0 \qquad A_t = 0.3^t$$

$$B_0 = 1, \quad B_1 = 0.06 \cdot 1 + 0.24 \cdot 1 = 0.3, \qquad B_t = C_1\,0.3^t + C_2\,0.3^t$$

$$C_1 + C_2 = 1, \quad 0.3C_1 + 0.3C_2 = 0.0074, \qquad C_1 = 1, \quad C_2 = 0 \qquad B_t = 0.3^t$$

Exercise 15.4.5.

$$\textbf{a}.\quad w_0 \;=\; 3 \qquad w_1 \;=\; 1 \qquad w_{t+2} - 5w_{t+1} + 6w_t \;=\; 0$$

$$r^2 - 5r + 6 = 0, \qquad r_1 = 2, \quad r_2 = 3 \qquad w_t = C_1 2^t + C_2 3^t,$$

$$C_1 + C_2 = 3, \quad 2C_1 + 3C_2 = 1, \qquad C_1 = 8, \quad C_2 = -5 \qquad w_t = 8 \cdot 2^t - 5 \cdot 3^t$$

$$\textbf{c}.\quad w_0 = 2 \qquad w_1 = 1 \qquad w_{t+2} - 2w_{t+1} - 8w_t = 0$$

$$r^2 - 2r - 8 = 0, \qquad r_1 = -2, \quad r_2 = 4 \qquad w_t = C_1(-2)^t + C_2 4^t,$$

$$C_1 + C_2 = 2, \quad -2C_1 + 4C_2 = 1, \qquad C_1 = 7/6, \quad C_2 = 5/6$$

$$w_t = (7/6)\cdot(-2)^t + 5/6\cdot 4^t$$

Exercise 15.4.6. Because $q = r_1 \cdot r_2$, if $q = 0$ then one of r_1 or r_2 is zero.

$$A_{t+1} - pA_t = 0, \qquad A_1 \quad \text{known, implies that} \qquad A_t = A_1 p^{t-1}$$

$$\begin{array}{rrclrcl} \textbf{a}. & A_0 & = & 2 & A_{t+1} & = & 0.3A_t + 0.6B_t \\ & B_0 & = & 1 & B_{t+1} & = & 0.2A_t + 0.4B_t \end{array}$$

$$p = 0.7, \quad q = 0.3\cdot 0.4 - 0.2\cdot 0.6 = 0, \qquad A_1 = 0.3\cdot 2 + 0.6\cdot 1 = 1.2,$$

$$A_0 = 2, \quad A_t = 1.2\cdot 0.7^t \quad \text{for} \quad t \geq 1.$$

Exercise 15.4.8.

$$\begin{array}{rclrcl} A_0 & = & \text{Given} & A_{t+1} & = & a_{1,1}A_t + a_{1,2}B_t \\ B_0 & = & \text{Given} & B_{t+1} & = & a_{2,1}A_t + a_{2,2}B_t \end{array}$$

$$\begin{aligned} A_{t+1} &= a_{1,1}A_t + a_{1,2}B_t \\ B_{t+1} &= a_{2,1}A_t + a_{2,2}B_t \\ a_{2,1}A_{t+1} &= a_{2,1}a_{1,1}A_t + a_{2,1}a_{1,2}B_t \\ a_{1,1}B_{t+1} &= a_{1,1}a_{2,1}A_t + a_{1,1}a_{2,2}B_t \\ a_{1,1}B_{t+1} - a_{2,1}A_{t+1} &= a_{1,1}a_{2,2}B_t - a_{2,1}a_{1,2}B_t \\ B_{t+2} &= a_{2,1}A_{t+1} + a_{2,2}B_{t+1} \\ -a_{2,1}A_{t+1} &= -B_{t+2} + a_{2,2}B_{t+1} \\ a_{1,1}B_{t+1} - B_{t+2} + a_{2,2}B_{t+1} &= a_{1,1}a_{2,2}B_t - a_{2,1}a_{1,2}B_t \\ B_{t+2} - (a_{1,1} + a_{2,2})B_{t+1} + (a_{1,1}a_{2,2} - a_{2,1}a_{1,2})B_t &= 0 \\ B_{t+2} - p\,B_{t+1} + q\,B_t &= 0, \end{aligned}$$

where $p = a_{1,1} + a_{2,2}$ and $q = a_{1,1}a_{2,2} - a_{2,1}a_{1,2}$.

Suppose r_1 and r_2 are distinct real roots of $r^2 - p\,r + p = 0$.

Let $B_t = C_1\,r_1^t + C_2\,r_2^t$. Then the algebra is identical to that on page 222.

Then

$$B_t = \frac{r_2\,B_0 - B_1}{r_2 - r_1}\,r_1^t + \frac{B_1 - r_1\,B_0}{r_2 - r_1}\,r_2^t$$

Exercise 15.4.10.

$$\begin{array}{rlrcl} a. & A_0 = 10 & A_{t+1} & = & 0.50A_t + 0.2B_t \\ & B_0 = 0 & B_{t+1} & = & 0.15B_t + 0.7B_t \end{array}$$

$p = 0.5 + 0.7 = 1.2$, $q = 0.5 \cdot 0.7 - 0.15 \cdot 0.2 = 0.32$, $r^2 - 1.2r + 0.32 = 0$,
$r_1 = 0.8$, $r_2 = 0.4$.

$$A_t = C_1\, 0.8^t + C_2\, 0.4^t \quad A_0 = 10, \quad A_1 = 0.50 \cdot 10 + 0.2 \cdot 0 = 5$$

$$C_1 = \frac{A_1 - r_2 A_0}{r_1 - r_2} = \frac{5 - 0.4 \cdot 10}{0.8 - 0.4} = 2.5$$

$$C_2 = \frac{r_1 A_0 - A_1}{r_1 - r_2} = \frac{0.8 \cdot 10 - 5}{0.8 - 0.4} = 7.5$$

$$A_t = 2.5 \cdot 0.8^t + 7.5 \cdot 0.4^t$$

$$B_t = C_1\, 0.8^t + C_2\, 0.4^t \quad B_0 = 0, \quad B_1 = 0.15 \cdot 10 + 0.7 \cdot 0 = 1.5$$

$$C_1 = \frac{B_1 - r_2 B_0}{r_1 - r_2} = \frac{1.5 - 0.4 \cdot 0}{0.8 - 0.4} = 3.75$$

$$C_2 = \frac{r_1 B_0 - B_1}{r_1 - r_2} = \frac{0.8 \cdot 0 - 1.5}{0.8 - 0.4} = -3.75$$

$$B_t = 3.75 \cdot 0.8^t - 3.75 \cdot 0.4^t$$

Exercise 15.5.1.

Suppose $r_1^2 - pr_1 + q = 0$ and $p^2 - 4q = 0$. Let

$$A_t = t \times r_1^t.$$

We will show that A_t solves $A_{t+2} - pA_{t+1} + qA_t = 0$.

Because $p^2 - 4q = 0$, $r_1 = (p \pm \sqrt{p^2 - 4q})/2 = p/2$.

$$\begin{aligned}
A_{t+2} - pA_{t+1} + qA_t &= (t+2) \times r_1^{t+2} - p(t+1) \times r_1^{t+1} + qt \times r_1^t \\
&= r_1^t \left((t+2)\, r_1^2 - p(t+1)r_1 + qt\right) \\
&= r_1^t \left(t\,(r_1^2 - pr_1 + q) + 2r_1^2 - pr_1\right) \\
&= r_1^t(2r_1^2 - pr_1) = r_1^t(2r_1 - p) \\
&= r_1^t \times 0 = 0.
\end{aligned}$$

Theorem 15.4.1 states that if u_t and v_t both solve $A_{t+2} - pA_{t+1} + qA_t = 0$ then for any number pair C_1 and C_2, $C_1\, u_t + C_2\, v_t$ also solves $A_{t+2} - pA_{t+1} + qA_t = 0$. We know that r_1^t and tr_1^t solve $A_{t+2} - pA_{t+1} + qA_t = 0$, so $C_1 r_1^t + C_2 t r_1^t$ solves $A_{t+2} - pA_{t+1} + qA_t = 0$.

Exercise 15.5.2.

$$\begin{aligned}
A_0 = 1 \quad A_{t+1} &= A_t - B_t \\
B_0 = 0 \quad B_{t+1} &= A_t + B_t
\end{aligned}$$

$$p = ????, \quad q = ????, \quad r_1 = ???? \quad r_2 = ????$$

$$a = ???? = 1 \qquad b = ???? = 1 \qquad \rho = ???? = \sqrt{2} \qquad \theta = ???? = \pi/4$$

Recall Equation 15.24,

$$A_t = C_1\rho^t \cos t\theta + C_2\rho^t \sin t\theta \qquad C_1 = ???? \qquad C_2 = \frac{A_1 - aA_0}{b}.$$

$A_1 = ???? = 1$ and $B_1 = ???? = 1$.

$$C_1 = 1 \qquad C_2 = \frac{A_1 - aA_0}{b} = \frac{1 - 1\cdot 1}{1} = 0$$

$$A_t = ???? = (\sqrt{2})^t \cos t\pi/4$$

Recall Equation 15.25,

$$B_t = C_1\rho^t \cos t\theta + B_2\rho^t \sin t\theta \qquad C_1 = B_0 \qquad C_2 = \frac{B_1 - aB_0}{b}.$$

$$C_1 = B_0 = 0, C_2 = \frac{B_1 - aB_0}{b} = \frac{1 - a\cdot 0}{1} = 1.$$

$$B_t = ???? = (\sqrt{2})^t \sin t\pi/4.$$

Exercise 15.5.3.

$$\begin{array}{lllll} \mathbf{a}.\ A_0 & = & 1 & A_{t+1} & = & 0.5A_t + 0.04B_t \\ B_0 & = & 2 & B_{t+1} & = & 0.24A_t + 0.4B_t \end{array}$$

$$p = 0.9, \quad q = 0.1904, \qquad r^2 - 0.9r + 0.1904 = 0$$

$$r_1 = 0.56, \qquad r_2 = 0.34$$

$$A_1 = 0.50\cdot 1 + 0.04\cdot 2 = 0.58, \qquad B_1 = 0.24\cdot 1 + 0.4\cdot 2 = 1.04.$$

$$\begin{array}{lllll} C_1 & = & \frac{A_1 - r_2A_0}{r_1 - r_2} & = & \frac{0.58 - 0.34\cdot 1}{0.22} = 12/11 \\ C_2 & = & \frac{r_1A_0 - A_1}{r_1 - r_2} & = & \frac{0.56\cdot 1 - 0.58}{0.22} = -1/11 \\ D_1 & = & \frac{B_1 - r_2B_0}{r_1 - r_2} & = & \frac{1.04 - 0.34\cdot 2}{0.22} = 18/11 \\ D_2 & = & \frac{r_1B_0 - B_1}{r_1 - r_2} & = & \frac{0.56\cdot 2 - 1.04}{0.22} = 4/11 \\ & & A_t & = & \tfrac{12}{11}\,0.56^t - \tfrac{1}{11}\,0.34^t \\ & & B_t & = & \tfrac{18}{11}\,0.56^t + \tfrac{4}{11}\,0.34^t \end{array}$$

$$\begin{array}{llllll} \mathbf{c}.\ A_0 & = & 3 & A_{t+1} & = & 0.3A_t - 0.5B_t \\ B_0 & = & 2 & B_{t+1} & = & 0.2A_t + 0.1B_t \end{array}$$

$$p = 0.4, \quad q = 0.13, \quad p^2 - 4q = -0.36 < 0$$

$$r_1 = a + b\,i = 0.2 + 0.3\,i, \quad r_2 = 0.2 - 0.3i,$$

$$\rho = \sqrt{a^2+b^2} = \sqrt{0.13} \qquad \theta = \arccos\frac{a}{\rho} = \arccos\frac{0.2}{\sqrt{13}}$$

$$A_1 = 0.3*3 - 0.5*2 = -0.1 \qquad B_1 = 0.2*3 + 0.1*2 = 0.8$$

$$C_1 = A_0 = 3 \qquad C_2 = \frac{A_1 - aA_0}{b} = \frac{-0.1-0.2\cdot 3}{0.3} = -7/3$$

$$D_1 = B_0 = 2 \qquad D_2 = \frac{B_1 - aB_0}{b} = \frac{0.8-0.2\cdot 2}{0.3} = 4/3$$

$$\begin{aligned} A_t &= C_1\rho^t\cos t\theta + C_2\rho^t\sin t\theta &=& \; 3(\sqrt{13})^t\cos(t\arccos\frac{0.2}{\sqrt{13}}) \\ &&& -\tfrac{7}{3}(\sqrt{13})^t\sin(t\arccos\frac{0.2}{\sqrt{13}}) \end{aligned}$$

$$\begin{aligned} B_t &= D_1\rho^t\cos t\theta + D_2\rho^t\sin t\theta &=& \; 2(\sqrt{0.13})^t\cos(t\arccos\frac{0.2}{\sqrt{13}}) \\ &&& +\tfrac{4}{3}(\sqrt{0.13})^t\sin(t\arccos\frac{0.2}{\sqrt{13}}) \end{aligned}$$

e.
$$\begin{aligned} A_0 &= 4 & A_{t+1} &= 3.0A_t - 0.5B_t \\ B_0 &= 5 & B_{t+1} &= 2.0A_t + 1.0B_t \end{aligned}$$

$$A_t = 4\cdot 2^t + 0.75\cdot t\cdot 2^t \qquad B_t = 5\cdot 2^t + 1.5\cdot t\cdot 2^t$$

g.
$$\begin{aligned} A_0 &= 0 & A_{t+1} &= 0.62A_t - 0.08B_t \\ B_0 &= -1 & B_{t+1} &= 0.48A_t + 0.18B_t \end{aligned}$$

$$A_t = 0.4\cdot 0.5^t - 0.4\cdot 0.3^t \qquad B_t = 0.6\cdot 0.5^t - 1.6\cdot 0.3^t$$

i.
$$\begin{aligned} A_0 &= 1 & A_{t+1} &= 0.5A_t - 1.00B_t \\ B_0 &= 3 & B_{t+1} &= 0.1A_t + 0.30B_t \end{aligned}$$

$$r = 0.4+0.3i \quad a = 0.4 \quad b = 0.3 \quad \rho = 0.5 \quad \theta = \arccos\frac{a}{\rho} = \arccos 0.8 = 0.6435$$

$$A_1 = -2.5 \quad C_1 = 1 \quad C_2 = \frac{-2.5-0.4\cdot 1}{0.3} = 9\frac{2}{3}$$

$$B_1 = 1 \quad D_1 = 3 \quad D_2 = -6/3$$

$$A_t = -2.5\cdot 0.5^t\cos(t\cdot 0.6435) - \frac{29}{3}\sin(t\cdot 0.6435)$$

$$B_t = 3\cdot 0.5^t\cos(t\cdot 0.6435) - \frac{29}{3}\sin(t\cdot 0.6435)$$

Exercise 15.5.4. For a. the output is

```
0.0484 0.5600 0.3400 1.0909 -0.0909 1.6364 0.3636 0.1061 0.1658
```

$$
\begin{aligned}
A_t &= 1.0909\,0.56^t - 0.0909\,0.34^t \\
B_t &= 1.634\,0.56^t + 0.3636\,0.34^t
\end{aligned}
$$

For c. the output is

```
-0.3600 0.3606 0.9828 3.0000 -2.3333 2.0000 1.3333 -0.0077 -0.0398
```

$$
\begin{aligned}
A_t &= C_1\rho^t \cos t\theta + C_2\rho^t \sin t\theta \\
&= 3(0.3606)^t \cos(t\,0.9828) - 2.3333(0.3606)^t \sin(t\,0.9828).
\end{aligned}
$$

Exercise 15.5.5. Substitute $A_t = \rho^t \sin t\theta$ into $A_{t+2} - pA_{t+1} + qA_t$ and get

$$
\begin{aligned}
\rho^{t+2}\sin((t+2)\theta) - p\rho^{t+1}\sin((t+1)\theta) + q\rho^t\sin(t\theta) &= \\
\rho^t \times \Big(\rho^2 \sin\left((t+1)\theta + \theta\right) - p\rho\sin((t+1)\theta) + q\sin\left((t+1)\theta - \theta\right)\Big) &= \\
\rho^t \times \Big(\rho^2 \sin\left((t+1)\theta + \theta\right) - p\rho\sin((t+1)\theta) + \rho^2\sin\left((t+1)\theta - \theta\right)\Big) &= \\
???? &= \\
???? &= \\
\rho^t \times \left(\rho^2 2\sin\left((t+1)\theta\right)\frac{p}{2\rho} - p\rho\sin((t+1)\theta)\right) &= 0
\end{aligned}
$$

Exercise 15.6.3.

a. $p = 4 \quad q = 1 \quad r_1 = 2 + \sqrt{15}/2 \quad r_2 = 2 - \sqrt{15}/2 \quad A^{-1} = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}.$

c. $p = 5 \quad q = 10 \quad r_1 = 2.5 + i\cdot\sqrt{15}/2 \quad r_2 = 2.5 - i\cdot\sqrt{15}/2 \quad A^{-1} = \begin{bmatrix} 0.4 & 0.3 \\ -0.2 & 0.1 \end{bmatrix}.$

Exercise 15.6.4.

$$
\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \qquad r^2 - (a+d)\,r + ad = (r-a)(r-b) = 0, \quad r_1 = a, \quad r_2 = b
$$

Exercise 15.6.5.

a. $A+B=\left[\begin{array}{cc} 9 & 7 \\ 3 & 4 \end{array}\right]$ c. $A+C=\text{Zip}$ e. $A+I=\left[\begin{array}{cc} 4 & 4 \\ -1 & 3 \end{array}\right]$

g. $B\times A=\left[\begin{array}{cc} 15 & 30 \\ 10 & 20 \end{array}\right]$ i. $C\times A=\text{Zip}$ k. $A\times A^{-1}=\left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right]$

m. $A^{-1}\times C=\left[\begin{array}{c} 0.8 \\ -0.1 \end{array}\right]$ o. $A^2=\left[\begin{array}{cc} 5 & 20 \\ -5 & 0 \end{array}\right]$ q. $e^A=\left[\begin{array}{cc} -1.419 & 23.50 \\ -5.875 & -7.24 \end{array}\right]$

s. $e^I=\left[\begin{array}{cc} e & 0 \\ 0 & e \end{array}\right]$

Exercise 15.6.6.

a.

$$p=1.4 \quad q=0.48 \quad r^2-1.4r+q=(r-0.8)(r-0.6) \quad r_1=0.8 \quad r_2=0.6$$

and these are the diagonal terms of L.

b.

$$\begin{aligned} B\,C &= \left[\begin{array}{cc} 1.2 & 0.4 \\ 0.8 & 0.4 \end{array}\right]\left[\begin{array}{rr} 2.5 & -2.5 \\ -5.0 & 7.5 \end{array}\right] \\ &= \left[\begin{array}{cc} 1.2(2.5)+0.4(-5) & 1.2(-2.5)+0.4(7.5) \\ 0.8(2.5)+0.4(-5) & 0.8(-2.5)+0.4(7.5) \end{array}\right]=\left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right] \end{aligned}$$

c.

$$\begin{aligned} B\times L\times C &= \left[\begin{array}{cc} 1.2 & 0.4 \\ 0.8 & 0.4 \end{array}\right]\left[\begin{array}{rr} 0.8 & 0 \\ 0 & 0.6 \end{array}\right]\left[\begin{array}{rr} 2.5 & -2.5 \\ -5.0 & 7.5 \end{array}\right] \\ &= \left[\begin{array}{cc} 1.2(0.8)+0.4(0) & 1.2(0)+0.4(0.6) \\ 0.8(0.8)+0.4(0) & 0.8(0)+0.4(0.6) \end{array}\right]\left[\begin{array}{rr} 2.5 & -2.5 \\ -5.0 & 7.5 \end{array}\right] \\ &= \left[\begin{array}{cc} 0.96 & 0.24 \\ 0.64 & 0.24 \end{array}\right]\left[\begin{array}{rr} 2.5 & -2.5 \\ -5.0 & 7.5 \end{array}\right]= \end{aligned}$$

$$\left[\begin{array}{rr} 0.96(2.5)+0.24(-5.0) & 0.96(-2.5)+0.24(7.5) \\ 0.64(2.5)+(0.24)(-5.0) & 0.64(-2.5)+0.24(7.5) \end{array}\right]=\left[\begin{array}{rr} 1.2 & -0.6 \\ 0.4 & 0.2 \end{array}\right]=A$$

d.

$$\begin{aligned} L^2 &= \left[\begin{array}{rr} 0.8 & 0 \\ 0 & 0.6 \end{array}\right]\left[\begin{array}{rr} 0.8 & 0 \\ 0 & 0.6 \end{array}\right] \\ &= \left[\begin{array}{cc} 0.8(0.8)+0(0) & 0.8(0)+0(0.6) \\ 0(0.8)+0.6(0) & 0(0)+0.6(0.6) \end{array}\right]=\left[\begin{array}{rr} 0.64 & 0 \\ 0 & 0.36 \end{array}\right] \end{aligned}$$

e.

$$B \times L^2 \times C = \begin{bmatrix} 1.2 & 0.4 \\ 0.8 & 0.4 \end{bmatrix} \begin{bmatrix} 0.64 & 0 \\ 0 & 0.36 \end{bmatrix} \begin{bmatrix} 2.5 & -2.5 \\ -5.0 & 7.5 \end{bmatrix} =$$

$$\begin{bmatrix} 1.2(0.64) + 0.4(0) & 1.2(0) + 0.4(0.36) \\ 0.8(0.64) + 0.4(0) & 0.8(0) + 0.4(0.36) \end{bmatrix} \begin{bmatrix} 2.5 & -2.5 \\ -5.0 & 7.5 \end{bmatrix} =$$

$$\begin{bmatrix} 0.768 & 0.144 \\ 0.512 & 0.144 \end{bmatrix} \begin{bmatrix} 2.5 & -2.5 \\ -5.0 & 7.5 \end{bmatrix} =$$

$$\begin{bmatrix} 0.76(2.5) + 0.144(-5.0) & 0.76(-2.5) + 0.144(7.5) \\ 0.76(2.5) + (0.144)(-5.0) & 0.76(-2.5) + 0.144(7.5) \end{bmatrix} = \begin{bmatrix} 1.2 & -0.84 \\ 0.56 & -0.2 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1.2 & -0.6 \\ 0.4 & 0.2 \end{bmatrix} \begin{bmatrix} 1.2 & -0.6 \\ 0.4 & 0.2 \end{bmatrix} =$$

$$\begin{bmatrix} 1.2(1.2) + (-0.6)(0.4) & 1.2(-0.6) + (-0.6)(0.2) \\ 0.4(1.2) + 0.2(0.4) & 0.4(-0.6) + 0.2(0.2) \end{bmatrix} = \begin{bmatrix} 1.2 & -0.84 \\ 0.56 & -0.2 \end{bmatrix}$$

f. Using a calculator

$$B \times L^5 \times C = \begin{bmatrix} 0.82752 & -0.74976 \\ 0.49984 & -0.42208 \end{bmatrix}$$

and

$$A^5 = \begin{bmatrix} 0.82752 & -0.74976 \\ 0.49984 & -0.42208 \end{bmatrix}$$

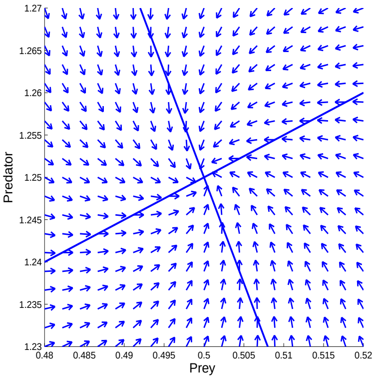
Chapter 16

Nonlinear Dynamical Systems; Stable and Unstable Equilibria

Where are we going?

Models of symbiotic, predator-prey, and competing species involve quadratic terms and are therefore nonlinear. Equilibrium points and the distinction between stable and unstable equilibrium points are crucial to understanding the biology of dynamical systems.

The phase plane for a predator-prey model shows a general movement toward the equilibrium point at (0.5,1.25), but tends to spiral around it.



16.1 Equilibria of Pairs of Difference Equations.

Penicillin kinetics. Suppose you initiate constant penicillin infusion in a patient with a two liter serum pool at the rate of 64.8 milligrams every five minutes. What will be the serum penicillin concentration as time progresses? (Does your patient explode with too much penicillin?) Assume the kinetics of penicillin are as in Equations 15.6

$$\begin{array}{llcl} A_0 = 0 & A_{t+1} - A_t & = & -0.23 \times A_t + 0.10 \times B_t \\ B_0 = 0 & B_{t+1} - B_t & = & 0.068 \times A_t - 0.10 \times B_t \end{array} \qquad (16.1)$$

In these equations, A_t denotes serum penicillin concentration in milligrams per μliter and B_t denotes tissue penicillin concentration in milligrams per μliter in a 'tissue' compartment.

Equations 16.1 account for body processing of penicillin, but do not account for your infusion of 64.8 milligrams (32.4 μg/ml-serum) into the serum. Because all of the infused penicillin enters the serum pool, we write a model:

> **Mathematical Model.** Every five minutes, 6.8% of the serum penicillin diffuses to tissue, 10% of the tissue penicillin diffuses to serum, the kidneys remove 16.2 (=23 - 6.8) percent of the serum penicillin, and 32.4 μg/ml-serum penicillin is added to the serum.

We assume your patient had no penicillin in his body when you initiated treatment, and write:

Initial Conditions		**InternalBody Kinetics**	**Infusion**
$A_0 = 0$	$A_{t+1} - A_t$	$= -(0.162 + 0.068) \times A_t + 0.10 \times B_t \ +$	32.4
$B_0 = 0$	$B_{t+1} - B_t$	$= 0.068 \times A_t - 0.10 \times B_t$	

(16.2)

The **equilibrium** state of your patient, if such exists, will be described by two numbers, A_e and B_e, measured in milligrams, such that

$$A_t \rightarrow A_e \qquad \text{and} \qquad B_t \rightarrow B_e \qquad \text{as} \qquad t \rightarrow \infty$$

Explore 16.1.1 Assume that your patient has reached an equilibrium state so that the levels of serum and tissue penicillin are constants, A_e and B_e. Which of the following would you expect,

$$A_e < B_e, \qquad A_e = B_e, \qquad \text{or} \qquad A_e > B_e?$$

■

Because $A_t \rightarrow A_e$, it must be that $A_{t+1} \rightarrow A_e$ also, and similarly $B_t \rightarrow B_e$ implies $B_{t+1} \rightarrow B_e$. From the kinetic equations, we can write

$$\begin{aligned} A_e - A_e &= -0.23 A_e + 0.1 B_e + 32.4 \\ B_e - B_e &= 0.068 A_e - 0.10 B_e \end{aligned}$$

$$\begin{aligned} 0 &= -0.23 A_e + 0.1 B_e + 32.4 \\ 0 &= 0.068 A_e - 0.10 B_e \end{aligned}$$

$$\begin{aligned} A_e &= 200 \quad \text{milligrams} \\ B_e &= 136 \quad \text{milligrams} \end{aligned}$$

The penicillin *concentrations* at equilibrium in the two compartments are

$$AC_e = \frac{200 \quad \text{milligrams}}{1 \quad \text{liter}} = 200\ \mu\text{g/ml}, \qquad AC_e = \frac{136 \quad \text{milligrams}}{0.68 \quad \text{liter}} = 200\ \mu\text{g/ml},$$

and are equal.

Exercises for Section 16.1, Equilibria of Pairs of Difference Equations.

Exercise 16.1.1 Determine the equilibrium state, A_e,B_e, of your patient for each of the following modifications of the penicillin system. Contrast these results with the results of the original model and discuss how they affect your thinking about treatment of your patient.

a. Your patient had been receiving penicillin before your treatment so that $A_0 = 100$ and $B_0 = 68$.

b. Your patient's kidneys are only 50% effective so that every five minutes they remove 8.1% of the penicillin in the serum (and all other kinetics are the same).

c. The model of penicillin kinetics is

> **Mathematical Model.** Every five minutes, 15% of the serum penicillin diffuses to tissue, 10% of the tissue penicillin diffuses to serum, the kidneys remove 8 percent of the serum penicillin, and 32 milligrams of penicillin are added to the serum.

Exercise 16.1.2 Consider the SIR equations for epidemics.

$$\begin{aligned} S_{t+1} &= S_t - \beta \times S_t \times I_t \\ I_{t+1} &= I_t + \beta \times S_t \times I_t - \gamma \times S_{t-d} \times I_{t-d} \\ R_{t+1} &= R_t + \gamma \times S_{t-d} \times I_{t-d}. \end{aligned}$$

Find the possible equilibrium values (S_e, I_e, R_e).

16.2 Stability of the equilibria of linear systems.

There is a radical difference between

$$\text{Case 1.} \qquad A_t = 5 \times \left(\frac{2}{3}\right)^t + 8 \times \left(\frac{4}{5}\right)^t \qquad \text{for which} \qquad \lim_{t\to\infty} A_t = 0$$

$$\text{Case 2.} \qquad A_t = 5 \times \left(\frac{3}{2}\right)^t + 8 \times \left(\frac{4}{5}\right)^t \qquad \text{for which} \qquad \lim_{t\to\infty} A_t = \infty$$

If A_t is the post treatment amount of penicillin in your patient, you expect Case 1 (at least not Case 2). If A_t is the size of your fish population, you hope for Case 2 (at least not Case 1).

Definition 16.2.1 An equilibrium point of a linear dynamical system

$$\begin{aligned} A_0 &= \text{ Given } \quad A_{t+1} = m_{1,1}A_t + m_{1,2}A_t + H_1 \\ B_0 &= \text{ Given } \quad B_{t+1} = m_{2,1}B_t + m_{2,2}B_t + H_2 \end{aligned} \tag{16.3}$$

is a point (A_e, B_e) such that

$$\begin{aligned} A_e &= m_{1,1}A_e + m_{1,2}A_e + H_1 \\ B_e &= m_{2,1}B_e + m_{2,2}B_e + H_2 \end{aligned} \tag{16.4}$$

The question of this section is whether (A_e, B_e) is *stable*.

Definition 16.2.2 An equilibrium point (A_e, B_e) of the linear dynamical system 16.3 is *stable* means that for *any* (A_0, B_0) the sequences $A_0, A_1, A_2, \cdots$ and $B_0, B_1, B_2 \cdots$ defined by Equations 16.3 converge respectively to A_e and B_e.

By subtracting Equations 16.4 from Equations 16.3 we get

$$A_{t+1} - A_e = m_{1,1} \times (A_t - A_e) + m_{1,2} \times (B_t - B_e)$$

$$B_{t+1} - B_e = m_{2,1} \times (A_t - A_e) + m_{2,2} \times (B_t - B_e).$$

With the substitution $x_t = A_t - A_e$ and $y_t = B_t - B_e$, we get

$$\begin{aligned} x_{t+1} &= m_{1,1}x_t + m_{1,2}x_t \\ y_{t+1} &= m_{2,1}y_t + m_{2,2}y_t \end{aligned} \tag{16.5}$$

Then $(A_t, B_t) \to (A_e, B_e)$ of Equations 16.3 if and only if $(x_t, y_t) \to (0, 0))$ or, (A_e, B_e) is a stable equilibrium point of Equations 16.3 if and only if (0,0) is a stable equilibrium point of the simpler set of Equations 16.5. The Equations 16.5 are said to be *homogeneous* where as Equations 16.3 are *nonhomogeneous* if $H_1 \neq 0$ or $H_2 \neq 0$.

The following four examples demonstrate the primary dynamics of two first order difference equations near equilibrium points. In all four cases, the origin is the equilibrium point, the initial conditions, $x_0 = 0.25$ and $y_0 = 0.5$ are identical, and the equation $y_{n+1} = 0.1 \times x_n + 0.86 \times y_n$ are the same. Furthermore, the coefficients in the equations for x_{n+1} are similar, but methodically changed between examples.

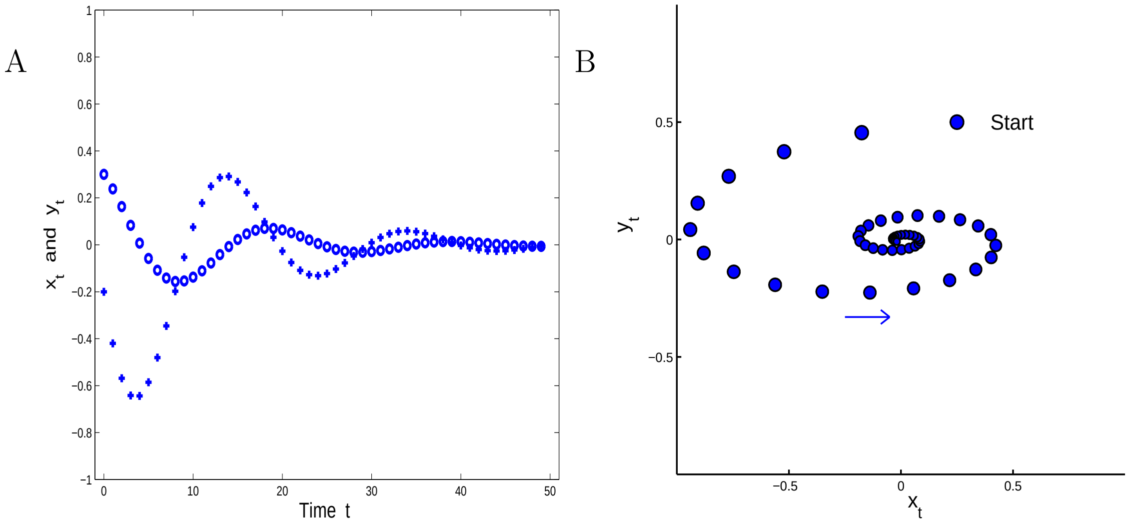


Figure 16.1: Trajectories of (x_t, y_t) for the linear dynamical systems A and B in Equations 16.6. The origin, (0,0), is a stable equilibrium for both of these systems.

$$
\begin{array}{lllllll}
\text{A.} & x_0 & = & 0.25 & x_{n+1} & = & 0.9 \times x_n + 0.04 \times y_n \\
 & y_0 & = & 0.5 & y_{n+1} & = & 0.1 \times x_n + 0.86 \times y_n \\
 & & & & & & \\
\text{B.} & x_0 & = & 0.25 & x_{n+1} & = & 0.9 \times x_n - 0.8 \times y_n \\
 & y_0 & = & 0.5 & y_{n+1} & = & 0.1 \times x_n + 0.86 \times y_n \\
 & & & & & & \\
\text{C.} & x_0 & = & 0.25 & x_{n+1} & = & 0.9 \times x_n + 0.4 \times y_n \\
 & y_0 & = & 0.5 & y_{n+1} & = & 0.1 \times x_n + 0.86 \times y_n \\
 & & & & & & \\
\text{D.} & x_0 & = & 0.25 & x_{n+1} & = & 1.15 \times x_n - 0.8 \times y_n \\
 & y_0 & = & 0.5 & y_{n+1} & = & 0.1 \times x_n + 0.86 \times y_n
\end{array}
\qquad (16.6)
$$

The graphs in Figure 16.1 are *trajectories in the phase plane of* (x_t, y_t), plots of y_t versus x_t, a common graph when looking at equilibria. You may be more familiar with graphs of x_t versus t and y_t versus t, and the graph in Figure 16.2 shows such a plot for Equations 16.6B.

In Examples A and B, the trajectories move towards the equilibrium point (0,0). From all initial conditions the trajectories converge to (0,0), as suggested by the phase plane plot in Figure 16.3. Note that following the arrows from any point in the plane will lead to (0,0); (0,0) is a stable equilibrium point. In Examples C and D the trajectories diverge to infinity, and will do so for any starting point different from (0,0). For Examples C and D the equilibrium point (0,0) is not stable. All paths not starting at exactly (0,0) spiral outward in Figure 16.3. Because (0,0) is an equilibrium point of all four dynamical systems of Equations 16.6, any trajectory that starts at (0,0) stays at (0,0) for all four of them. The question of stability is whether trajectories that start at a point different from the equilibrium point (0,0) converge to (0,0).

Unstable equilibria rarely occur in nature because there are enough random perturbations of biological systems to insure that even if a system is exactly in an unstable equilibrium state, it will soon

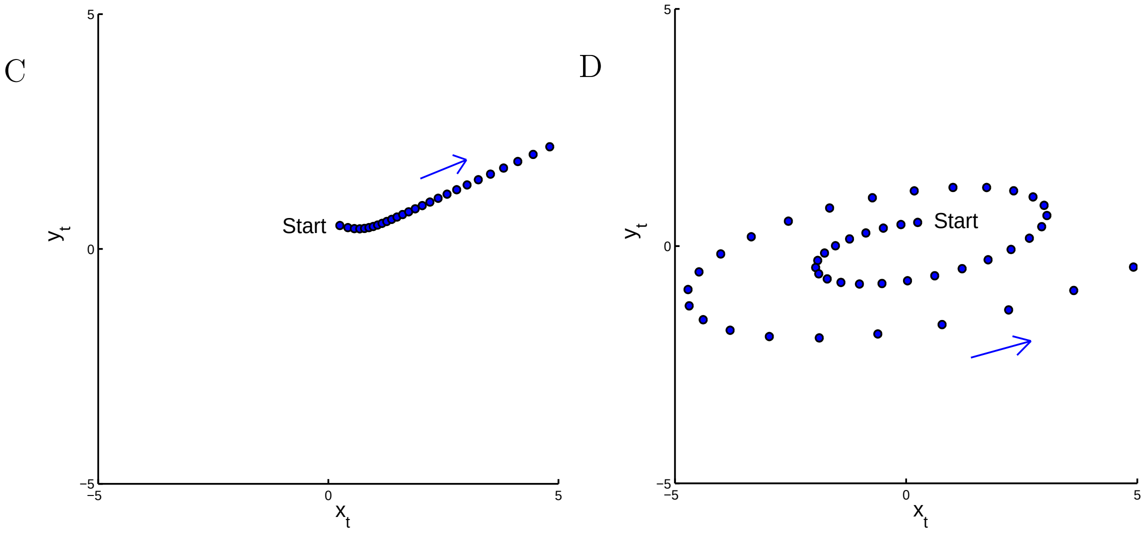


Figure 16.1: (Continued) Trajectories of (x_t, y_t) for the linear dynamical systems C and D in Equations 16.6. Both of these systems are unstable, $(x_t, y_t) \to (\infty, \infty)$.

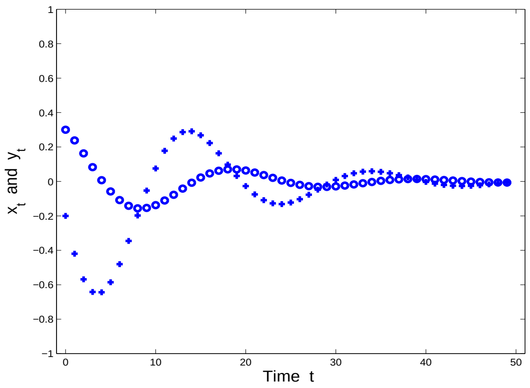


Figure 16.2: Graphs of x_t (+) and y_t (o) for the dynamical system of Equations 16.6B.

be perturbed and move away from the unstable equilibrium.

Stable equilibria, however, are very important in nature. Once a system is at or near a stable equilibrium, it will tend to return to or stay near the equilibrium condition when random perturbations move it a small distance away from equilibrium.

Trajectories A and C differ noticeably from trajectories B and D, which spiral around the equilibrium point. The spirals are due to the complex roots of the characteristic polynomials associated with the systems of Examples B and D.

What happens to $\mathbf{x_t}$ as $\mathbf{t} \to \infty$?

The characteristic equation (see Equation 15.16) of the linear dynamical system, Equations 16.5, is

$$z^2 - p \times z + q = 0, \quad \text{where} \quad p = m_{1,1} + m_{2,2} \quad \text{and} \quad q = m_{1,1}m_{2,2} - m_{2,1}m_{1,2} \tag{16.7}$$

and x_t and y_t satisfy the second order linear difference equations

$$x_{t+2} - p \times x_{t+1} + qx_t = 0 \qquad \text{and} \qquad y_{t+2} - p \times y_{t+1} + qy_t = 0.$$

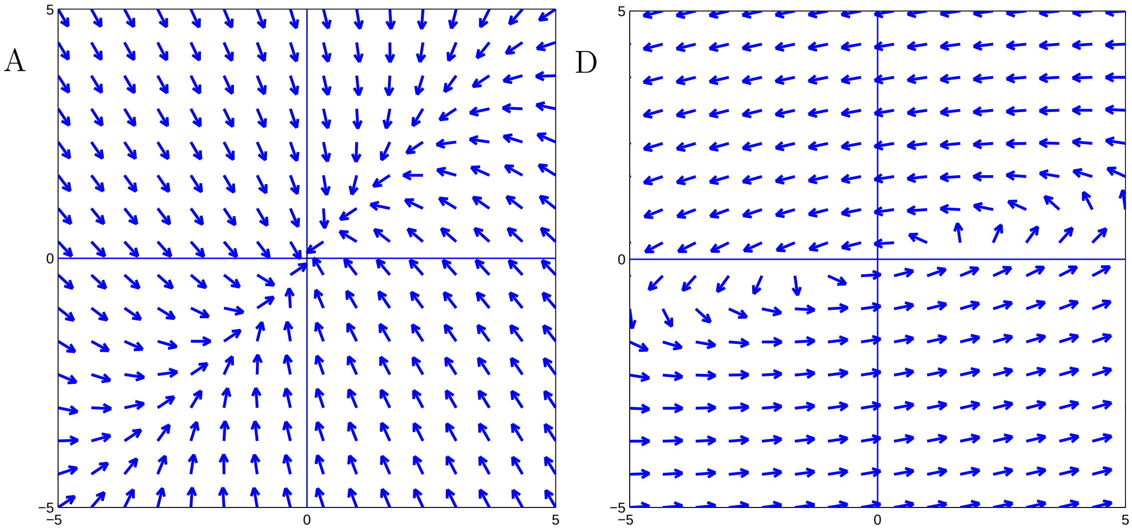


Figure 16.3: Direction arrows for the linear systems A and D of Equations 16.6. At each of several points assumed to be (x_t, y_t), an arrow pointing from (x_t, y_t) toward (x_{t+1}, y_{t+1}) is constructed. Starting at any point in the plane, by following a sequence of arrows one can get an idea of eventual destination of the path starting at the given point. It is obvious in A that all paths lead to (0,0); it is less obvious in D, but all paths spiral outward.

(See Sections 15.4 and 15.5)

The three forms of the solution to $x_{t+2} - px_{t+1} + qx_t = 0$ are

$$\begin{aligned} x_t &= C_1 \times r_1^t + C_2 \times r_2^t \qquad r_1 \text{ and } r_2 \qquad \text{distinct real roots.} \\ y_t &= D_1 \times r_1^t + D_2 \times r_2^t \end{aligned} \tag{16.8}$$

$$\begin{aligned} x_t &= C_1 \times r_1^t + C_2 \times t \times r_1^t \qquad r_1 \text{ a repeated real root.} \\ y_t &= D_1 \times r_1^t + D_2 \times t \times r_1^t \end{aligned} \tag{16.9}$$

$$\begin{aligned} x_t &= C_1 \times \rho^t \sin t\theta + C_2 \times \rho^t \cos t\theta \qquad r = a + bi \qquad \text{a complex root,} \\ y_t &= D_1 \times \rho^t \sin t\theta + D_2 \times \rho^t \cos t\theta \qquad \rho = \sqrt{a^2+b^2}, \quad \theta = \arccos \frac{a}{\rho}. \end{aligned} \tag{16.10}$$

Theorem 16.2.1 For the three forms of solutions in Equations 16.8, 16.9 and 16.10, $x_t \to 0$ and $y_t \to 0$ for all choices of x_0 and y_0 if and only if

$$|r_1| < 1 \text{ and } |r_2| < 1; \qquad \text{or} \quad |r_1| < 1; \qquad \text{or} \quad |\rho| < 1, \tag{16.11}$$

depending on which of the three formulas describe x_t.

Thus the dynamical systems 16.3 are stable only if the corresponding conditions are met.

Proof. The results are valid because $\lim_{t\to\infty} r^t = 0$ if $|r| < 1$ and $\lim_{t\to\infty} t \times r^t = 0$ if $|r| < 1$. (For $\lim_{t\to\infty} t \times r^t = 0$ if $|r| < 1$, see Exercise 16.2.11.) End of proof.

Explore 16.2.1 In the distinct real root case, $x_t = C_1 \times r_1^t + C_2 \times r_2^t$, one might have $r_1 < 1$ and $r_2 > 1$ and $C_2 = 0$. Then $x_t = C_1 \times r_1^t \to 0$ even though the condition $|r_1| < 1$ and $|r_2| < 1$ is not met. Why is this not a contradiction to Theorem 16.2.1? ■

The roots of the dynamical systems A - D in Equations 16.6 are

$$\begin{array}{llllllllllll} \text{A}: & r_1 & \doteq & 0.946, & r_2 & \doteq & 0.814 & \text{B}: & r_1 & \doteq & 0.880 + 0.282i, & \rho \doteq 0.924 \\ & & & & & & & & & & & \theta \doteq 0.310 \\ \\ \text{C}: & r_1 & \doteq & 1.08, & r_2 & \doteq & 0.679 & \text{D}: & r_1 & \doteq & 1.005 + 0.243i, & \rho \doteq 1.03 \\ & & & & & & & & & & & \theta \doteq 0.221 \end{array}$$

Thus A. is stable because $|r_1| = 0.946$ and $|r_2| = 0.814$ are both less than 1. B. is stable because $\rho = 0.924$ is less than 1. C. is not stable because $|r_1| = 1.08$ is greater than one; that $|r_2| = 0.679$ is less than one is not redeeming. D. is not stable because $\rho = 1.03$ is greater than 1.

Without actually computing the roots of the characteristic Equation 16.5, one can determine whether the dynamical systems Equations 16.5 are stable by examining the coefficients.

Theorem 16.2.2 The fate of $\mathbf{x_t}$. The dynamical systems of Equations 16.3 and Equations 16.5, with characteristic equation

$$z^2 - (m_{1,1} + m2, 2) \times z + m_{1,1}m_{2,2} - m_{2,1}m_{1,2} = z^2 - pz + q = 0,$$

are stable if and only if

$$0 \le |p| < 1 + q < 2.$$

Proof. **Danger: Obnubilation Zone.** This argument is tedious and reading it can be delayed – a very long time.

We show that if $0 \le |p| < 1 + q < 2$ then $\lim_{t\to\infty} x_t = 0$. In the case of complex roots, $\rho^2 = q$ (Equation 15.26) and because $0 < 1 + q < 2$, $|q| < 1$ so $|\rho| < 1$. In the case of a repeated root, r_1, $p^2 - 4q = 0$ and $r_1 = \frac{p}{2}$. Because $|p| < 2$, $|r_1| < 1$. Now suppose the roots r_1 and r_2 are real and distinct ($p^2 - 4q > 0$) and $0 \le |p| < 1 + q < 2$. Then

$$\begin{array}{ccccccc} & & |p| & < & 1+q & & \\ \\ 1+q & > & -p & \text{and} & p & < & 1+q \\ \\ 1+p & > & -q & \text{and} & -q & < & 1-p \\ \\ 4+4p+p^2 & > & p^2-4q & \text{and} & p^2-4q & < & 4-4p+p^2 \end{array}$$

Because $|p| < 2$, both $2+p$ and $2-p$ are positive. Then

$$\begin{array}{ccccccc} 2+p & > & \sqrt{p^2-4q} & \text{and} & \sqrt{p^2-4q} & < & 2-p \\ -2-p & < & -\sqrt{p^2-4q} & < & \sqrt{p^2-4q} & < & 2-p \\ -1 & < & \dfrac{p-\sqrt{p^2-4q}}{2} & < & \dfrac{p+\sqrt{p^2-4q}}{2} & < & 1 \end{array}$$

Thus the roots, r_1 and r_2 are between -1 and 1 and $\lim_{t\to\infty} A_t = 0$.

Now suppose $\lim_{t\to\infty} x_t = 0$. Then if the roots are real and distinct they must lie between -1 and 1 and the steps of the previous argument may be reversed to show that $|p| < q+1 < 2$.

If the root r_1 is repeated, then $|r_1| < 1$ and $r_1^2 = q < 1$ and $q+1 < 2$. Furthermore, $p^2 - 4q = 0$, so $|p| = 2\sqrt{q}$. Now,

$$(1-\sqrt{q})^2 > 0, \qquad 1 - 2\sqrt{q} + q > 0, \qquad 1 + q > 2\sqrt{q} = p$$

If the roots are complex, then $\rho < 1$ and $\rho^2 = q$ (Equation 15.26) and $q < 1$ and $q+1 < 2$. Furthermore, $p^2 - 4q < 0$, so $|p| < 2\sqrt{q} < q+1$, as above.

Exercises for Section 16.2, Stability of the equilibria of linear systems.

Exercise 16.2.1 Write the characteristic equations 16.7, for the dynamical systems A - D in Equations 16.6 and compute their characteristic roots.

Exercise 16.2.2 Use the values of p and q from the characteristic equations, 16.7 for the dynamical systems A - D in Equations 16.6 and Theorem 16.2.2 to test whether the systems are stable.

Exercise 16.2.3 Compute the first three iterates of the dynamical systems of Equations 16.6. Plot the iterates on a phase plane and see that they match the trajectories shown in Figure 16.1

The calculator and MATLAB programs shown below will compute the values for trajectory A. By changing coefficients, they can be modified to compute trajectories B, C and D.

```
PROGRAM:LIN
:Fix 5                                      MATLAB
:{50,2} -> dim FF                           close all;clc;clear
:0.25 -> FF(1,1)                            M=[0.9 0.04; 0.1 0.86];
:0.5 -> FF(1,2)                             P(:,1)=[0.25; 0.5];
:For(N,1,5)                                 for i = 1:5
:0.9*FF(N,1)+0.04*FF(N,2) -> FF(N+1,1)          P(:,i+1)=M*P(:,i);
:0.1*FF(N,1)+0.86*FF(N,2) -> FF(N+1,2)      end
:End                                        P.'
:Pause FF                     plot(P(1,:),P(2,:),'x','linewidth',2)
:Fix 9
```

Exercise 16.2.4 The following MATLAB program produces Figure 16.3 A. Modify the program to compute a direction field for Equation 16.6C. Is (0,0) a stable equilibrium point?

```
close all;clc;clear
A=0.9; B=0.04; C=0.1; D=0.86;

wdht=0.7;
lx=-wdht; ux=wdht; ly=-wdht; uy=wdht;
plot([lx ux],[0 0],'linewidth',1)
hold
plot([0 0],[ly uy],'linewidth',1)

axis([lx ux ly uy]); axis('square')

d = 0.03; xv=[-d d;0 0];
av=[0.2*d d;0.4*d 0]; bv=[0.2*d d;-0.4*d 0];

itx=(ux-lx)/15;ity=(uy-ly)/15;
for x0=lx:itx:ux;
    for y0=ly:ity:uy
        x1=A*x0+B*y0;    y1=C*x0+D*y0;

        dxy=sqrt((x1-x0)^2+(y1-y0)^2);
        dx=dd*(x1-x0)/dxy;  dy=dd*(y1-y0)/dxy;

        alph=atan2(dy,dx);

        rot=[cos(alph) -sin(alph);sin(alph) cos(alph)];
        vp = rot*xv;   va = rot*av;   vb = rot*bv;

        plot(x0+dx+va(1,:),y0+dy+va(2,:),'linewidth',2)
        plot(x0+dx+vb(1,:),y0+dy+vb(2,:),'linewidth',2)
        plot([x0 x0+2*dx],[y0 y0+2*dy],'linewidth',2)
    end
end
```

Exercise 16.2.5 For the dynamical system 16.6C,

$$\begin{array}{rclrcl} x_0 & = & 0.25 & x_{n+1} & = & 0.9 \times x_n + 0.04 \times y_n \\ y_0 & = & 0.5 & y_{n+1} & = & 0.1 \times x_n + 0.86 \times y_n \end{array}$$

the characteristic roots are $r_1 = 0.946$ and $r_2 = 0.814$. x_t and y_t are given by

$$x_t = C_1 \times r_1^t + C_2 \times r_2^t \qquad (16.12)$$

$$y_t = D_1 \times r_1^t + D_2 \times r_2^t \tag{16.13}$$

Where C_1, C_2, D_1 and D_2 are computed from

$$\begin{aligned} x_0 &= C_1 + C_2 & \qquad y_0 &= D_1 + D_2 \\ x_1 &= C_1 \times r_1 + C_2 \times r_2 & \qquad y_1 &= D_1 \times r_1 + D_2 \times r_2 \end{aligned}$$

Compute C_1, C_2, D_1 and D_2 and use these values in Equations 16.12 and 16.13 to compute x_2, x_3, and y_2, y_3.

Exercise 16.2.6 For the dynamical system 16.6D,

$$\begin{aligned} x_0 &= 0.25 & \qquad x_{n+1} &= 1.15 \times x_n - 0.8 \times y_n \\ y_0 &= 0.5 & \qquad y_{n+1} &= 0.1 \times x_n + 0.86 \times y_n \end{aligned}$$

the characteristic roots are

$$r_1 \doteq 1.005 + 0.243i, \qquad r_2 \doteq 1.005 - 0.243i$$

x_t and y_t are given by

$$\begin{aligned} x_t &= C_1\, \rho^t \cos t\theta + C_2\, \rho^t \sin t\theta \\ y_t &= D_1\, \rho^t \cos t\theta + D_2\, \rho^t \sin t\theta \end{aligned} \tag{16.14}$$

where $\rho = \sqrt{1.005^2 + 0.243^2}$, $\theta = \arccos(1.005/\rho)$ and C_1, C_2, D_1 and D_2 are computed from

$$\begin{aligned} x_0 &= C_1 & \qquad y_0 &= D_1 \\ x_1 &= C_1\, \rho \cos\theta + C_2\, \rho \sin\theta & \qquad y_1 &= D_1\, \rho \cos\theta + D_2\, \rho \sin\theta \end{aligned}$$

Compute ρ, θ and C_1, C_2 and D_1, D_2 and use these values in Equations 16.14 and 16.14 to compute x_2, x_3, and y_2, y_3.

Exercise 16.2.7 Determine whether the following systems are stable.

$$\begin{array}{llll} \text{a.} & \begin{aligned} x_{t+1} &= 0.48x_t + 0.2y_t \\ y_{t+1} &= 0.128x_t + 0.72y_t \end{aligned} & \text{b.} & \begin{aligned} x_{t+1} &= 0.8x_t - 0.5y_t \\ y_{t+1} &= 0.2x_t + 0.9y_t \end{aligned} \\ \text{c.} & \begin{aligned} x_{t+1} &= 1.1x_t - 0.5y_t \\ y_{t+1} &= 0.2x_t + 0.9y_t \end{aligned} & \text{d.} & \begin{aligned} x_{t+1} &= 0.8x_t - 0.1y_t \\ y_{t+1} &= 0.1x_t + 0.6y_t \end{aligned} \\ \text{e.} & \begin{aligned} x_{t+1} &= 1.2x_t - 0.1y_t \\ y_{t+1} &= 0.9x_t + 0.6y_t \end{aligned} & \text{f.} & \begin{aligned} x_{t+1} &= 0.56x_t + 0.4y_t \\ y_{t+1} &= 0.256x_t + 1.04y_t \end{aligned} \end{array}$$

Exercise 16.2.8 The following systems have one or two characteristic roots equal to 1 or complex roots with $\rho = 1$. Compute the characteristic roots, the first four points of the trajectory and plot them on a phase diagram.

Note: The MATLAB program in Exercise 16.2.3 will compute trajectory points and plot the phase diagrams.

$$\begin{array}{llllllll}
\text{a.} & x_0 & = & 2 & x_{t+1} & = & x_t & \\
 & y_0 & = & 3 & y_{t+1} & = & & y_t \\
\\
\text{b.} & x_0 & = & 2 & x_{t+1} & = & & y_t \\
 & y_0 & = & 3 & y_{t+1} & = & x_t & \\
\\
\text{c.} & x_0 & = & 2 & x_{t+1} & = & & y_t \\
 & y_0 & = & 3 & y_{t+1} & = & -x_t & \\
\\
\text{d.} & x_0 & = & 2 & x_{t+1} & = & 0.5x_t & \\
 & y_0 & = & 2.5 & y_{t+1} & = & & 2y_t \\
\\
\text{e.} & x_0 & = & 2 & x_{t+1} & = & 0.5x_t & \\
 & y_0 & = & 0 & y_{t+1} & = & & 2y_t \\
\\
\text{f.} & x_0 & = & 2 & x_{t+1} & = & 0.5x_t & \\
 & y_0 & = & 1 & y_{t+1} & = & & y_t \\
\\
\text{g.} & x_0 & = & 2 & x_{t+1} & = & 0.5x_t & + \ 0.5y_t \\
 & y_0 & = & 3 & y_{t+1} & = & 0.5x_t & + \ 0.5y_t
\end{array}$$

$$\begin{array}{llllllll}
\text{h.} & x_0 & = & 2 & x_{t+1} & = & \frac{\sqrt{2}}{2}x_t & - \ \frac{\sqrt{2}}{2}y_t \\
 & y_0 & = & 3 & y_{t+1} & = & \frac{\sqrt{2}}{2}x_t & + \ \frac{\sqrt{2}}{2}y_t \\
\\
\text{i.} & x_0 & = & 1 & x_{t+1} & = & 0.5x_t & + \ y_t \\
 & y_0 & = & 2 & y_{t+1} & = & 0.25x_t & + \ 0.5y_t \\
\\
\text{j.} & x_0 & = & 1 & x_{t+1} & = & 0.5x_t & - \ y_t \\
 & y_0 & = & 2 & y_{t+1} & = & 0.25x_t & + \ 0.5y_t \\
\\
\text{k.} & x_0 & = & 1 & x_{t+1} & = & x_t\cos\alpha & - \ y_t\sin\alpha \\
 & y_0 & = & 0 & y_{t+1} & = & x_t\sin\alpha & + \ y_t\cos\alpha
\end{array}$$

Exercise 16.2.9 Use Theorem 16.2.2 and the condition, $0 \leq |p| < 1 + q < 2$ to determine for which of the equations it is sure that $\lim_{t\to\infty} A_t = 0$.

$$\begin{array}{llcllll}
\text{a.} & A_{t+2} - 0.5A_{t+1} + 0.5A_t & = & 0 \qquad & \text{b.} & A_{t+2} - 0.5A_{t+1} - 0.5A_t & = \ 0 \\
\\
\text{c.} & A_{t+2} - 1.4A_{t+1} + 0.5A_t & = & 0 & \text{d.} & A_{t+2} - 1.4A_{t+1} - 0.5A_t & = \ 0 \\
\\
\text{e.} & A_{t+2} - 0.5A_{t+1} + 1.5A_t & = & 0 & \text{f.} & A_{t+2} - 0.5A_{t+1} - 1.5A_t & = \ 0 \\
\\
\text{g.} & A_{t+1} = 0.5A_{t+1} + 0.5B_t & = & 0 & \text{h.} & A_{t+1} = 0.5A_{t+1} - 0.5B_t & = \ 0 \\
 & B_{t+1} = 0.5A_{t+1} + 0.5B_t & = & 0 & & B_{t+1} = 0.5A_{t+1} + 0.5B_t & = \ 0
\end{array}$$

Exercise 16.2.10 From Theorem 16.2.1 for the case of distinct real roots,

$$\begin{aligned} x_t &= C_1 r_1^t + C_2 r_2^t \\ y_t &= D_1 r_1^t + D_2 r_2^t, \end{aligned}$$

if $|r_1| < 1$ and $|r_2| < 1$ then $x_t \to 0$ and $y_t \to 0$. However, x_t and y_t may initially increase in magnitude. They may not decrease monotonically to zero. For the following system, compute the characteristic roots, r_1 and r_2 and confirm that the system is stable. Then compute iterates (x_t, y_t) until the length of (x_t, y_t), defined to be $\sqrt{x_t^2 + y_t^2}$, is less than the length of (x_0, y_0).

$$\begin{aligned} x_0 &= 0.81 \qquad & x_{t+1} &= 0.82x_t + 0.09y_t \\ y_0 &= 0.58 \qquad & y_{t+1} &= 0.36x_t + 0.78y_t. \end{aligned}$$

Exercise 16.2.11 In Equation 16.11, we claim that for

$$x_t = C_1\, r_1^t + C_2 \times t \times r_1^t, \qquad \lim_{t\to\infty} A_t = 0 \qquad \text{if} \qquad |r_1| < 1.$$

Use L'Hospital's Theorem 14.5.1 to show that $\lim_{t\to\infty} t \times r^t = 0$ if $|r| < 1$.

Exercise 16.2.12 Show that the characteristic equation of

$$\begin{aligned} x_{n+1} &= p \cdot x_n - q \cdot y_n \\ y_{n+1} &= x_n \end{aligned}$$

is

$$x_{n+2} - px_{n+1} + qx_n = 0.$$

Exercise 16.2.13 (Only for the adventurous.) For the dynamical system 16.6 B,

$$\begin{aligned} x_0 &= 0.25 \qquad & x_{n+1} &= 0.9 * x_n - 0.8 * y_n \\ y_0 &= 0.5 \qquad & y_{n+1} &= 0.1 * x_n + 0.86 * y_n \end{aligned}$$

the characteristic equation is

$$\rho^2 - 1.76\rho + 0.854 \;=\; 0$$

and the two characteristic roots are

$$\begin{aligned} r_1 &= 0.88 + 0.28213\, i \\ r_2 &= 0.88 - 0.28213\, i \end{aligned}$$

Observe that

$$\begin{aligned} x_0 &= 0.25 \\ x_1 &= 0.9 \cdot 0.25 - 0.8 \cdot 0.5 \\ &= -0.175 \end{aligned}$$

The equations for C_1 and C_2 in $x_t = C_1 \cdot r_1^t + C_2 \cdot r_2^t$ for $t = 0$ and $t = 1$ are

$$\begin{aligned} 0.25 &= C_1 \times r_1^0 + C_2 \times r_2^0 \\ 0.25 &= C_1 + C_2 \\ -0.175 &= C_1 \times r_1^1 + C_2 \times r_2^1 \\ -0.175 &= (0.88 + 0.28213\, i) \times C_1 + (0.88 - 0.28213\, i) \cdot C_2 \end{aligned}$$

Then using complex algebra and arithmetic (actually ordinary algebra), show that

$$C_1 = 0.125 + 0.700\,i \qquad \text{and} \qquad C_2 = 0.125 - 0.700\,i$$

and

$$x_t = (0.125 + 0.700\,i) \times (0.88 + 0.28213\,i)^t + (0.125 - 0.700\,i) \times (0.88 - 0.28213\,i)^t.$$

a. Use your technology to compute x_t for $t = 0, \cdots, 5$. The complex number $a + b\,i$ may be entered as $a + bi$ or as (a,b).

b. Write a corresponding equation for y_t.

16.3 Asymptotic stability of equilibria of nonlinear systems.

Biological systems tend to be nonlinear and we wish to determine when the equilibrium points of nonlinear dynamical systems are asymptotically stable. The principle way of doing so is to show that the *local linear approximation* to the dynamical system at an equilibrium is stable. We define the local linear approximation below; it is a linear dynamical system and its stability is the stability of a linear system as defined in Definition 16.2.2.

Definition 16.3.1 Asymptotical Stability. An *equilibrium point* of a dynamical system,

$$\begin{aligned} x_{t+1} &= F(x_t, y_t) \\ y_{t+1} &= G(x_t, y_t), \end{aligned} \tag{16.15}$$

is a point (a_e, b_e) satisfying

$$\begin{aligned} a_e &= F(a_e, b_e) \\ b_e &= G(a_e, b_e). \end{aligned} \tag{16.16}$$

The system 16.15 is *asymptotically stable* at (a_e, b_e) if there is a positive number δ such that if (x_0, y_0) is within a distance δ of (a_e, b_e) then the iterates of Equations 16.15,

$$(x_t, y_t) \quad \longrightarrow \quad (a_e, b_e).$$

The local linear approximation to a a *one-dimensional* iteration

$$x_{t+1} = F(x_t)$$

at an equilibrium point, $a_e = F(a_e)$, is the linear iteration

$$x_{t+1} = F(a_e) + F'(a_e)(x_t - a_e) \tag{16.17}$$

(see Figure 16.4). The linear iteration is stable (Exercise 16.3.2) if and only if $\mid F'(a_e) \mid < 1$.

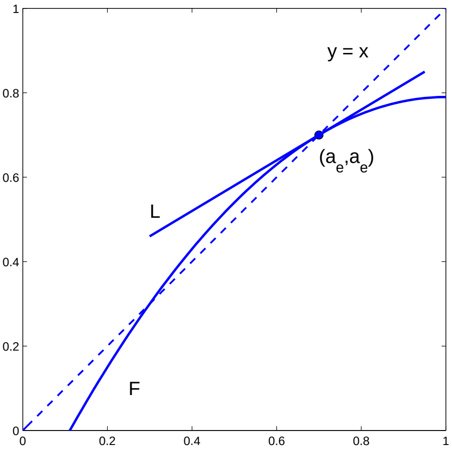


Figure 16.4: Graphs of a function F and its tangent line, L at an equilibrium point (a,a). L is the local linear approximation to F.

Furthermore, by Theorem 14.4.1, the one dimensional dynamical system

$$x_{t+1} = F(x_t)$$

is asymptotically stable at an equilibrium $a_e = F(a_e)$ if $\mid F'(a_e) \mid < 1$. Thus the nonlinear $x_{t+1} = F(x_t)$ is asymptotically stable if its linear approximation $x_{t+1} = L(x_t)$ is stable. We will find a similar result for the two dimensional dynamical system 16.15.

Definition 16.3.2 *Local Linear Approximation.*
Suppose F and G are functions of two variables and (a,b) is in the domain of both F and G and the partial derivatives F_1, F_2, G_1, and G_2 are all continuous on the interior of a circle with center at (a,b). The local linear approximation to the two dimensional dynamical system

$$\begin{aligned} x_{t+1} &= F(x_t, y_t) \\ y_{t+1} &= G(x_t, y_t) \end{aligned} \tag{16.18}$$

at a an equilibrium point (a_e, b_e) is

$$\begin{aligned} x_{t+1} &= a_e + F_1(a_e,b_e) \times (x_t - a_e) + F_2(a_e,b_e) \times (y_t - b_e) \\ y_{t+1} &= b_e + G_1(a_e,b_e) \times (x_t - a_e) + G_2(a_e,b_e) \times (y_t - b_e) \end{aligned} \tag{16.19}$$

With $\xi_t = x_t - a_e$ and $\eta_t = y_t - b_e$ Equations 16.19 become

$$\begin{aligned} \xi_{t+1} &= F_1(a_e,b_e) \times \xi_t + F_2(a_e,b_e) \times \eta_t \\ \eta_{t+1} &= G_1(a_e,b_e) \times \xi_t + G_2(a_e,b_e) \times \eta_t \end{aligned} \tag{16.20}$$

which are homogeneous linear equations.

The matrix

$$\left[\begin{array}{cc} F_1(a,b) & F_2(a,b) \\ G_1(a,b) & G_2(a,b) \end{array}\right] \tag{16.21}$$

is called the *Jacobian matrix* at (a,b) of the transformation

$$\begin{array}{rcl} u &=& F(x,y) \\ v &=& G(x,y) \end{array} \tag{16.22}$$

from the (x,y)-plane to the (u,v)-plane.

Example 16.3.1 In the next section we consider two populations that have a symbiotic relationship, a special case of which is

$$\begin{array}{rclcl} x_{t+1} &=& x_t + \frac{5}{98}x_t(1+\frac{4}{10}y_t - x_t) &=& F(x_t,y_t) \\ \\ y_{t+1} &=& y_t + \frac{7}{120}y_t(1+\frac{5}{7}x_t - y_t) &=& G(x_t,y_t). \end{array} \tag{16.23}$$

An equilibrium point of the system is (1.96,2.4) and the Jacobian matrix at (1.96, 2.4) is computed by

$$\begin{array}{rclcrcl} F(x,y) &=& \frac{103}{98}x + \frac{2}{98}x\times y - \frac{5}{98}x^2 & & G(x,y) &=& \frac{127}{120}y + \frac{1}{24}xy - \frac{7}{120}y^2 \\ \\ F_1(x,y) &=& \frac{103}{98} + \frac{2}{98}y - \frac{10}{98}x & & F_2(x,y) &=& \frac{2}{98}x \\ \\ G_1(x,y) &=& \frac{1}{24}y & & G_2(x,y) &=& \frac{127}{120} + \frac{1}{24}x - \frac{14}{120}y \end{array}$$

$$\begin{array}{rclcrcl} F_1(1.96,2.4) &=& 0.9 & & F_2(1.96,2.4) &=& 0.04 \\ \\ G_1(1.96,2.4) &=& 0.1 & & G_2(1.96,2.4) &=& 0.86 \end{array}$$

Then the Jacobian matrix and homogeneous local linear approximation to Equations 16.23 at the equilibrium point (1.96,2.4) are

$$\left[\begin{array}{cc} 0.9 & 0.04 \\ 0.1 & 0.86 \end{array}\right] \qquad \begin{array}{rcl} \xi_{t+1} &=& 0.9\xi_t + 0.04\eta_t \\ \\ \eta_{t+1} &=& 0.1\xi_t + 0.86\eta_t \end{array} \tag{16.24}$$

The alert reader may recognize this linear dynamical system as being that of Equations 16.6A for which the characteristic roots are approximately 0.946 and 0.814. The homogeneous linear dynamical system 16.24 is stable.

Because the local linear approximation 16.24 to the nonlinear dynamical system 16.23 at the equilibrium point (1.96,2.4) is stable, the nonlinear dynamical system 16.23 is asymptotically stable at (1.96,2.4).

The basis for the previous paragraph is in Theorem 16.3.1. The idea of the theorem and of local linear approximation can be seen by an algebraic rearrangement of the nonlinear system 16.23

$$\begin{aligned} x_{t+1} - 1.96 &= 0.9(x_t - 1.96) + 0.04(y_t - 2.4) \\ &\qquad + \frac{2}{98}(x_t - 1.96)(y_t - 2.4) - \frac{5}{98}(x_t - 1.96)^2 \\ y_{t+1} - 2.4 &= 0.1(x_t - 1.96) + 0.86(y_t - 2.4) \\ &\qquad + \frac{5}{120}(x_t - 1.96)(y_t - 2.4) - \frac{7}{120}(y_t - 1.96)^2 \end{aligned}$$

The linear terms are those of the local linear approximation. The idea of Theorem 16.3.1 is that if (x_t, y_t) is close to the equilibrium (1.96,2.4) so that $x_t - 1.96$ and $y_t - 2.4$ are 'small' then the quadratic terms $(x_t - 1.96)(y_t - 2.4)$, $(x_t - 1.96)^2$ and $(y_t - 1.96)^2$ are '(small)^2', even smaller, and contribute very little in computing the trajectory.

Explore 16.3.1 Using Equations 16.23, let $(x_0, y_0) = (1, 22)$ and compute (x_1, y_1). If you compute (x_3, y_3) and (x_4, y_4) you will likely conclude that (x_t, y_t) does not converge to the equilibrium point (1.96,2.4). Does this mean that System 16.23 is not asymptotically stable? ■

Theorem 16.3.1 Stability of local linear approximation implies asymptotic stability of the nonlinear system. Suppose (a_e, b_e) is an equilibrium point of a dynamical system

$$\begin{aligned} x_{t+1} &= F(x_t, y_t) \\ y_{t+1} &= G(x_t, y_t) \end{aligned} \tag{16.25}$$

and the homogeneous local linear approximation to 16.25 at (a_e, b_e)

$$\begin{aligned} \xi_{t+1} &= F_1(a_e, b_e)\xi_t + F_2(a_e, b_e)\eta_t \\ \eta_{t+1} &= G_1(a_e, b_e)\xi_t + G_2(a_e, b_e)\eta_t \end{aligned} \tag{16.26}$$

is stable. Then the system 16.25 is asymptotically stable. We assume that F and G and their partial derivatives are continuous and the domains of F and G are all number pairs.

The proof of Theorem 16.3.1 is beyond the scope of this text. We will assume that it is true and apply it to four examples in the next section.

Exercises for Section 16.3, Local linear approximations to nonlinear systems.

Exercise 16.3.1 Find the local linear approximations to the following two-dimensional dynamical systems at the indicated points. Check to see whether the points given are equilibrium points of the system. If the point is an equilibrium point of a system, determine whether it is a asymptotically stable equilibrium point of the system.

a. $$\begin{aligned} x_{t+1} &= x_t^2 + y_t^2 \\ y_{t+1} &= x_t \times y_t \end{aligned} \qquad \text{at the point} \quad (0,0)$$

b. $$\begin{aligned} x_{t+1} &= x_t^2 + y_t^2 \\ y_{t+1} &= x_t \times y_t \end{aligned} \qquad \text{at the point} \quad (1,1)$$

c. $$\begin{aligned} x_{t+1} &= x_t(1-x_t)(1-y_t) + 1/6 \\ y_{t+1} &= y_t(1-y_t)(1-x_t) + 1/8 \end{aligned} \qquad \text{at the point} \quad (1/3, 1/4)$$

d. $$\begin{aligned} x_{t+1} &= 2x_t(1-x_t)(1-y_t) + 5/9 \\ y_{t+1} &= y_t(1-y_t)(1-x_t) + 11/16 \end{aligned} \qquad \text{at the point} \quad (2/3, 3/4)$$

e. $$\begin{aligned} x_{t+1} &= x_t + 0.1y_t(1-x_t) - 0.018 \\ y_{t+1} &= y_t + 0.2x_t(1-y_t) - 0.016 \end{aligned} \qquad \text{at the point} \quad (0.1, 0.2)$$

f. $$\begin{aligned} x_{t+1} &= x_t + 0.1y_t(1-x_t) - 0.018 \\ y_{t+1} &= y_t + 0.2x_t(1-y_t) - 0.016 \end{aligned} \qquad \text{at the point} \quad (0.8, 0.9)$$

g. $$\begin{aligned} x_{t+1} &= \sqrt{3}\cos(\tfrac{\pi}{3}y_t) \\ y_{t+1} &= \frac{1}{\sqrt{2}}\cos(\frac{7\pi}{6}x_t) \end{aligned} \qquad \text{at the point} \quad (3/2, 1/2)$$

Exercise 16.3.2 Show that if M and H are numbers and and $|M| < 1$ then the one-dimensional linear dynamical system

$$x_{t+1} = M \times x_t + H$$

is stable.

Exercise 16.3.3 The dynamics of the SIR model of Section 15.1

$$\begin{aligned} S_{t+1} &= S_t - \beta \times S_t \times I_t \\ I_{t+1} &= I_t + \beta \times S_t \times I_t - \gamma \times S_{t-d} \times I_{t-d} \\ R_{t+1} &= R_t + \gamma \times S_{t-d} \times I_{t-d}. \end{aligned}$$

is determined by the first two equations; R_t does not affect the spread of the disease. Let M be the total population size and

$$x_t = \frac{S_t}{M}, \qquad y_t = \frac{I_t}{M}.$$

Then the first two of the SIR equations are

$$\begin{aligned} x_{t+1} &= x_t - \beta\, M \times x_t \times y_t \\ y_{t+1} &= y_t + \beta\, M \times x_t \times y_t - \gamma \times y_t \end{aligned} \qquad (16.27)$$

a. Show that (1,0) is an equilibrium point of Equations 16.27.

b. One of the characteristic roots is 1. Find conditions on β M and γ sufficient to insure that the other characteristic root is less than 1.

This is an important condition in that it distinguishes epidemics from non-epidemic introduction of disease.

Exercise 16.3.4 Examine the behavior of x_t for $t = 2, 3, \cdots, 20$ of the difference equation

$$\begin{aligned} x_{t+1} &= \alpha_0 x_t + \alpha_1 x_{t-1} \\ x_0 &= 0.1, \qquad x_1 = 0.2 \end{aligned}$$

for

a. $\alpha_0 = \frac{5}{6}, \qquad \alpha_1 = -\frac{1}{2}.$

b. $\alpha_0 = \frac{3}{2}, \qquad \alpha_1 = -\frac{1}{2}.$

c. $\alpha_0 = \frac{5}{2}, \qquad \alpha_1 = -1.$

16.4 Four examples of nonlinear dynamical systems.

16.4.1 Deer Population dynamics.

We now consider a population of deer growing in an environment that will support 500 female animals. Assume the low density growth rate to be 0.16 and that 15 female deer are harvested from the population each year. Because deer reach sexual maturity in their second year, the equation for population change is

$$\begin{array}{rcll} & & \text{Population Kinetics} & \text{Harvest} \\ w_{t+1} - w_t & = & 0.16 \times w_{t-1} \times \left(1 - \frac{w_{t-1}}{500}\right) & -15 \end{array} \tag{16.28}$$

where w_t is the number of adult female deer in year t. The number 0.16 that we have called the low density growth rate accounts for both the survival for two years of the population measured by w_{t-1} and the survival of the fauns born to those deer.

It is helpful to divide each term by 500 and let $u_t = \frac{w_t}{500}$. Then

$$u_{t+1} - u_t = 0.16 \times u_{t-1} \times (1 - u_{t-1}) - 0.03 \tag{16.29}$$

Equation 16.29 may be treated as a second order nonlinear dynamical system in one variable, or as we next show, as a first order nonlinear dynamical system in two variables.

Let $v_t = u_{t-1}$. Then $v_{t+1} = u_t$ and Equation 16.29 may be written as

$$\begin{aligned} u_{t+1} &= u_t + 0.16 v_t (1 - v_t) - 0.03 \\ v_{t+1} &= u_t \end{aligned} \tag{16.30}$$

The equilibrium points of Equation 16.30 are (u_e, v_e) where

$$\begin{aligned} u_e &= u_e + 0.16v_e(1 - v_e) - 0.03 \\ v_e &= u_e \end{aligned}$$

which is equivalent to

$$\begin{aligned} u_e &= u_e + 0.16u_e(1 - u_e) - 0.03 \\ 0 &= -0.16u_e^2 + 0.16u_e - 0.03 \qquad (16.31) \\ 0 &= -0.16(u_e^2 - u_e + 0.188) = -0.16(u_e - 0.25)(u_e - 0.75) \end{aligned}$$

The equilibrium points of the iteration equation are (0.25,0.25) and (0.75,0.75).

Let

$$F(u, v) = u + 0.16v(1 - v) - 0.03 \qquad G(u, v) = u$$

Then

$$\begin{aligned} F_1(u, v) &= 1 & F_2(u, v) &= 0.16 - 0.32v \\ F_1(u, v) &= 1 & G_2(u, v) &= 0 \end{aligned}$$

At the equilibrium point (0.25,0.25) the homogeneous local linear approximation and Jacobian matrix are

$$\begin{aligned} x_{t+1} &= x_t + 0.08y_t \\ y_{t+1} &= x_t \end{aligned} \qquad \begin{bmatrix} 1 & 0.08 \\ 1 & 0 \end{bmatrix}$$

The characteristic equation of the linear dynamical system is

$$r^2 - r - 0.08 = 0 \qquad \text{with roots} \quad r_1 = 1.074, \quad r_2 = -0.74$$

Because $r_1 = 1.074 > 1$ the nonlinear system 16.30 is not asymptotically stable at the equilbrium point (0.25, 0.25). That the dynamical system 16.30 is asymptotically stable at (0.75,0.75) is in Exercise 16.4.2.

16.4.2 Symbiosis Systems.

Imagine two interacting populations, X and Y, with population numbers at time n, X_n and Y_n. Assume that they have a mutualistic symbiotic relationship, in which the presence of each species enhances the growth of the other species. For example, corals are hosts to some algae that provide the coral with energy; clown fish protect anemone from fish that eat anemone and the stinging tentacles of anemone protect the clown fish from its predators. X has a low density growth rate, r_x, and in isolation, the maximum supportable X population is M. The presence of Y, however, expands the supportable X population. If Y_n is at its isolation maximum of N, then X can grow to $(1 + \alpha)M$ (α and β are positive). In turn, the presence of X expands the Y universe.

$$\begin{aligned} X_{n+1} - X_n &= r_x \cdot X_n \left(1 + \alpha \tfrac{Y_n}{N} - \tfrac{X_n}{M}\right) \\ Y_{n+1} - Y_n &= r_y \cdot Y_n \left(1 + \beta \tfrac{X_n}{M} - \tfrac{Y_n}{N}\right) \end{aligned} \qquad (16.32)$$

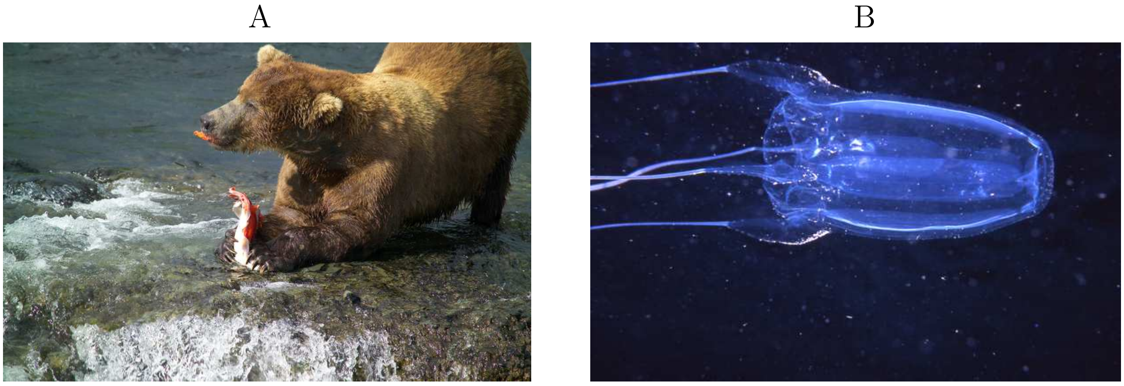


Figure 16.5: Predators. A. A Brown Bear eating fat rich skin of a salmon in Katmai National Park and Reserve, Photo by Peter Hamel, http://www.nps.gov/katm/photosmultimedia/Bears-Gallery.htm. B. Predator. Sea Wasp, a Box Jelly Fish. See cover of this book for color versions.

As often happens it is useful to divide by maximum supportable populations and use variables that are fractions of maximum supportable populations. Divide the first equation of Equations 16.32 by M and the second equation by N. Then

$$\begin{aligned}\frac{X_{n+1}}{M}-\frac{X_n}{M} &= r_x\cdot\frac{X_n}{M}\left(1+\alpha\frac{Y_n}{N}-\frac{X_n}{M}\right)\\ \frac{Y_{n+1}}{N}-\frac{Y_n}{N} &= r_y\cdot\frac{Y_n}{N}\left(1+\beta\frac{X_n}{M}-\frac{Y_n}{N}\right)\end{aligned}$$

Then we let $x_n=\frac{X_n}{M}$ and $y_n=\frac{Y_n}{N}$ and write

$$\begin{aligned}x_{n+1}-x_n &= r_x\cdot x_n\left(1+\alpha y_n-x_n\right)\\ y_{n+1}-y_n &= r_y\cdot y_n\left(1+\beta x_n-y_n\right)\end{aligned}\qquad(16.33)$$

Equations 16.33 involve only four parameters whereas the original Equations 16.32 involved six. The dynamics and stability conditions are identical, however.

Let (x_e,y_e) denote an equilibrium point of Equations 16.33. Then x_n and y_n are not changing when $x_n=x_e$ and $y_n=y_e$ so that

$$\begin{aligned}x_e-x_e &= r_x\cdot x_e\left(1+\alpha y_e-x_e\right) &= 0\\ y_e-y_e &= r_y\cdot y_e\left(1+\beta x_e-y_e\right) &= 0\end{aligned}\qquad(16.34)$$

There are four solutions:

$$(x_e,y_e)=(0,0),\quad (x_e,y_e)=(1,0),\quad (x_e,y_e)=(0,1),\quad \text{or}\quad \left\{\begin{aligned}x_e &= \frac{1+\alpha}{1-\alpha\beta}\\ y_e &= \frac{1+\beta}{1-\alpha\beta}\end{aligned}\right.$$

The equilibria are marked on Figure 16.6 along with the lines $1+\alpha y-x=0$ and $1+\beta x-y=0$. The arrows point from (x_n,y_n) toward (x_{n+1},y_{n+1}). If you follow a path of arrows, you will move toward the equilibrium point,

$$(x_e,y_e)=\left(\frac{1+\alpha}{1-\alpha\beta},\frac{1+\beta}{1-\alpha\beta}\right).$$

and away from the points (0,0), (0,1), and (1,0). In the shaded region, $x_{n+1} - x_n > 0$ because $1 + \alpha y_n - x_n > 0$ and $y_{n+1} - y_n > 0$ because $1 + \beta x_n - y_n > 0$. Therefore all of the arrows point up and to the right.

Explore 16.4.1 . Two points are marked in Figure 16.6, A on the line $1 + \alpha y - x = 0$ and B on the line $1 + \beta x - y = 0$. In which directions would the arrows A and B be pointing if they were marked? ■

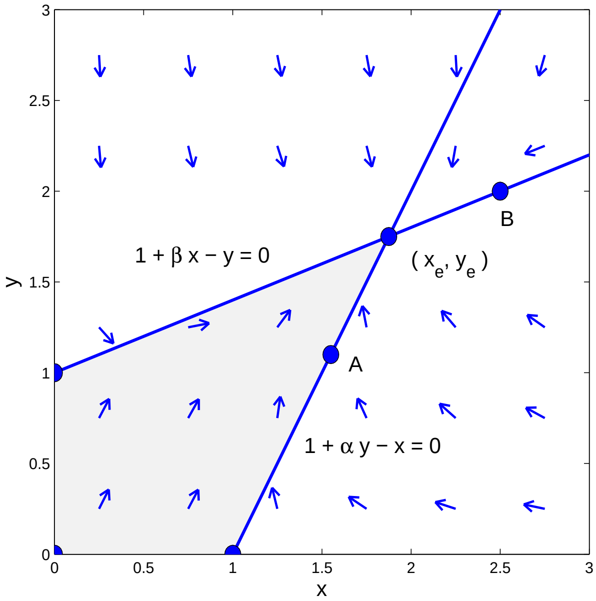


Figure 16.6: A phase plane for a symbiosis system, $x_{n+1} - x_n = r_x \cdot x_n\,(1 + \alpha y_n - x_n)$; $y_{n+1} - y_n = r_y \cdot y_n\,(1 + \beta x_n - y_n)$

Instability of the equilibrium point (1,0). Let

$$F(x,y) = x + r_x \times x \times (1 + \alpha y - x) \qquad G(x,y) = y + r_y \times y \times (1 + \beta x - y)$$

Then Equations 16.33 are

$$x_{n+1} = F(x_n, y_n) \qquad y_{n+1} = G(x_n, y_n).$$

The Jacobian matrix of the nonlinear system is computed and evaluated at (1,0) as

$$\begin{array}{rclrcl} F_1(x,y) & = & 1 + r_x + r_x\alpha y - 2r_x x & F_2(x,y) & = & r_x\alpha x \\ G_1(x,y) & = & r_y\beta y & G_2(x,y) & = & 1 + r_y + r_y\beta x - 2r_y y \\ F_1(1,0) & = & 1 - r_x & F_2(0,1) & = & r_x\alpha \\ G_1(1,0) & = & 0 & G_2(1,0) & = & 1 + r_y + r_y\beta \end{array} \qquad (16.35)$$

The Jacobian matrix

$$\begin{bmatrix} 1 - r_x & r_x\alpha \\ 0 & 1 + r_y + r_y\beta \end{bmatrix}$$

is in *upper triangular form* (only zeros below the diagonal) from which it follows that the roots of the characteristic equation are the diagonal entries, $1 - r_x$ and $1 + r_y + r_y\beta$ (see Exercise 15.6.4). The root $1 + r_y + r_y\beta$ is greater than 1 ($r_y > 0$ and $\beta r_y > 0$) so the local linear approximation to the nonlinear system 16.33 is not stable and we conclude that the nonlinear system is not asymptotically stable at the equilibrium point (1,0).

At the equilibrium point (1,0), the population Y is not present and the population X is at its maximum supportable population M in the absence of Y. If a small number of Y is introduced and $r_y > 0$ the Y population will grow exponentially for a short while and the system will move to the equilibrium point $x_e = \frac{1+\alpha}{1-\alpha\beta}$, $y_e = \frac{1+\beta}{1-\alpha\beta}$.

Shown in Figure 16.7 are computations that illustrate the introduction of 0.01 N of the Y population, and a comparison of the trajectory of the nonlinear equations with the trajectory of the local linear approximations

$$\begin{aligned} \xi_{n+1} &= (1-r_x)\xi_n \quad + \quad r_x\alpha\eta_n \\ \eta_{n+1} &= \qquad\qquad\qquad (1+r_y+r_y\beta)\eta_n. \end{aligned} \tag{16.36}$$

Parameter values used are

$$\begin{array}{llll} r_x = 0.1 & \alpha = 0.5 & x_0 = 1 & \xi_0 = 0 \\ r_y = 0.5 & \beta = 0.4 & y_0 = 0.01 & \eta_0 = 0.01 \end{array}$$

The nonlinear equations are

$$\begin{aligned} x_{n+1} - x_n &= 0.1 \cdot x_n\,(1 + 0.5y_n - x_n) \\ y_{n+1} - y_n &= 0.5 \cdot y_n\,(1 + 0.4x_n - y_n) \end{aligned}$$

and the local linear approximation at (1,0) is

$$\begin{aligned} \xi_{n+1} &= (1-0.1)\,\xi_n + 0.1 \cdot 0.5\,\eta_n &= 0.9\,\xi_n \quad + \quad 0.05\,\eta_n \\ \eta_{n+1} &= \qquad\qquad 1.52 \cdot 0.5\,\eta_n &= \qquad\qquad 0.076\,\eta_n. \end{aligned}$$

The initial X population $x_0 = 1$ was at its equilibrium value for no Y present; Y was set $y_0 = 0.01$. If X is initially isolated from Y and a small amount (0.01) of Y is introduced into the environment, both populations increase. The point (1,0) is a *nonstable* equilibrium point.

Stability of the equilibrium point ($x_e = \frac{1+\alpha}{1-\alpha\beta}$, $y_e = \frac{1+\beta}{1-\alpha\beta}$). We refer to this as the two-species equilibrium point. In the other three equilibrium points only one or no species were present.

This point (x_e, y_e) is chosen so that

$$(1 + \alpha y_e - x_e) = 0 \qquad \text{and} \qquad (1 + \beta x_e - y_e) = 0$$

in Equations 16.34.

Partial derivatives are computed from from Equations 16.35, as

$$\begin{aligned} F_1(x,y) &= 1 + r_x + r_x\alpha y - 2r_x x & F_2(x,y) &= r_x\alpha x \\ G_1(x,y) &= r_y\beta y & G_2(x,y) &= 1 + r_y + r_y\beta x - 2r_y y \end{aligned}$$

n	x_n	y_n	ξ_n	η_n
0	1.000	0.010	0.000	0.010
1	1.001	0.011	0.001	0.011
2	1.001	0.011	0.001	0.011
3	1.001	0.012	0.001	0.012
4	1.002	0.013	0.002	0.013
5	1.002	0.014	0.002	0.014
10	1.005	0.020	0.005	0.020
20	1.011	0.038	0.011	0.039
30	1.022	0.073	0.022	0.076
40	1.042	0.138	0.044	0.150

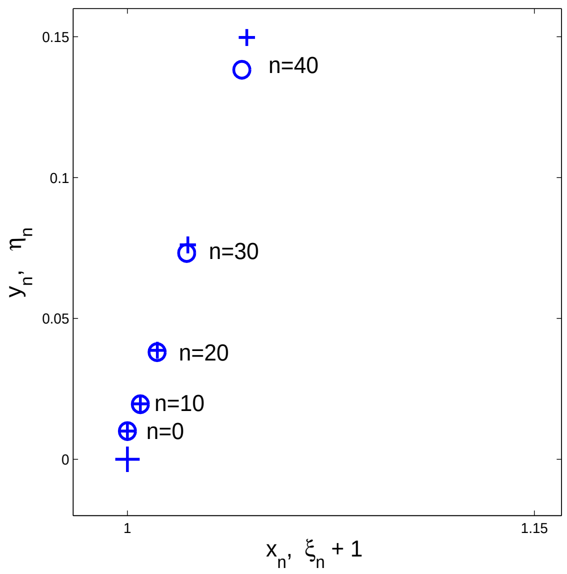


Figure 16.7: Comparison of the trajectories of the nonlinear dynamical system 16.33, with the trajectory of the local linear approximation to the nonlinear system near the equilibrium point (1,0), Equations 16.36. The starting point ($n = 0$) is (1.0,0.01) and the points ($x_n - 1, y_n$) plotted with 'o', move away from the equilibrium point (1,0). 1 has been added to ξ_n and ($\xi_n + 1, \eta_n$) is plotted with '+'.

Note that $1 + \alpha y_e - x_e = 0$ and $1 + \beta x_e - y_e = 0$. The Jacobian at (x_e, y_e) is

$$\begin{aligned} F_1(x_e, y_e) &= 1 - r_x x_e & \qquad F_2(x_e, y_e) &= r_x \alpha x_e \\ G_1(x_e, y_e) &= r_y \beta y_e & \qquad G_2(x_e, y_e) &= 1 - r_y \beta y_e, \end{aligned} \tag{16.37}$$

and the local linear approximation at (x_e, y_e) is

$$\begin{aligned} \xi_{n+1} &= (1 - r_x\, x_e)\, \xi_n &+& \quad r_x\, \alpha\, x_e\, \eta_n \\ \eta_{n+1} &= r_y\, \beta\, y_e\, \xi_n &+& \quad (1 - r_y\, \beta\, y_e)\, \eta_n. \end{aligned} \tag{16.38}$$

For a system in which r_x, r_y, α, and β are explicitly known, it is relatively easy to compute the equilibrium point

$$(x_e, y_e) = (\frac{1+\alpha}{1-\alpha\beta}, \frac{1+\beta}{1-\alpha\beta})$$

and iteration matrix

$$M = \left[\begin{array}{cc} 1 - r_x\, x_e & r_x\, \alpha\, x_e \\ r_y\, \beta\, y_e & 1 - r_y\, \beta\, y_e \end{array} \right].$$

Then the characteristic roots of M can be computed. A MATLAB program to do this follows.

```
close all;clc;clear
rx = 0.2; ry = 0.3; alpha=0.4; beta= 0.1;
xe=(1+alpha)/(1-alpha*beta)
ye=(1+beta)/(1-alpha*beta)
J=[1-rx*xe rx*alpha*xe;rx*beta*ye  1-ye*beta*ye]
R=eig(J)  %%The roots appear in R.
```

The following qualitative conditions guarantee that the equilibrium point

$$(x_e, y_e) = (\frac{1+\alpha}{1-\alpha\beta}, \frac{1+\beta}{1-\alpha\beta})$$

has positive coordinates and that the system is locally asymptotically stable at (x_e, y_e). The theorem is valid for both positive and negative A and B. For $\alpha > 0$ and $\beta > 0$, the case:

$$\begin{array}{llll} A = \alpha, & B = \beta & & \text{represents Symbiosis,} \\ A = -\alpha, & B = \beta & & \text{represents Predator-Prey, and} \\ A = -\alpha, & B = -\beta & & \text{represents Competition.} \end{array}$$

Theorem 16.4.1 Asymptotic stability of two-species equilibrium points. If

$$0 < |A| < 1, \qquad 0 < |B| < 1,$$

then the dynamical system

$$\begin{aligned} x_{n+1} - x_n &= r_x \cdot x_n\,(1 + Ay_n - x_n) \\ y_{n+1} - y_n &= r_y \cdot y_n\,(1 + Bx_n - y_n) \end{aligned} \tag{16.39}$$

has an equilibrium point

$$(x_e, y_e) = (\frac{1+A}{1-AB}, \frac{1+B}{1-AB})$$

for which both coordinates are positive, and if

$$0 < |A| < 1, \qquad 0 < |B| < 1, \qquad r_x\,x_e < 1, \qquad \text{and} \qquad r_y\,y_e < 1. \tag{16.40}$$

the system is asymptotically stable at that equilibrium point.

Explore 16.4.2 Show that the conditions $0 < |A| < 1$ and $0 < |B| < 1$ guarantee that $x_e > 0$.

The proof of asymptotic stability appears on page 293.

Explore 16.4.3 For the values

$$\alpha = 0.7, \qquad \beta = 0.8, \qquad r_x = 0.3, \qquad r_y = 0.2$$

the equilibrium point of Equations 16.33 is

$$(x_e, y_e) = (\frac{1+\alpha}{1-\alpha\beta}, \frac{1+\beta}{1-\alpha\beta}) = (\frac{85}{22}, \frac{65}{28}).$$

Are the inequalities of Theorem 16.4.1,

$$|A| < 1, \qquad |B| < 1, \qquad r_x x_e < 1, \qquad \text{and} \qquad r_y y_e < 1,$$

satisfied?

Is the equiibrium point, (85/22, 65/22) of Equations 16.33 locally asymptotically stable? ■

n	x_n	y_n	$\hat{\xi}_n$	$\hat{\eta}_n$
0	1.860	2.200	-0.100	-0.200
1	1.862	2.217	-0.098	-0.182
2	1.864	2.231	-0.095	-0.166
3	1.867	2.244	-0.093	-0.153
4	1.870	2.256	-0.089	-0.140
5	1.873	2.266	-0.086	-0.130
10	1.890	2.306	-0.069	-0.091
20	1.917	2.348	-0.041	-0.049
30	1.935	2.370	-0.024	-0.028
40	1.945	2.383	-0.014	-0.016

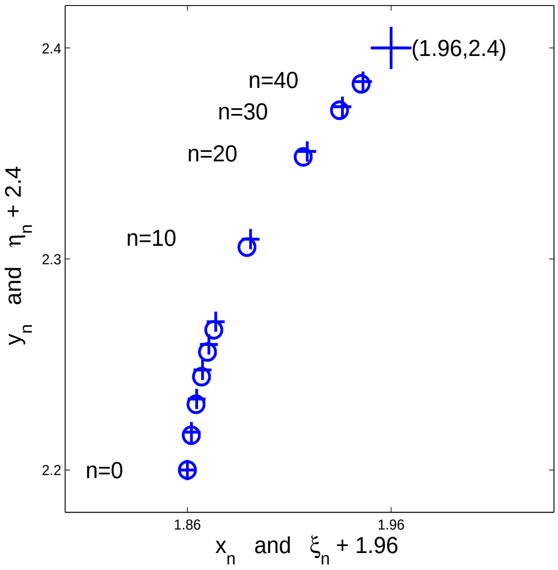


Figure 16.8: Comparison of the trajectories of the nonlinear dynamical system 16.33 marked with 'o', with the trajectory of the local linear approximation to the nonlinear system at the equilibrium point $(1+\alpha)/(1-\alpha\beta),(1+\beta)/1-\alpha\beta) = (1.96, 2.4)$ Equations 16.48 ξ_n and η_n have been translated back to (1.96,2.4) and $(\xi_n + 1.96, \eta_n + 2.4)$ is plotted with '+'.

An example computation for the special case

$$\begin{array}{ll} r_x = \frac{5}{98} & \alpha = 0.4 \\ r_y = \frac{7}{120} & \beta = \frac{5}{7} \end{array}$$

is shown in Figure 16.8. The local linear approximation, Equations 16.48, are

$$\begin{aligned} \xi_{n+1} &= 0.9\xi_n + 0.04\eta_n \\ \eta_{n+1} &= 0.1\xi_n + 0.86\eta_n \end{aligned}$$

As was shown in Example 16.3.1, this is Equations 16.6A for which the characteristic roots are approximately 0.946 and 0.814, and the system is stable. Computations with this example are shown in Figure 16.8. The equilibrium point is $((1+\alpha)/(1-\alpha\beta),(1+\beta)/(1-\alpha\beta)) = (1.96, 2.4)$ and starting values were $x_0 = 1.86$, $y_0 = 2.2$, and $\xi_0 = -0.1$, $\eta_0 = -0.2$.

Summary of this section for use in later sections. We have shown that for the dynamical system 16.32,

$$\begin{aligned} X_{n+1} - X_n &= r_x \cdot X_n \left(1 + \alpha\frac{Y_n}{N} - \frac{X_n}{M}\right) \\ Y_{n+1} - Y_n &= r_y \cdot Y_n \left(1 + \beta\frac{X_n}{M} - \frac{Y_n}{N}\right), \end{aligned}$$

the normalized system is

$$\begin{aligned} x_{n+1} - x_n &= r_x \cdot x_n \left(1 + \alpha y_n - x_n\right) \\ y_{n+1} - y_n &= r_y \cdot y_n \left(1 + \beta x_n - y_n\right) \end{aligned}$$

If $1 - \alpha\beta \neq 0$ then an equilibrium point, e, of the normalized system and Jacobian, J_e, evaluated at that equilibrium point are

$$e = \begin{bmatrix} \dfrac{1+\alpha}{1-\alpha\beta} \\ \dfrac{1+\beta}{1-\alpha\beta} \end{bmatrix} \qquad J_e = \begin{bmatrix} 1 - r_x x_e & r_x\alpha x_e \\ r_y\beta y_e & 1 - r_y y_e \end{bmatrix} \tag{16.41}$$

The same algebra gives the same formula for all α and β, even when one or both of them are negative. It needs to be checked that the equilibrium point, e, has both coordinates positive and is therefore biologically relevant.

16.4.3 Predator-Prey Systems

We now analyze a dynamical system that is often used to discuss models of two species one of which is a predator of the other.

$$\begin{aligned} X_{n+1} - X_n &= r_x \cdot X_n \left(1 - \alpha \tfrac{Y_n}{N} - \tfrac{X_n}{M}\right) \\ Y_{n+1} - Y_n &= r_y \cdot Y_n \left(1 + \beta \tfrac{X_n}{M} - \tfrac{Y_n}{N}\right) \end{aligned} \tag{16.42}$$

The system is obviously similar to the model of symbiosis, Equations 16.32, the only difference being the minus sign in front of α in the first equation. Y is the predator, and has a low density growth rate, r_y, and in isolation, the maximum supportable Y-population is N. However, presence of the prey X expands the supportable Y-population. If X_n is at its isolation maximum of M, then Y can grow to $(1+\beta)N$ (α and β are positive). X has a low density growth rate of r_x and in the absence of Y has a maximum supportable population of M. However, If Y is at its isolation maximum of N, the maximum X is reduced to $(1-\alpha)M$.

Because of the similarity of the predator-prey equations 16.42 to the symbiosis equations 16.32, most of the work of this subsection is in the form of exercises.

16.4.4 Competition.

Imagine that there are two populations, X and Y, competing for the same resource. In the absence of Y, X has a maximum sustainable population, M, but the presence of Y reduces the resources available to X and reduces the supportable X population. Similarly, in the absence of X, Y has a maximum sustainable population N and the presence of X reduces the supportable Y population. Equations descriptive of this competition are

$$\begin{aligned} X_{n+1} - X_n &= r_x \, X_n \left(1 - \alpha \frac{Y_n}{N} - \frac{X_n}{M}\right) \\ Y_{n+1} - Y_n &= r_y \, Y_n \left(1 - \beta \frac{X_n}{M} - \frac{Y_n}{N}\right) \end{aligned} \tag{16.43}$$

Continued in Exercise 16.4.20

Discussion. In nature the parameters, r_x, r_y, α, and β, are not constant in time, they may change with seasons, for example, and it is unusual to have only two species interacting. Nevertheless, the models we study are suggestive of natural behavior. Some interesting data in Figure 16.9 is redrawn from Parsons et al[1] regarding oceanic phytoplankton (prey) and zooplankton (predator). Perhaps plants are not usually considered prey, but the model works in this setting. Observe first the periodic variation in total phytoplankton and zoopankton in the graphs. The Artic suggests variation in r_x with the season,

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[1] "Biological Oceanographic Processes, 2nd Edition, T. R. Parsons, M. Takahashi, and B. Hargrave, Pergamon Press Ltd., Oxford, 1977, Figure 17, p 30.

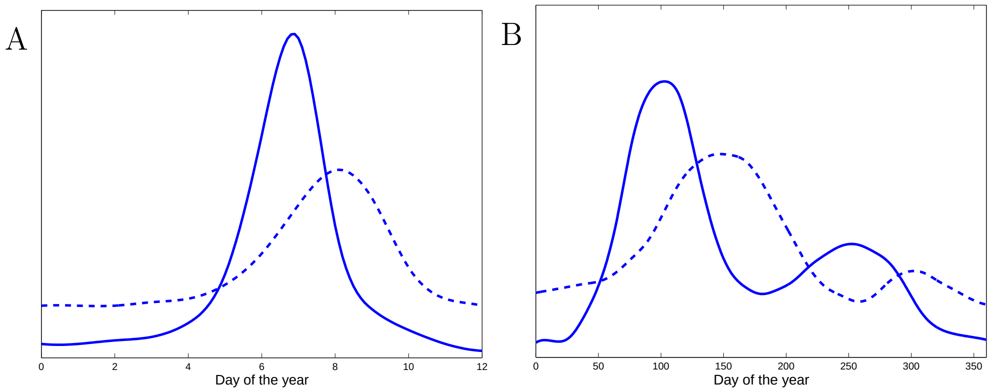

Figure 16.9: Oceanic phytoplankton (solid line) and zooplankton (dashed line) in A The Arctic and B. The North Atlantic. Redrawn from Parsons et al who modified a figure from A. K. Heinrich, 1962, The life histories of plankton animals and seasonal cycles of plankton communities in the ocean, J. Cons. Int. Explor. Mer, 27, 15-24. We have permission to use the figure from Heinrich.

and thus reflects external forces affecting the system. But the within year oscillations of the North Atlantic is suggestive of complex roots associated with an equilibrium point, a property internal to the system.

Exercises for Section 16.4, Four examples of nonlinear dynamical systems.

Exercise 16.4.1 Read the coordinates of 13 points of each of the curves phytoplankton and zooplankton for the North Atlantic in Figure 16.9B and plot a phase graph for these two groups.

Exercise 16.4.2 The point (0.75,0.75) was shown to be an equilibrium point of the deer dynamical system, Equations 16.30. Determine whether the system is asymptotically stable at (0.75,0.75).

Exercise 16.4.3 Suppose 20 deer were harvested annually from the deer population of 16.28. Show that (0.5,0.5) is an equilibrium point of the dynamical system. Determine whether the system is asymptotically stable at (0.5,0.5).

Exercise 16.4.4 Do Explore 16.4.3

Exercise 16.4.5 For the symbiosis system,

$$\begin{aligned} x_{n+1} - x_n &= \frac{5}{98} \cdot x_n \left(1 + 0.4 y_n - x_n\right) \\ y_{n+1} - y_n &= \frac{7}{120} \cdot y_n \left(1 + \frac{5}{7} x_n - y_n\right) \end{aligned}$$

(1.76,2.0) is an equilibrium point. Choose the initial value, $(x_0, y_0) = (1.76, 2.1)$ and compute (x_1, y_1), (x_2, y_2), (x_3, y_3).

Exercise 16.4.6 Choose an initial value, $(\xi_0, \eta_0) = (-0.2, 0.3)$ and

$$\begin{aligned} \hat{\xi}_{n+1} &= 0.9\hat{\xi}_n + 0.04\hat{\eta}_n \\ \hat{\eta}_{n+1} &= 0.1\hat{\xi}_n + 0.86\hat{\eta}_n \end{aligned}$$

to compute $\hat{\xi}_1, \hat{\eta}_1)$, $\hat{\xi}_2, \hat{\eta}_2)$, and $\hat{\xi}_3, \hat{\eta}_3)$. These are Equations 16.6A.

Exercise 16.4.7 Note: Use Equations 16.41.

For the following systems, find all of the equilibrium points (x_e, y_e).

a. For each equilibrium point with $x_e > 0$ and $y_e > 0$, determine the stability of the system at that equilibrium point.

b. Draw a phase plane and in each region of the phase plane bounded by null lines, draw a vector pointing from a point (x_n, y_n) toward (x_{n+1}, y_{n+1}).

$$\begin{array}{lrcl} \text{a.} & x_{n+1} - x_n & = & 0.1 * x_n * (1 - 0.5y_n - x_n) \\ & y_{n+1} - y_n & = & 0.05 * y_n * (1 + 0.2x_n - y_n) \end{array}$$

$$\begin{array}{lrcl} \text{b.} & x_{n+1} - x_n & = & 0.05 * x_n * (1 - 0.2y_n - x_n) \\ & y_{n+1} - y_n & = & 0.1 * y_n * (1 - 0.5x_n - y_n) \end{array}$$

$$\begin{array}{lrcl} \text{c.} & x_{n+1} - x_n & = & 0.2 * x_n * (1 + 0.4y_n - x_n) \\ & y_{n+1} - y_n & = & 0.4 * y_n * (1 + 0.8x_n - y_n) \end{array}$$

$$\begin{array}{lrcl} \text{d.} & x_{n+1} - x_n & = & 0.05 * x_n * (1 - 0.2y_n - x_n) \\ & y_{n+1} - y_n & = & 0.1 * y_n * (1 - y_n) \end{array}$$

Exercise 16.4.8 Use the parameter values

$$\alpha = 0.9, \qquad \beta = 0.8, \qquad r_x = 0.3, \qquad r_y = 0.2$$

in

$$\begin{aligned} x_{n+1} - x_n &= r_x \cdot x_n \ (1 + \alpha\, y_n - x_n) \\ y_{n+1} - y_n &= r_y \cdot y_n \ (1 + \beta\, x_n - y_n) \end{aligned}$$

a. Find the two-species equilibrium point ($x_e > 0$ and $y_e > 0$).

b. Write the local linear approximation at (x_e, y_e).

c. Write the characteristic equation of the linear system.

d. Compute the roots to the characteristic equation.

e. Is the system stable?

Exercise 16.4.9 Show that the local linear approximation to the symbiosis dynamical system at a two-species equilibrium point (that is, $x_e > 0$ and $y_e > 0$) has real roots.

Predator-Prey exercises follow.

Exercise 16.4.10 Normalization of the Equations 16.42,

$$\begin{aligned} X_{n+1} - X_n &= r_x \cdot X_n \left(1 - \alpha \tfrac{Y_n}{N} - \tfrac{X_n}{M}\right) \\ Y_{n+1} - Y_n &= r_y \cdot Y_n \left(1 + \beta \tfrac{X_n}{M} - \tfrac{Y_n}{N}\right), \end{aligned}$$

reduces the complexity. Divide the first equation of Equations 16.42 by M and the second equation by N. Let $x_n = \frac{X_n}{M}$ and $y_n = \frac{Y_n}{N}$ and show that

$$\begin{aligned} x_{n+1} - x_n &= r_x \cdot x_n \left(1 - \alpha y_n - x_n\right) \\ y_{n+1} - y_n &= r_y \cdot y_n \left(1 + \beta x_n - y_n\right) \end{aligned} \qquad (16.44)$$

Discussion. Equations 16.44 involve only four parameters whereas the original Equations 16.42 involved six. The dynamics and stability conditions are identical.

Exercise 16.4.11 Let (x_e, y_e) denote an equilibrium point of Equations 16.44. Then x_n and y_n are not changing when $x_n = x_e$ and $y_n = y_e$ so that

$$\begin{aligned} r_x x_e \left(1 - \alpha y_e - x_e\right) &= 0 \\ r_y y_e \left(1 + \beta x_e - y_e\right) &= 0 \end{aligned}$$

Show that the four possibilities are

$$(x_e, y_e) = (0,0), \quad (x_e, y_e) = (1,0), \quad (x_e, y_e) = (0,1), \quad \text{and} \quad \left\{ \begin{aligned} x_e &= \frac{1-\alpha}{1+\alpha\beta} \\ y_e &= \frac{1+\beta}{1+\alpha\beta} \end{aligned} \right.$$

Exercise 16.4.12 Draw a phase plane similar to that of Figure 16.6 for Equations 16.44,

$$\begin{aligned} x_{n+1} - x_n &= r_x \cdot x_n \left(1 - \alpha y_n - x_n\right) \\ y_{n+1} - y_n &= r_y \cdot y_n \left(1 + \beta x_n - y_n\right) \end{aligned}$$

and the parameters, $r_x = 0.04$, $r_y = 0.01$, $\alpha = 0.5$ and $\beta = 0.4$.

You should draw the graphs of $1 - 0.5y - x = 0$ and $1 + 0.4x - y = 0$ and identify the point of intersection of the two lines. Shade the region where both $1 - 0.5y - x$ and $1 + 0.4x - y = 0$ are positive. Draw at least one arrow pointing from a potential (x_n, y_n) toward (x_{n+1}, y_{n+1}) in each of the four regions marked by $1 - 0.5\,y - x = 0$ and $1 + 0.4\,x - y = 0$.

Exercise 16.4.13 We (and you) analyze here the two-species predator-prey equilibrium point.

Use Equations 16.41 to write the equilibrium point and the Jacobian matrix for the nonlinear dynamical system, Equations 16.44, at the equilibrium point,

Exercise 16.4.14 State explicitly the version of Theorem 16.4.1 that applies to the predator-prey model, Equations 16.44.

n	x_n	y_n	ξ_n	η_n
0	0.3333	1.0000	-0.0833	-0.1667
3	0.3648	1.0198	-0.0440	-0.1848
6	0.3901	1.0388	-0.0128	-0.1758
9	0.4093	1.0566	0.0094	-0.1497
12	0.4229	1.0731	0.0231	-0.1153
15	0.4317	1.0880	0.0295	-0.0793
18	0.4369	1.1013	0.0305	-0.0463
21	0.4394	1.1129	0.0276	-0.0190

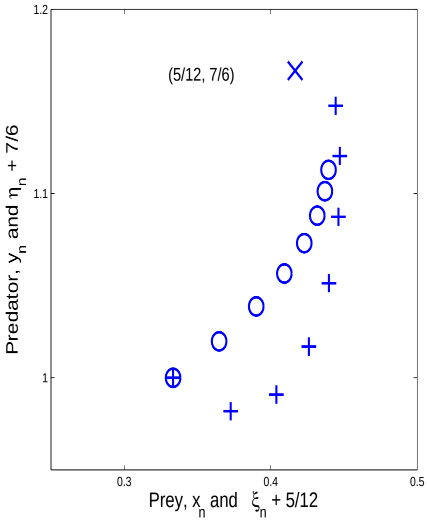

Figure 16.10: The trajectory of the nonlinear dynamical system 16.45 (symbol 'o') and the trajectory of the local linear approximation 16.46 (symbol '+') translated from the origin to the equilibrium point, (5/12,7/6).

Example 16.4.1 It is helpful to consider a special case. For the parameter values

$$\begin{array}{ll} r_x = 0.2 & \alpha = 0.5 \\ r_y = 0.05 & \beta = 0.4 \end{array}$$

The predator-prey equations are

$$\begin{aligned} x_{n+1} - x_n &= 0.2 \cdot x_n \,(1 - 0.5 y_n - x_n) \\ y_{n+1} - y_n &= 0.05 \cdot y_n \,(1 + 0.4 x_n - y_n) \end{aligned} \tag{16.45}$$

and the two-species equilibrium point is

$$\begin{aligned} x_e = \frac{1-\alpha}{1+\alpha\beta} &= 5/12 \\ y_e = \frac{1+\beta}{1+\alpha\beta} &= 7/6 \end{aligned}$$

The local linear approximation is

$$\begin{aligned} \xi_{n+1} &= (1 - r_x x_e)\xi_n - \alpha r_x x_e \eta_n &&= (11/12)\xi_n - (1/24)\eta_n \\ \eta_{n+1} &= \beta r_y y_e \xi_n + (1 - r_y y_e)\eta_n &&= -(7/300)\xi_n + (113/120)\eta_n \end{aligned} \tag{16.46}$$

Computations with Equations 16.45 and 16.46 with

$$\begin{array}{llll} r_x = 0.2 & \alpha = -0.5 & x_0 = 1/3 & \hat{\xi}_0 = -5/12 \\ r_y = 0.05 & \beta = 0.4 & y_0 = 1 & \hat{\eta}_0 = -1/6 \end{array}$$

are shown in Figure 16.10

Exercise 16.4.15 Choose the initial value, $(x_0, y_0) = (0.10.2)$ and use

$$\begin{aligned} x_{n+1} - x_n &= 0.3 \cdot x_n \,(1 - 0.2 y_n - x_n) \\ y_{n+1} - y_n &= 0.2 \cdot y_n \,(1 + 0.1 x_n - y_n) \end{aligned}$$

to compute (x_1, y_1), (x_2, y_2), $\cdots$ (x_{20}, y_{20}). Plot your points on a graph similar to that in Figure 16.10.

Exercise 16.4.16 Choose an initial value, $(\xi_0, \eta_0) = (-0.7, -0.9)$ and

$$\begin{aligned} \xi_{n+1} &= 0.7647\xi_n + -0.04706\eta_n \\ \eta_{n+1} &= 0.02157\xi_n + 0.7843\eta_n \end{aligned}$$

to compute $\xi_1, \eta_1)$, $\xi_2, \eta_2)$, $\cdots$ $\xi_8, \eta_8)$. Add (0.7843,1.0784) to each point and plot the resulting points on the axes you used for Exercise 16.4.15.

Exercise 16.4.17 Compute the characteristic values of

$$\begin{aligned} \xi_{n+1} &= 0.7647\xi_n + -0.04706\eta_n \\ \eta_{n+1} &= 0.02157\xi_n + 0.7843\eta_n \end{aligned}$$

and determine whether both of them are of magnitude less than 1.

Exercise 16.4.18 For each of the following systems, find an equilibrium point which has both coordinates positive, if there is one, and determine whether it is stable.

$$\begin{array}{lrclrcl} \text{a.} & x_{n+1} &=& x_n + 0.7x_n(1 - 0.5y_n - x_n) & y_{n+1} &=& y_n + 0.3y_n(1 + 0.2x_n - y_n) \\ \text{b.} & x_{n+1} &=& x_n + 0.7x_n(1 - 0.8y_n - x_n) & y_{n+1} &=& y_n + 0.3y_n(1 + 0.5x_n - y_n) \\ \text{c.} & x_{n+1} &=& x_n + 1.2x_n(1 - 0.9y_n - x_n) & y_{n+1} &=& y_n + 0.8y_n(1 + 0.2x_n - y_n) \\ \text{d.} & x_{n+1} &=& x_n + 1.8x_n(1 - 1.2y_n - x_n) & y_{n+1} &=& y_n + 1.2y_n(1 + 0.5x_n - y_n) \\ \text{e.} & x_{n+1} &=& x_n + 1.4x_n(1 - 1.0y_n - x_n) & y_{n+1} &=& y_n + 0.8y_n(1 + 0.5x_n - y_n) \\ \text{f.} & x_{n+1} &=& x_n + 1.4x_n(1 - 1.1y_n - x_n) & y_{n+1} &=& y_n + 0.8y_n(1 + 0.5x_n - y_n) \end{array}$$

Exercise 16.4.19 Show that with the parameters $\alpha = 0.8$, $\beta = 0.4$, $r_x = 1.5$ and $r_y = 0.2$ the equilibrium point in the predator-prey system 16.44 has positive coordinates but the system is not asymptotically stable at that equilibrium point.

Exercise 16.4.20 **Competition.** Analyze Equations 16.43,

$$\begin{aligned} X_{n+1} - X_n &= r_x\, X_n \left(1 - \alpha\frac{Y_n}{N} - \frac{X_n}{M}\right) \\ Y_{n+1} - Y_n &= r_y\, Y_n \left(1 - \beta\frac{X_n}{M} - \frac{Y_n}{N}\right), \end{aligned}$$

that describe interaction of two competing species.

The steps will include:

a. Normalize the equations to obtain equations that have only four parameters.

Normalize the competition Equations 16.43. (16.47)

b. Show that there are four equilibrium points, one of which is

$$\left(\frac{1-\alpha}{1-\alpha\,\beta}, \frac{1-\beta}{1-\alpha\,\beta}\right)$$

c. Draw the phase plane for Equations 16.47 with $r_x = 0.2$, $r_y = 0.1$, $\alpha = 0.4$ and $\beta = 0.1$, including the lines $1 - 0.3\,y - x = 0$ and $1 - 0.4\,x - y = 0$, the coordinates of the two-species equilibrium point, and arrows in each of four regions bounded by the lines.

d. Determine whether previous system is locally stable at the two-species equilibrium point.

e. Draw the phase plane and analyze Equations 16.47 with with $r_x = 0.2$, $r_y = 0.1$, $\alpha = 1.5$ and $\beta = 2$.

f. State and prove a theorem that gives sufficient conditions for the two-species equilibrium point of Equations 16.47 to be locally stable.

Exercise 16.4.21 For each of the following systems, find an equilibrium point which has both coordinates positive, if there is one, and determine whether it is stable.

a. $x_{n+1} = x_n + 0.7x_n(1 - 0.5y_n - x_n) \qquad y_{n+1} = y_n + 0.3y_n(1 - 0.2x_n - y_n)$

b. $x_{n+1} = x_n + 0.7x_n(1 - 0.8y_n - x_n) \qquad y_{n+1} = y_n + 0.3y_n(1 - 0.5x_n - y_n)$

c. $x_{n+1} = x_n + 1.2x_n(1 - 0.9y_n - x_n) \qquad y_{n+1} = y_n + 0.8y_n(1 - 0.2x_n - y_n)$

d. $x_{n+1} = x_n + 1.8x_n(1 - 1.2y_n - x_n) \qquad y_{n+1} = y_n + 1.2y_n(1 - 0.5x_n - y_n)$

e. $x_{n+1} = x_n + 1.4x_n(1 - 1.0y_n - x_n) \qquad y_{n+1} = y_n + 0.8y_n(1 - 0.5x_n - y_n)$

f. $x_{n+1} = x_n + 1.4x_n(1 - 0.9y_n - x_n) \qquad y_{n+1} = y_n + 0.8y_n(1 - 0.5x_n - y_n)$

16.5 Chapter 16 Suggested solutions.

Explore 16.1.1. It is yours to guess. The answer follows in the text.

Exercise 16.1.1. c. Now the model and equilibrium equations are

$$\begin{aligned}
A_{t+1} - A_t &= -(0.08 + 0.15)A_t + 0.1B_t + 32\\
B_{t+1} - B_t &= 0.15A_t - 0.10B_t\\
A_e - A_e &= -(0.08 + 0.15)A_e + 0.1B_e + 32\\
B_e - B_e &= 0.15A_e - 0.10B_e,
\end{aligned}$$

from which

$$A_e = 40 \quad \mu\text{g/ml} \qquad B_e = 60 \quad \mu\text{g/ml}$$

Exercise 16.1.2. If $I_e = 0$, S_e and R_e can be any number.

If $I_e \neq 0$ then ????.

Exercise 16.2.1. a. $r^2 - 1.76r + 0.77 = 0$, $r_1 = 0.916$, $r_2 = 0.814$.

c. $r^2 - 1.76r + 0.734 = 0$, $r_1 = 1.081$, $r_2 = 0.679$.

Exercise 16.2.2. a. $p = 1.76$, $q = 0.77$, $0 < 1.76 < 1 + 0.77 < 2$, Stable.

c. $p = 1.76$, $q = 0.734$, $0 < 1.76 \not< 1 + 0.734 < 2$, Not Stable.

Exercise 16.2.3.

A		C	
x_t	y_t	x_t	y_t
0.2500	0.5000	0.2500	0.5000
0.2450	0.4550	0.4250	0.4550
0.2387	0.4158	0.5645	0.4338
0.2315	0.3815	0.6816	0.4295

Exercise 16.2.5. $x_2 = 0.5645$, $x_3 = 0.6816$, $y_2 = 0.4338$, $y_3 = 0.4295$.

Exercise 16.2.6. $x_2 = -0.4934$, $x_3 = -0.8714$, $y_2 = 0.3801$, $y_3 = 0.2775$.

Exercise 16.2.7. a. $p = 1.2$, $q = 0.3430$, $r_1 = 0.7302$, $r_2 = 0.4698$, Stable. c. $p = 2$, $q = 1.09$, $r_1 = 1.000 + 0.3i$, $\rho = 1.044$, Unstable. e. $p = 1.8$, $q = 0.8100$, $r_1 = 0.9$, $r_2 = 0.9$, Stable.

Exercise 16.2.8.

a. $r_1 = 1$, $r_2 = 1$, Not Stable.

t	0	1	2	3	4
x_t	2	2	2	2	2
y_t	3	3	3	3	3

c. $r_1 = i$, $r_2 = -i$, $\rho = 1$ Not Stable.

t	0	1	2	3	4
x_t	2	3	-2	-3	2
y_t	3	-2	-3	2	3

e. $r_1 = 2$, $r_2 = 0.5$, Not Stable.

t	0	1	2	3	4
x_t	2	1	0.5	0.25	0.125
y_t	0	0	0	0	0

g. $r_1 = 1$, $r_2 = 0$, Not Stable.

t	0	1	2	3	4
x_t	2.0	2.5	2.5	2.5	2.5
y_t	3.0	2.5	2.5	2.5	2.5

i. $r_1 = 1$, $r_2 = 0$, Not Stable.

t	0	1	2	3	4
x_t	1.0	2.50	2.50	2.50	2.50
y_t	2.0	1.25	1.25	1.25	1.25

k. $r_1 = \cos(\theta) + i\sin\theta$, $\rho = 1$, Not Stable.

t	0	1	2	3	4
x_t	1	$\cos\theta$	$\cos 2\theta$	$\cos 3\theta$	$\cos 4\theta$
y_t	0	$\sin\theta$	$\sin 2\theta$	$\sin 3\theta$	$\sin 4\theta$

Exercise 16.2.9. $0 < |p| < 1 + q < 2$. a. $|p| = 0.5$, $q = 0.5$, $0 < 0.5 < 1 + 0.5 < 2$. Stable.

c. $|p| = 1.4$, $q = 0.5$, $0 < 1.4 < 1 + 0.5 < 2$. Stable.

e. $|p| = 0.5$, $q = 1.5$, $0 < 0.5 < 2.5 \not< 2$. Not Stable

g. $|p| = 1.0$, $q = 0$, $0 < 1.0 \not< 1 < 2$. Not Stable.

Exercise 16.2.10. $p = 1.6$, $q = 0.6072$, $r_1 \doteq 0.61889$, $r_2 \doteq -0.98111$. Stable

t	0	1	2	3	4	5	6	7
x_t	0.8100	0.7164	0.6544	0.6121	0.5819	0.5594	0.5418	0.5271
y_t	0.5800	0.7440	0.8382	0.8894	0.9141	0.9225	0.9209	0.9134
$\|[x_t, y_t]\|$	0.9962	1.0328	1.0634	1.0797	1.0836	1.0789	1.0685	1.0546

t	8	9	10	11
x_t	0.5145	0.5031	0.4925	0.4826
y_t	0.9022	0.8889	0.8745	0.8594
$\|[x_t, y_t]\|$	1.0386	1.0214	1.0036	0.9856

Exercise 16.2.11. Suppose $r < 1$. Then

$$\lim_{t\to\infty} t\, r^t = \lim_{t\to\infty} \frac{t}{(1/r)^t} \overset{a}{=} \lim_{t\to\infty} \frac{1}{(1/r)^t\, \ln(1/r)} = \lim_{t\to\infty} \frac{r^t}{-\ln r} \overset{b}{=} \frac{0}{-\ln r} = 0$$

At step a, L'Hospital's Rule is used. At step b, the argument is similar to that of Exercise 14.5.8 to show that $\lim\limits_{t\to\infty} 1/2^t = 0$.

Exercise 16.2.13.

$$\begin{aligned} x_0 &= 0.25 & x_{n+1} &= 0.9 * x_n - 0.8 * y_n \\ y_0 &= 0.5 & y_{n+1} &= 0.1 * x_n + 0.86 * y_n \end{aligned}$$

$$\begin{aligned} y_1 &= 0.1\, x_0 + 0.86\, y_0 = 0.1 \cdot 0.25 + 0.86 \cdot 0.5 = 0.455 \\ 0.5 &= D_1 + D_2 \\ 0.455 &= D_1 \times r_1^1 + D_2 \times r_2^1 \\ 0.455 &= (0.88 + 0.28213\, i) \times D_1 + (0.88 - 0.28213\, i) \cdot D_2 \\ (0.88 - 0.28213\, i)0.5 &= (0.88 - 0.28213\, i)D_1 + (0.88 - 0.28213\, i)D_2 \\ 0.015 + 0.14107\, i &= 0.56426\, i \cdot D_1 \\ D_1 &= (0.015 + 0.14107\, i)/0.56426\, i = 0.25 - 0.026583\, i \\ D_2 &= 0.5 - D_1 = 0.25 + 0.026583\, i \end{aligned}$$

$$\begin{aligned} y_t &= (0.25 + 0.026583\, i) \cdot (0.88 + 0.28213\, i)^t \\ &\quad + (0.25 - 0.026583\, i) \cdot (0.88 - 0.28213\, i)^t. \end{aligned}$$

t	0	1	2	3	4	5	6	7	8
x_t	0.250	-0.175	-0.522	-0.768	-0.907	-0.940	-0.880	-0.7460	-0.561
y_t	0.500	0.455	0.374	0.269	0.155	0.042	-0.058	-0.138	-0.193

MATLAB and TI-86 programs to solve the system follow.

```
close all;clc;clear
m=[0.9 -0.8;0.1 0.86];
z(:,1)=[0.25;0.5]; z(:,2)=m*z(:,1);
p=trace(m);             q=det(m);
r1=(p+sqrt(p^2-4*q))/2;
r2=(p-sqrt(p^2-4*q))/2;
C1=(z(1,2)-r2*z(1,1))/(r1-r2);
C2=(r1*z(1,1)-z(1,2))/(r1-r2);
D1=(z(2,2)-r2*z(2,1))/(r1-r2);
D2=(r1*z(2,1)-z(2,2))/(r1-r2);
for i = 2:8
    z(1,i+1)=C1*r1^i+C2*r2^i;
    z(2,i+1)=D1*r1^i+D2*r2^i;
end
['p      ' 'q      ' 'r1     ' 'r2']
[p  q  r1 r2]
['C1     ' 'C2     ' 'D1     ' 'D2']
[C1 C2 D1 D2]
z
```

```
PROGRAM:LIN
:Fix 5
:(0.125,0.70003) -> C1
:(0.125,-0.70003) -> C2
:(0.88,0.28213) -> R1
:(0.88,-0.28213) -> R2
:For(N,0,5)
:XN=C1*R1^N+C2*R2^N
:Display XN
:End
:Fix 9
```

Exercise 16.3.1.

$$\text{a.}\quad \begin{bmatrix} x^2+y^2 \\ x \times y \end{bmatrix}_{0,0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and (0,0) is an equilibrium point.

$$J = \begin{bmatrix} 2x & 2y \\ y & xy \end{bmatrix}_{0,0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$r_1 = r_2 = 0$, and the system is asymptotically stable at (0,0).

$$\text{c.}\quad \begin{bmatrix} x(1-x)(1-y)+1/6 \\ y(1-y)(1-x)+1/8 \end{bmatrix}_{1/3,1/4} = \begin{bmatrix} 1/3 \\ 1/4 \end{bmatrix}$$

and (1/3,1/4) is an equilibrium point.

$$J = \begin{bmatrix} (1-2x)(1-y) & -x(1-x) \\ -y(1-y) & (1-2y)(1-x) \end{bmatrix}_{1/3,1/4} = \begin{bmatrix} 1/4 & -2/9 \\ -3/16 & 1/3 \end{bmatrix}$$

$r_1 \doteq 0.6477$ and $r_2 \doteq -0.0643$, and the system is asymptotically stable at (1/3,1/4).

$$\text{e.}\quad \begin{bmatrix} x+0.1y(1-x)-0.018 \\ y+0.2x(1-y)-0.016 \end{bmatrix}_{0.1,0.2} = \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix}$$

and (0.1,0.2) is an equilibrium point.

$$J = \begin{bmatrix} 1-0.1y & 0.1(1-x) \\ 0.2(1-y) & 1-0.2x \end{bmatrix}_{0.1,0.2} = \begin{bmatrix} 0.98 & 0.09 \\ 0.16 & 0.98 \end{bmatrix}$$

$r_1 = 1.1$ and $r_2 = 0.9$, and the system is not stable at (0.1,0.2).

g. (3/2,1/2) is an equilibrium point and the system is not stable at (3/2,1/2).

Exercise 16.3.3. (1,0) is an equilibrium point and $r_1 = 1$ and $r_2 = 1 + \beta\, M - \gamma$.

To be stable we must have

$$1 + \beta\, M - \gamma < 1, \qquad \text{then}\beta\, M < \gamma, \qquad \text{or} \qquad \frac{\beta M}{\gamma} < 1.$$

$$\text{The number} \quad \frac{\beta M}{\gamma} \quad \text{is a critical number for control of epidemics.}$$

Exercise 16.3.4. a. $\lim_{t\to\infty}\, x_t = 0$.

c. $\lim_{t\to\infty}\, x_t = \infty$.

Exercise 16.4.4. The inequality $r_x\, x_e = 0.3 \cdot 85/22 = 1.1591 < 1$ is not satisfied.

The Jacobian,

$$J|_{85/22,65/22} = \begin{bmatrix} (1 - r_x x_e) & \alpha r_x x_e \\ \beta r_y y_e & (1 - r_y y_e) \end{bmatrix}_{85/22,65/22} \doteq \begin{bmatrix} -0.1591 & 0.8114 \\ 0.6545 & 0.1818 \end{bmatrix}$$

has characteristic equation $r^2 - 0.227r - 0.5600 = 0$ has roots $r_1 \doteq 0.7598$ and $r_2 \doteq -0.7371$ and the system is stable.

Explore 16.4.2. . *Proof.* It will suffice to show that the linear approximations

$$\begin{aligned} \xi_{n+1} &= (1 - r_x x_e)\xi_n + A r_x x_e \eta_n \\ \eta_{n+1} &= B r_y y_e \xi_n + (1 - r_y y_e)\eta_n \end{aligned} \tag{16.48}$$

are stable. Let K be the maximum of $1 - r_x x_e + |A|\, r_x x_e$ and $|B| r_y y_e + 1 - r_y y_e$. (Observe that $0 < K < 1$). Then for any (ξ_n, η_n)

$$\begin{aligned} |\xi_{n+1}| &= |(1 - r_x x_e)\xi_n + A r_x x_e \eta_n| \\ &\le |(1 - r_x x_e)|\xi_n| + |A r_x x_e \eta_n| \\ &\le (1 - r_x x_e)\max(|\xi_n|, |\eta_n|) + |A| r_x x_e \max(|\xi_n|, |\eta_n|) \\ &\le (1 - r_x x_e + |A| r_x x_e)\max(|\xi_n|, |\eta_n|) \\ &\le K \max(|\xi_n|, |\eta_n|) \end{aligned}$$

Similarly

$$|\eta_{n+1}| \le K \max(|\xi_n|, |\eta_n|), \quad \text{so that} \quad \max(|\xi_{n+1}|, |\eta_{n+1}|) \le K \max(|\xi_n|, |\eta_n|).$$

By cascading

$$\max(|\xi_n|, |\eta_n|) \le K^n \max(|\xi_0|, |\eta_0|) \qquad \text{and} \qquad (\xi_n, \eta_n) \to (0, 0)$$

The local linear approximation 16.48 is stable. End of proof.

Exercise 16.4.2. The system is locally asymptotically stable at (0.75,0.75).

Exercise 16.4.3. Equation 16.28 now becomes

$$w_{t+1} - w_t = 0.16 \times w_{t-1} \times \left(1 - \frac{w_{t-1}}{500}\right) - 20$$

and the two dimensional dynamical system corresponding to 16.30 is

$$\begin{aligned} u_{t+1} &= u_t + 0.16 v_t (1 - v_t) - 0.04 \\ v_{t+1} &= u_t \end{aligned}$$

The equilibrium point is $(v_e, u_e) = (0.5, 0.5)$ and the roots are $r_1 = 1$ and $r_2 = 0$.

Exercise 16.4.5.

n	x_n	y_n
0	1.7600	2.1000
1	1.7672	2.1193

Exercise 16.4.7. The equilibrium point, Jacobian at the equilibrium point, and roots of the Jacobian are denoted by e, J, and r, respectively. See the graphs in Figure 16.11.

$$\text{a.} \quad e = \begin{bmatrix} 0.4545 \\ 1.0909 \end{bmatrix} \qquad J = \begin{bmatrix} 0.9545 & -0.0227 \\ 0.0109 & 0.9455 \end{bmatrix} \qquad r = \begin{bmatrix} 0.9500 + 0.0151i \\ 0.9500 - 0.0151i \end{bmatrix}$$

$$\text{c.} \quad e = \begin{bmatrix} 2.0588 \\ 2.6471 \end{bmatrix} \qquad J = \begin{bmatrix} 0.5882 & 0.1647 \\ 0.8471 & -0.0588 \end{bmatrix} \qquad r = \begin{bmatrix} 0.7589 \\ -0.2294 \end{bmatrix}$$

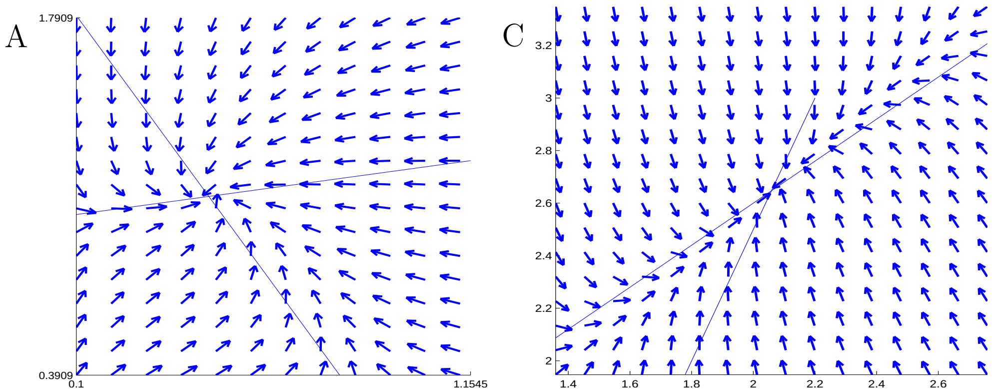

Figure 16.11: Phase plane graphs for A. $x_{n+1} - x_n = 0.1 * x_n * (1 - 0.5y_n - x_n)$, $y_{n+1} - y_n = 0.05 * y_n * (1 + 0.2x_n - y_n)$; and C. $x_{n+1} - x_n = 0.2 * x_n * (1 + 0.4y_n - x_n)$ $y_{n+1} - y_n = 0.4 * y_n * (1 + 0.8x_n - y_n)$.

Exercise 16.4.8. $(x_e, y_e) = (47.5/7, 45/7)$ and $r_1 = 0.7624$, $r_2 = -2.0838$.

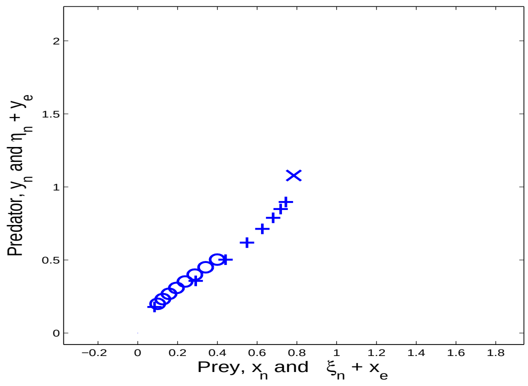

Figure 16.12: Exercise 16.4.16. Graph for Exercises 16.4.15 (o) and 16.4.16 (+).

Exercise 16.4.9. The Jacobian of the local linear approximation to the symbiosis dynamical system at a two-species equilibrium point is

$$\begin{bmatrix} 1 - r_x\, x_e & r_x\, \alpha\, x_e \\ r_y\, \beta\, y_e & 1 - r_y\, y_e \end{bmatrix}$$

$$p = (1 - r_x\, x_e) + (1 - r_y\, y_e), \qquad q = (1 - r_x\, x_e) \cdot (1 - r_y\, y_e) - (r_y\, \beta\, y_e)(r_x\, \alpha\, x_e)$$

Show that $p^2 - 4q > 0$.

Exercise 16.4.15 and 16.4.16. See Figure 16.12.

Exercise 16.4.17. The roots are $0.7745 + 0.0303i$ and $0.7747 - 0.0303i$. The magnitudes are $\sqrt{0.7745^2 + 0.0303^2} = 0.7751$. The system is stable.

Exercise 16.4.18. a. $r_1 = 0.6772 + 0.1019i$, $r_2 = 0.6772 - 0.1019i$. Stable.

c. $r_1 = 0.8767$, $r_2 = 2080$. Stable.

e. No equilibrium with positive coordinates.

Exercise 16.4.19. With the parameters $\alpha = 0.8$, $\beta = 0.4$, $r_x = 1.5$ and $r_y = 0.2$ the equilibrium point is $(1.36\overline{36}, 0.45\overline{45})$. The characteristic roots are $r_1 \doteq 0.87816$ and $r_2 \doteq -1.01452$. Because $|r_2| > 1$, the system will be unstable.

Exercise 16.4.20.

a. Normalize:

$$\begin{aligned} \frac{X_{n+1}}{M} - \frac{X_n}{M} &= r_x \frac{X_n}{M}\left(1 - \alpha\frac{Y_n}{N} - \frac{X_n}{M}\right) \\ \frac{Y_{n+1}}{N} - \frac{Y_n}{N} &= r_y \frac{Y_n}{N}\left(1 - \beta\frac{X_n}{M} - \frac{Y_n}{N}\right) \\ x_{n+1} - x_n &= r_x\, x_n\,(1 - \alpha\, y_n - x_n) \\ y_{n+1} - y_n &= r_y\, y_n\,(1 - \beta\, x_n - y_n) \end{aligned}$$

b. Find the equilibria:

$$\begin{aligned} x_e - x_e &= r_x\, x_e\, (1 - \alpha\, y_e - x_e) \\ y_e - y_e &= r_y\, y_e\, (1 - \beta\, x_e - y_e) \end{aligned}$$

$$\begin{aligned} 0 &= x_e\, (1 - \alpha\, y_e - x_e) \qquad && x_e = 0 \quad \text{or} \quad 1 - \alpha\, y_e - x_e = 0 \\ 0 &= y_e\, (1 - \beta\, x_e - y_e) \qquad && y_e = 0 \quad \text{or} \quad 1 - \beta\, x_e - y_e = 0 \end{aligned}$$

$$\begin{array}{lll} x_e = 0 & \text{and} \;\; y_e = 0, & (x_e, y_e) = (0, 0). \\ x_e = 0 & \text{and} \;\; 1 - \beta\, x_e - y_e = 0, & (x_e, y_e) = (0, 1). \\ 1 - \alpha\, y_e - x_e = 0 & \text{and} \;\; y_e = 0, & (x_e, y_e) = (1, 0). \\ 1 - \alpha\, y_e - x_e = 0 & \text{and} \;\; 1 - \beta\, x_e - y_e = 0, & (x_e, y_e) = \left(\dfrac{1-\alpha}{1-\alpha\,\beta}, \dfrac{1-\alpha}{1-\alpha\,\beta}\right). \end{array}$$

c. Parameters are $r_x = 0.2$, $r_y = 0.1$, $\alpha = 0.4$ and $\beta = 0.1$. See Figure 16.13A.

d. The Jacobian at (5/8,15/16) of the competition equations is

$$\begin{aligned} F(x, y) &= x + r_x \cdot x(1 - \alpha y - x) \\ G(x, y) &= y + r_y \cdot y(1 - \beta x - y) \\ J_{1,1} = F_1(x, y) &= 1 + r_x \cdot (1 - \alpha y - x) - r_x x \\ J_{1,2} = F_2(x, y) &= -\alpha r_x x \\ J_{2,1} = G_1(x, y) &= -\beta r_y y \\ J_{2,2} = G_2(x, y) &= 1 + r_y \cdot (1 - \beta x - y) - r_y y \\ F_1(x_e, y_e) &= 1 + r_x \cdot (1 - \alpha y_e - x_e) - r_x x_e = 1 - r_x x_e \\ F_2(x_e, y_e) &= -\alpha r_x\, x_e \\ G_1(x_e, y_e) &= -\beta r_y\, y_e \\ G_2(x_e, y_e) &= 1 + r_y \cdot (1 - \beta x_e - y_e) - r_y\, y_e = 1 - r_y\, y_e \end{aligned}$$

$$\begin{aligned} J &= \begin{bmatrix} 1 - r_x x_e & -\alpha r_x\, x_e \\ -\beta r_y\, y_e & 1 - r_y\, y_e \end{bmatrix} = \begin{bmatrix} 1 - 0.2 \cdot 0.625 & -0.4 \cdot 0.2 \cdot 0.625 \\ -0.1 \cdot 0.1 \cdot 0.9375 & 1 - 0.1 \cdot 0.9375 \end{bmatrix} \\ &= \begin{bmatrix} 0.875 & -0.026 \\ -0.009375 & 0.90625 \end{bmatrix} \end{aligned}$$

$p = 1.78125$, $q = 0.79272$, $p^2 - 4 * q = 0.001951563$. The roots are real and $r_1 \doteq 0.9128$ and $r_2 \doteq 0.8684$. The system is stable.

e. The parameters are $r_x = 0.2$, $r_y = 0.1$, $\alpha = 1.5$ and $\beta = 2$. See Figure 16.13A. The Jacobian is

$$J = \begin{bmatrix} 1 - r_x x_e & -\alpha r_x\, x_e \\ -\beta r_y\, y_e & 1 - r_y\, y_e \end{bmatrix} = \begin{bmatrix} 1 - 0.2 \cdot 0.25 & -1.5 \cdot 0.2 \cdot 0.25 \\ -2 \cdot 0.1 \cdot 0.5 & 1 - 0.1 \cdot 0.5 \end{bmatrix}$$

$$= \begin{bmatrix} 0.95 & -0.075 \\ -0.1 & 0.95 \end{bmatrix}$$

$p = 1.9$, $q = 0.895$, $r_1 \doteq 1.0366$ and $r_2 \doteq 0.8634$. Not Stable.

f. **Theorem 16.5.1** Asymptotic stability of the competition two-species equilibrium point. If

$$\alpha < 1, \qquad \beta < 1, \qquad r_x x_e < 1, \qquad \text{and} \qquad r_y y_e < 1$$

then the equilibrium point

$$(x_e, y_e) = (\frac{1-\alpha}{1-\alpha\beta}, \frac{1-\beta}{1-\alpha\beta})$$

of the system

$$\begin{aligned} x_{n+1} - x_n &= r_x \cdot x_n\,(1 - \alpha y_n - x_n) \\ y_{n+1} - y_n &= r_y \cdot y_n\,(1 - \beta x_n - y_n) \end{aligned}$$

has positive coordinates and the system is asymptotically stable at (x_e, y_e).

Proof. It will suffice to show that the linear approximation

$$\begin{aligned} \xi_{n+1} &= (1 - r_x x_e)\xi_n \quad + \quad -\alpha r_x x_e \eta_n \\ \eta_{n+1} &= -\beta r_y y_e \eta_n \qquad + \quad (1 - r_y y_e)\eta_n \end{aligned}$$

is stable. Because $0 < r_x x_e < 1$ and $0 < \alpha < 1$, $1 - r_x x_e + \alpha r_x x_e < 1$. Similarly, $\beta r_y y_e + 1 - r_y y_e < 1$. Let K be the maximum of $1 - r_x x_e + \alpha r_x x_e$ and $\beta r_y y_e + 1 - r_y y_e$. . Then for any (ξ_n, η_n)

$$\begin{aligned} |\xi_{n+1}| &= |(1 - r_x x_e)\xi_n - \alpha r_x x_e \eta_n| \\ &\leq (1 - r_x x_e)\max(|\xi_n|, |\eta_n|) + \alpha r_x x_e \max(|\xi_n|, |\eta_n|) \\ &\leq K \max(|\xi_n|, |\eta_n|) \end{aligned}$$

Similarly

$$|\eta_{n+1}| \leq K \max(|\xi_n|, |\eta_n|), \quad \text{so that} \quad \max(|\xi_{n+1}|, |\eta_{n+1}|) \leq K \max(|\xi_n|, |\eta_n|).$$

By cascading

$$\max(|\xi_n|, |\eta_n|) \leq K^n \max(|\xi_0|, |\eta_0|) \qquad \text{and} \qquad (\xi_n, \eta_n) \to (0, 0)$$

The local linear approximation 16.48 is stable. End of proof.

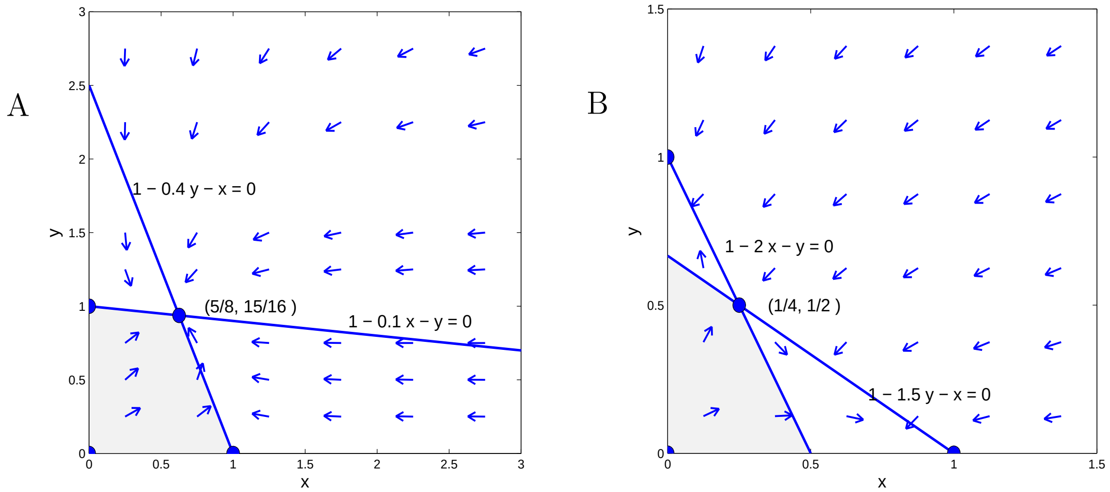

Figure 16.13: Exercise 16.4.20. Phase planes for the competition equations and two sets of parameters: A. Parameters are $r_x = 0.2$, $r_y = 0.1$, $\alpha = 0.4$ and $\beta = 0.1$ and the system is stable. B. The parameters are $r_x = 0.2$, $r_y = 0.1$, $\alpha = 1.5$ and $\beta = 2$ and the system is not stable. Two arrows in the regions upper left and lower right of the equilibrium suggest that the system may not be stable.

Exercise 16.4.21. a. $r_1 = 0.5535$, $r_2 = 0.7910$, Stable.

c. $r_1 = 0.8846$, $r_2 = 0.1886$, Stable.

e. No equilibrium point with both coordinates positive.

Chapter 17

Differential Equations

In This Chapter:

Continuous analogs of the following discrete models and additional continuous models are presented in this chapter.

We have presented several *discrete difference equation* models, including

Exponential Growth
$$P_{t+1} - P_t = R \times P_t$$

Logistic Growth
$$P_{t+1} - P_t = R \times P_t \times \left(1 - \frac{P_t}{M}\right)$$

Penicillin Clearance
$$\begin{array}{rcll} A_{t+1} - A_t & = & (-0.162 - 0.068)\,A_t + 0.1B_t & A_0 = 200 \\ B_{t+1} - B_t & = & 0.068A_t - 0.1B_t & B_0 = 0 \end{array}$$

Disease Spread
$$\begin{array}{rcllcr} S_{t+1} - S_t & = & -B \times S_t \times I_t & S_0 & = & 24,750 \\ I_{t+1} - I_t & = & B \times S_t \times I_t - G \times I_t & I_0 & = & 250 \\ R_{t+1} - R_t & = & G \times I_t & R_0 & = & 0 \end{array}$$

Whale Dynamics
$$N_{t+1} = 0.94N_t + N_{t-8}\left[0.06 + 0.0567\left\{1 - \left(\frac{N_{t-8}}{N_0}\right)^{2.39}\right\}\right] - 0.94C_t$$

We have also presented two *continuous time* models,

Exponential Growth
$$p(0) = p_0 \qquad p'(t) = r \times p(t)$$

Harmonic Oscillations
$$\begin{array}{rcl} y(0) & = & y_0 \\ y'(0) & = & y'_0 \end{array} \qquad y''(t) + \omega^2 \times y(t) = 0$$

For whale populations that are finite and have annual births, the discrete time model

$$N_{t+1} = 0.94N_t + N_{t-8}\left[0.06 + 0.0567\left\{1 - \left(\frac{N_{t-8}}{N_0}\right)^{2.39}\right\}\right] - 0.94C_t$$

is appropriate. Although bacterial populations are finite, they typically are so numerous and cell division so rapid that continuous time models with instantaneous rates of change describe well bacterial growth. Drug concentrations change continuously with continuous exchange between tissues and the vascular pool and with continuous kidney action, and continuous time models with instantaneous rates of change are preferred.

The discrete logistic growth, penicillin clearance, and disease spread models may be replaced by continuous time models

$$\text{Logistic Growth} \qquad p'(t) = r \times p(t) \times \left(1 - \frac{p(t)}{M}\right)$$

$$\text{Penicillin Clearance} \qquad \begin{array}{rclrcl} a'(t) &=& (-r_{0,1} - r_{2,1})\, a(t) + r_{1,2} b(t) & a(0) &=& 200 \\ b'(t) &=& r_{2,1} a(t) - r_{1,2} b(t) & b(0) &=& 0 \end{array}$$

$$\begin{array}{c}\text{Disease}\\ \text{Spread}\end{array} \qquad \begin{array}{rclrcl} s'(t) &=& -\beta \times s(t) \times i(t) & s(0) &=& 24,750 \\ i'(t) &=& \beta \times s(t) \times i(t) - \gamma \times i(t) & i(0) &=& 250 \\ r'(t) &=& \gamma \times i(t) & r(0) &=& 0 \end{array}$$

Rate coefficients of the discrete and continuous models are related but not equal. For logistic growth, for example, r in the continuous model and R in the discrete model are related by $r = \ln(1 + R)$.

Except for the harmonic equation, the derivative equations above are all **first order** differential equations because the first derivative and no higher order derivative appears. The harmonic equations

$$y''(t) + \omega^2 y(t) = 0 \qquad y''(t) + 3y'(t) + 4y(t) = 0$$

involve the second derivatives and are examples of a **second order** differential equations.

The equations are called **differential equations**, but would be better named *derivative* equations because they relate derivatives of functions to the functions. The words *differential equation* are in universal usage, however, and we will do the same. The expression derives from a frequent practice of writing, for example, the equation

$$y'(t) = 0.02 \times y \times (1 - \frac{y}{10}) \tag{17.1}$$

in Leibnitz notation

$$\frac{dy}{dt} = 0.02 \times y \times (1 - \frac{y}{10})$$

and then in *differential* form

$$dy = 0.02 \times y \times (1 - \frac{y}{10})\, dt$$

and finally even in the form

$$\frac{dy}{y \times (1 - y/10)} = 0.02\, dt$$

We have not defined the symbols, dy and dt. They are called differentials, and considered to be infinitesimals; dy is an infinitesimal change in y due to an infinitesimal change, dt, in t. Thus, y' is the slope, $\frac{\text{rise}}{\text{run}} = \frac{dy}{dt}$. This concept has been used successfully by many scientist, but has suffered from philosophical attack on the concept, infinitesimal. We used the notation $\int_a^b f(t)\,dt$ without separate definition of dt. We do not do so, but some scientists reason that to integrate f on $[a, b]$, one "sums the values of $f(t)$ times infinitesimal increases, dt, in t from $t = a$ to $t = b$."

17.1 Differential equation models of biological processes.

Differential equations were important early in the development of biological concepts.

17.1.1 Single species population models.

Thomas Malthus[1] described human population growth as

> **Malthus' Model of Population Growth.** The rate of population increase is proportional to the size of the population.

Let $p(t)$ denote population size at time t. Then Malthus asserted that

$$\begin{array}{lcccc} & \begin{array}{c}\text{Rate of}\\ \text{increase}\end{array} & \begin{array}{c}\text{is proportional}\\ \text{to}\end{array} & & \begin{array}{c}\text{population}\\ \text{size}\end{array} \\ \\ \textbf{Malthus} & p'(t) & = & r \quad \times & p(t)\end{array}$$

From this differential equation we have concluded (Property of Exponential Growth or Decay, Property 5.5.2, that

$$p(t) = p_0 e^{rt} \qquad (p_0 \text{ is founding population size})$$

which is the exponential growth Malthus predicted. Malthus also modeled the increase in food production as

> **Malthus' Model of Food Production.** The growth rate of food production is constant.

Let $f(t)$ denote the food produced at time t. Then

$$\begin{array}{lccc} & \begin{array}{c}\text{Rate of}\\ \text{increase}\end{array} & \text{is} & \text{constant} \\ \\ \textbf{Food Production} & f'(t) & = & C\end{array}$$

We know that if $f'(t) = C$ then

$$f(t) = Ct + f_0 \qquad (f_0 \text{ is initial food production.})$$

Malthus knew that every exponential graph eventually exceeds every linear graph and direly predicted wide spread poverty and degradation, Figure 17.1.

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[1] Anonymous publication in 1898, An essay on the Principle of Population as it Affects the Future Improvement of Society with Remarks on the Speculations of Mr. Godwin, M. Condorcet, and Other Writers.

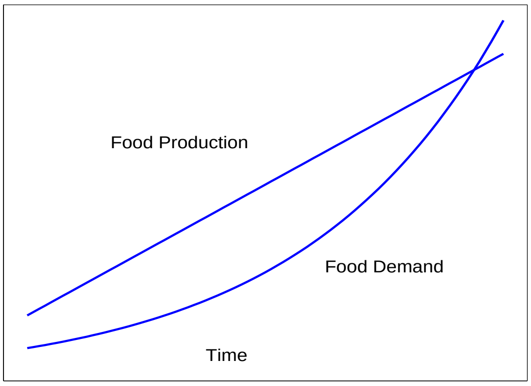

Figure 17.1: Malthus forecast that food demand would out strip food production.

In 1838 and 1845 P. F. Verhulst[2] connected the population to its environment and asserted that:

> **Mathematical Model of population growth in a limited environment** The growth rate of a population is proportional to the size of the population and to the fraction of the carrying capacity unused by the population.

By the double proportionality, the growth rate is proportional to the *product* of the population size and the unused fraction of the carrying capacity.

Let $p(t)$ be the size of a population growing in an environment in which the carrying capacity is M. Let $U(p(t))$ denote the unused fraction of the carrying capacity for population size $p(t)$, and write

Growth rate	Is proportional to		Population size	Times	Unused fraction of capacity
$p'(t)$	$= r$	$\times$	$p(t)$	$\times$	$U(p(t))$

Scientists have written various expression for $U(p(t))$. Verhulst used the most direct:

$$\begin{aligned} &\text{Fraction of carrying capacity used} & &= \frac{p(t)}{M} \\ &\text{Unused fraction of carrying capacity} & U(p(t)) &= 1 - \frac{p(t)}{M} \end{aligned}$$

With $U(p(t)) = 1 - \frac{p(t)}{M}$ we derive Verhulst's logistic equation

$$\textbf{Verhulst} \qquad p'(t) = r \times p(t) \times \left(1 - \frac{p(t)}{M}\right) \tag{17.2}$$

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[2] Pierre F. Verhulst (1838) Notice sur la loi que la population pursuit dans son accroissement, *Corresondence mathématique et physique* 10:113-121; (1845) Nouveau Memoirs de l'Academie Royal des Sciences et Bellas-Letters.

Other choices for $U(p(t))$ include

$$\begin{array}{llcll} \textbf{Ricker} & U(p(t)) & = & \dfrac{Ae^{-p(t)/\beta}-1}{A-1} & p'(t) = r\times p(t)\times\dfrac{Ae^{-p(t)/\beta}-1}{A-1} \\ & A & > & 1 & p'(t) = \alpha\times p(t)\times e^{-p(t)/\beta}-\gamma p(t) \\ \textbf{Beverton-Holt} & U(p(t)) & = & \dfrac{1}{1+p(t)/\beta} & p'(t) = \dfrac{r\times p(t)}{1+p(t)/\beta} \\ \textbf{Gompertz} & U(p(t)) & = & -\ln(p(t)/\beta) & p'(t) = -rp(t)\ln(p(t)/\beta) \end{array} \tag{17.3}$$

The Ricker and Beverton-Holt equations are patterned after discrete difference equation models used by Ricker and Beverton-Holt to model fish populations. It may be inappropriate to attach those names to a differential equation; fish models are generally discrete. The Gompertz equation is used to model tumor growth.

In the Verhulst, Ricker and Beverton-Holt equations, when $p(t)$ is close to zero, $U(p(t))$ is close to 1 and

$$p'(t) \doteq r\times p(t) \qquad\qquad \text{for } p(t) \text{ small}$$

For that reason, the number r is called the *low density growth rate.* At low population numbers, population growth in the Verhulst, Ricker and Beverton-Holt models appears to be as Malthus projected. The Gompertz equation is quite different at low population numbers as will be seen in Explore 17.1.1 but shares a similarity as seen in Exercise 17.1.3.

Explore 17.1.1 Low density growth in the Gompertz model is different from that in the Verhulst, Ricker, and Beverton-Holt models.

a. Show that in the Ricker model in Equations 17.3, for $p(t) \ll \beta$, $U(t) \doteq 1$ so that $p'(t) \doteq r\,p(t)$.

b. Show that in the Gompertz model in Equations 17.3, for $p(t) \ll \beta$, $U(p(t)) \gg 1$ so that $p'(t) \gg r\,p(t)$.

17.1.2 Competition between two species.

Alfred J. Lotka and Vito Volterra formulated models of competing species and predator prey interactions in the early 1900's[3].

Two species that occupy a given environment may compete for space, food, or light. They may reach an accommodation in which both species continue, but often one of the species is eventually excluded from the environment, depending on the environmental capacity for each species and the degree to which one species absorbs the resources of the other. Assume that for each species:

> **Mathematical Model of population growth in competition.** The growth rate of a population is proportional to the size of the population and to the fraction of the carrying capacity unused by the **two** populations.

---

[3] A prime reference is Lotka's 1924 book, *Elements of Mathematical Biology* now available as a Dover reprint, 1956.

This is the same as for a single species, except for the **two**.

Suppose two competing species have populations $p_1(t)$ and $p_2(t)$ in an environment with capacities M_1 and M_2 respectively for each population in the absence of the other.

Similar to the Verhulst model for a single species, for two species we write

$$\begin{aligned} \text{Population 1} \qquad p_1'(t) &= r_1 \times p_1(t) \times \left(1 - \frac{p_1(t) + \alpha_{1,2} p_2(t)}{M_1}\right) \\ \text{Population 2} \qquad p_2'(t) &= r_2 \times p_2(t) \times \left(1 - \frac{p_2(t) + \alpha_{2,1} p_1(t)}{M_2}\right) \end{aligned} \tag{17.4}$$

$\alpha_{1,2}$ is a measure of the influence of Population 2 on the environment of Population 1. If a member of Population 2 consumes twice the resources that a member of Population 1 consumes, then $\alpha_{1,2} = 2$. A similar interpretation is given to $\alpha_{2,1}$.

According to this model, the eventual presence or absence of Populations 1 and 2 depend on the values of M_1, M_2, $\alpha_{1,2}$ and $\alpha_{2,1}$ (and, curiously, not on r_1 nor r_2).

17.1.3 Predator-Prey models.

Lotka and Verhulst also wrote equations descriptive of predator and prey interaction. We described a model for predator and prey values close to equilibrium in Section 7.5. To extend this model assume

> **Mathematical Model of prey and predator:** We assume two species, a prey and a predator, and that
>
> > The rate of predation is proportional to the size of the prey population and proportional to the size of the predator population.
>
> Furthermore:
>
> a. Without the predator, the prey population increases at a rate proportional to the size of the prey population (as Malthus predicted).
>
> b. The prey growth rate is decreased proportional to the rate of predation.
>
> c. Without the prey, the predator decreases at a rate proportional to the size of the predator population (the predator has no alternate food source).
>
> d. The predator growth rate is increased proportional to the rate of predation.

As usual, the double proportionality of the predation rate implies the the predation rate is proportional to the product of the size of the prey population and the size of the predator population.

Let $u(t)$ and $v(t)$ denote the sizes of the prey and predator populations, respectively. We interpret parts 1 and 2 above to say that

$$\textbf{Prey} \qquad u'(t) = a \times u(t) - b \times u(t) \times v(t) \tag{17.5}$$

You are asked to write the Predator equation in Exercise 17.1.9

17.1.4 Susceptible, Infectious, Recovered (SIR) epidemic models.

In 1927 W. O. McKermack and A. G. McKendrick[4] modeled the spread of contagious diseases based on the assumption that

> **SIR Model of Epidemics.** The rate at which people get infected is proportional to the rate of 'infections contacts' between already infected people and people who are susceptible of becoming infected.
>
> The rate of infections contacts is proportional to the number of susceptible people and to the number of infected people.

Suppose people are either susceptible to becoming infected, infected and capable of transmitting the disease, or have been infected and recovered, with numbers $S(t)$, $I(t)$, and $R(t)$, respectively and with t measured in days.

Then the basic model is

$$\begin{array}{lrcl} \textbf{Susceptible} & S'(t) & = & -\beta \times S(t) \times I(t) \\[2ex] \textbf{Infectious} & I'(t) & = & \beta \times S(t) \times I(t) - \gamma \times I(t) \\[2ex] \textbf{Recovered} & R'(t) & = & \gamma \times I(t) \end{array}$$

17.1.5 Environmental pollution.

Suppose a lake has a river running through it and a factory is built next to the lake and begins releasing a chemical into the lake. The water running into the lake is free of the chemical and initially the lake is free of the chemical. What is the expected future content of the chemical in the lake?

We use a simple, but sometimes overlooked, model.

> **Mathematical Model.** The rate at which the amount of chemical in the lake increases is equal to the rate at which the chemical enters the lake minus the rate at which the chemical leaves the lake.

We also use

> The rate at which the chemical leaves the lake is the product of the concentration of the chemical in the lake and the rate at which water leaves the lake.

To get an equation we need some notation. Let V meters^3 be the volume of the lake. "A river running through the lake" means that there is a river running into the lake at a rate of R m^3/day and a river running out of the lake also at a rate of R m^3/day. Assume the factory releases the chemical into the lake at a rate of T kilograms/day. Finally, let $P(t)$ be the kilograms of chemical in the lake t days

---

[4] W. O. McKermack and A. G. McKendrick, Contributions to the mathematical theory of epidemics, *Roy. Stat. Soc. J.* **115** (1927) 700-721.

after the factory is built. Initially, the lake is free of the chemical, and at all times the water flowing into the lake is free of the chemical. Then we write

$$\begin{array}{cccccc} \text{Initial} & \text{Rate of} & \text{is} & \text{Rate of} & \text{minus} & \text{Rate} \\ \text{state} & \text{increase} & & \text{release} & & \text{flowing out} \\ & & & & & \\ P(0)=0 & P'(t) & = & T & - & \dfrac{P(t)}{V}\times R \end{array} \tag{17.6}$$

Exercises for Section 17.1, Differential equation models of biological processes.

Exercise 17.1.1 Let $M = \beta = 1$ and $A = 3$ draw the graphs of $U(p)$ *vs* p for each of the four models in Equations 17.2 - 17.3, for $0 < p \leq 1.2$.

$$\begin{array}{lrclrcl} \text{Verhulst} & U(p(t)) & = & 1-\dfrac{p(t)}{M} & \text{Ricker} & U(p(t)) & = & \dfrac{Ae^{-p(t)/\beta}-1}{A-1} \\ \text{Beverton-Holt} & U(p(t)) & = & \dfrac{1}{1+p(t)/\beta} & \text{Gompertz} & U(p(t)) & = & -\ln(p(t)/\beta) \end{array}$$

Exercise 17.1.2 Find the values of $P(t)$ for which $P'(t) = 0$ in each of the equations

$$\begin{array}{lrcl} \textbf{Verhulst} & p'(t) & = & r\times p(t)\times\left(1-\dfrac{P(t)}{M}\right) \\ & & & \\ \textbf{Ricker} & p'(t) & = & r\times p(t)\times\dfrac{Ae^{-p(t)/\beta}-1}{A-1} \\ & & & \\ A>1 & p'(t) & = & \alpha\times p(t)\times e^{-p(t)/\beta}-\gamma p(t) \\ & & & \\ \textbf{Beverton-Holt} & p'(t) & = & \dfrac{r\times p(t)}{1+p(t)/\beta} \\ & & & \\ \textbf{Gompertz} & p'(t) & = & -rp(t)\ln(p(t)/\beta) \end{array} \tag{17.7}$$

Let $M = \beta = 1$ and $A = 3$ draw the graphs of $U(p)$ *vs* p for each of the four models in Equations 17.2 - 17.3, for $0 < p \leq 1.2$.

$$\begin{array}{lrclrcl} \text{Verhulst} & U(p(t)) & = & 1-\dfrac{p(t)}{M} & \text{Ricker} & U(p(t)) & = & \dfrac{Ae^{-p(t)/\beta}-1}{A-1} \\ \text{Beverton-Holt} & U(p(t)) & = & \dfrac{1}{1+p(t)/\beta} & \text{Gompertz} & U(p(t)) & = & -\ln(p(t)/\beta) \end{array}$$

Exercise 17.1.3 Let $m(p_0)$ be $p'(0)$ in the Gompertz model, where

$$p'(t) = -rp(t)\ln(p(t)/\beta), \quad t \geq 0, \qquad p(0) = p_0.$$

Then

$$m(p_0) = -rp_0\ln(p_0/\beta).$$

For $p_0/\beta \ll 1$ it follows from Explore 17.1.1 that $m(p_0) = p'(0) \gg rp_0$. Show, however, that

$$\lim_{p_0\to 0} m(p_0) = 0.$$

Exercise 17.1.4 Newton's model of heat absorption by an object. Newton asserted that

> **Mathematical Model of Heat Absorption.** The rate at which heat is absorbed by a body is proportional to the difference between the temperature of the air surrounding the body and the temperature of the body.

a. Write an initial condition and differential equation that describes temperature of a clam exposed at low tide to 34° air when ocean temperature is 20°C. It will be necessary to assume the clam has a heat capacity, C, such that the amount of heat required to increase the temperature of the clam Δ degrees centigrade is $C \times \Delta$.

b. Write a solution to your differential equation.

c. For clams growing in an intertidal zone, would you expect the larger clams to be higher in the zone or lower in the zone?

Exercise 17.1.5 Release of nitrogen in the tissue of a SCUBA diver as she ascends from deep water has been compared to the release of carbon dioxide in a Coca-Cola© when it is opened. Write a mathematical model descriptive of release of carbon dioxide in a Coca-Cola©. From your model, write a differential equation descriptive of the partial pressure of carbon dioxide in a Coca-Cola© t minutes after opening the Coca-Cola©.

Exercise 17.1.6 Equations 17.4 for two competing species are based on Verhulst's logistic single species Equation 17.2. Write the corresponding pairs of equations that would describe competition between two species

a. Based on the Ricker single species model in Equation 17.3.

b. Based on the Beverton-Holt single species model in Equation 17.3.

c. Based on the Gompertz single species model in Equation 17.3.

Exercise 17.1.7 Describe the relation between two species of Equations 17.4 when $\alpha_{1,2} = 0$ and $\alpha_{2,1} = 0$.

Exercise 17.1.8 Suppose that without the predator, the prey growth in Equation 17.5 is logistic, as in the Verhulst single species Equation 17.2. Write a modification of Equation 17.5 descriptive of prey growth in the presence of the predator.

Exercise 17.1.9 Write an equation for the growth rate of the predator population based on parts 3 and 4 of the Mathematical Model of Prey and Predator on page 304.

Exercise 17.1.10 a. The units on S', I' and R' in the SIR model are people/day. In order for the units to balance on the equations, what must be the units on β and γ?

b. Suppose the infection typically lasts seven days. What is an appropriate value of γ?

Note: The answer we would expect you to give for γ is $1/7 \doteq 0.143$. We will find that a better answer is $-\ln(1 - 1/7) \doteq 0.154$.

Exercise 17.1.11 Suppose immunity is not permanent in the SIR model, and recovered people become susceptible after six months. Modify the meaning of R and the SIR equations to account for this possibility.

Exercise 17.1.12 In Equation 17.6, the units on $P'(t)$ are inherited from the difference quotient of $P'(t)$ which is the limit:

$$\lim_{h \longrightarrow 0} \frac{P(t+h) - P(t)}{h} \qquad \frac{\text{kilograms}}{\text{day}} \qquad = \qquad p'(t)\frac{\text{kilograms}}{\text{day}}$$

Show that the units are the same on the two sides of

$$P'(t) = T - \frac{P(t)}{V} \times R$$

Exercise 17.1.13 What is a reasonable value to assume for the initial condition of the lake, $P(0)$ in Equation 17.6?

Exercise 17.1.14 Suppose there is a massive chemical spill on a single day into a lake with a river running through it. Write an initial condition, $P(0)$, and a differential equation that will model the amount of pollution in the lake. Write a solution to your equation.

Exercise 17.1.15 As the first approximation of penicillin clearance by the kidney in Section **??**, you may have written the difference equation

$$\begin{array}{rcl} P_0 & = & 200\,\text{mg/ml} \\ P_{t+1} - P_t & = & -0.23 \times P_t. \end{array} \qquad \text{with solution} \qquad P_t = 200 * 0.77^t\,\text{mg/ml}. \qquad (17.8)$$

Time is measured in 5 minute intervals (so that time $t+1$ is five minutes later than time, t) and P_t is the concentration of penicillin in the serum at time t. The equation reflects the assumption that 23% of the penicillin in the serum is removed every 5 minutes. The data shown in Figure **??** shows that $P_0 = 200\,\text{mg/ml}$ and $P_2 = 118\,\text{mg/ml}$, and the values computed from Equation 17.8 closely match this data.

Penicillin clearance is a continuous process, however; is does not occur in 5 minute batches.

a. Write a mathematical model of continuous penicillin clearance following a single injection of penicillin. Consider only the vascular pool; we will include the tissue compartment later.

b. Write a differential equation with initial condition that will describe the concentration of penicillin in the vascular pool in which the initial concentration is 200 mg/ml.

c. Compute your projected value of penicillin concentration for time $t = 10$ minutes.

d. You might have gotten 126.3 as the answer to your previous question. Adjust your model so that it 'predicts' exactly 118 for $t = 10$ minutes.

Exercise 17.1.16 Suppose a patient is administered penicillin by continuous infusion that enters the vascular pool of 2 liters at the rate of 2 gm/hour. Consider only the vascular pool, and write a model of penicillin amount in the vascular pool. Write an initial condition and a differential equation that is descriptive of the amount of penicillin in the serum as a function of time.

17.2 Solutions to differential equations.

Important biological and physical processes are well described by differential equations. The next few sections are directed to finding the functions that are described by the differential equations.

A general form of the first order differential equation is

$$y'(t) = f\,(t, y(t)) \tag{17.9}$$

where f is a function of two variables, (t, y). Equation 17.9 is a statement about a function $y(t)$ and a **solution** to the equation is a function for which the statement is true. We check that it is a solution by *substitution.* Consider the algebraic equation

$$x^2 - 3x + 2 = 0,$$

for example, which is a statement about a number x. The statement that the number 2 is a solution to the equation means that when 2 is substituted for x in the equation the result is a true statement

$$2^2 - 3 \times 2 + 2 = 4 - 6 + 2 \quad = \quad 0 \qquad\qquad \textbf{It Checks!}$$

Note that $x = 1$ is also a solution to the algebraic equation. However, $x = 5$ is not a solution:

$$5^2 - 3 \times 5 + 2 = 25 - 15 + 2 = 12 \quad \neq \quad 0 \qquad \textbf{It does not Check.}$$

An example of a differential equation is

$$y'(t) = t \times y(t)$$

Is $y = t^2$ a solution? Let's check by substitution.

$$\begin{array}{ccc} \text{LHS: } y'(t) & & \text{RHS: } t \times y(t) \\ y'(t) = [t^2]' & & t \times y(t) = t \times t^2 \\ 2t & \neq & t^3 \qquad \textbf{It does not Check.} \end{array}$$

So $y = t^2$ is not a solution. But we claim that $y = e^{\frac{t^2}{2}}$ *is* a solution. Again, we check by substitution.

$$\begin{array}{ccc} \text{LHS: } y'(t) & & \text{RHS: } t \times y(t) \\ y'(t) = \left[e^{\frac{t^2}{2}}\right]' & & t \times y(t) = t \times e^{\frac{t^2}{2}} \\ e^{\frac{t^2}{2}} \times \left[\frac{t^2}{2}\right]' & & t \times e^{\frac{t^2}{2}} \\ e^{\frac{t^2}{2}} \times t & = & t \times e^{\frac{t^2}{2}} \quad \textbf{It Checks!} \end{array}$$

Explore 17.2.1 Show that each of the functions shown below is a solution to $y'(t) = t \times y(t)$:

a. $y = \frac{1}{2}e^{\frac{t^2}{2}}$ b. $y = \frac{1}{4}e^{\frac{t^2}{2}}$ c. $y = 6e^{\frac{t^2}{2}}$ d. $y = -3e^{\frac{t^2}{2}}$

e. Find yet another solution to $y'(t) = t \times y(t)$.

f. Show that $y = e^{t^2}$ is not a solution to $y'(t) = t \times y(t)$. ■

Initial Condition. There are several (actually infinitely many) solutions to $y'(t) = t \times y(t)$. In addition to the differential equation, one usually also knows an *initial condition* $y(0)$ (or, $y(a)$, the value of y at another specific value, a, of t). Then for most first order differential equations that describe biological and physical processes, there will be *only one function* that satisfies both the initial condition and the differential equation. For example, suppose

$$y(0) = 2 \qquad y'(t) = t \times y(t)$$

Then $y(t) = 2e^{t^2/2}$ satisfies both conditions. We check by substitution in both equations.

$$\begin{array}{cccc} \text{Initial Condition} & \text{LHS: } y'(t) & & \text{RHS: } t \times y(t) \\ y(0) = 2 & y'(t) = \left[2e^{t^2/2}\right]' & & t \times y(t) = t \times 2e^{t^2/2} \\ y(0) = 2e^{0^2/2} & 2e^{t^2/2}\,[t^2/2]' & & t \times 2e^{t^2/2} \\ =2 & 2e^{t^2/2}t & = & t \times 2e^{t^2/2} \\ \textbf{Check!} & & \textbf{Check!} & \end{array}$$

Therefore, $y = 2e^{t^2/2}$ satisfies the *initial condition* $y(0) = 2$ and the *differential equation* $y'(t) = t \times y(t)$. Conditions that insure that $y = 2e^{t^2/2}$ is the only solution satisfying both the initial condition and the differential equation are given in Theorem 17.6.1 Existence and Uniqueness of Solutions, but we do not include a proof of this theorem.

Exercises for Section 17.2, Solutions to differential equations.

Exercise 17.2.1 Show that each solution satisfies the initial condition and the differential equation.

	Solution	Initial Condition	Differential Equation
a.	$y(t) = e^{2t} + e^t$	$y(0) = 2$	$y'(t) - y(t) = e^{2t}$
b.	$y(t) = \frac{1}{3}e^t + \frac{2}{3}e^{-2t}$,	$y(0) = 1$,	$y'(t) + 2y(t) = e^t$
c.	$y(t) = te^t$	$y(0) = 0$	$y'(t) - y(t) = e^t$
d.	$y(t) = \frac{t^2}{3} + \frac{1}{t}$,	$y(1) = \frac{4}{3}$,	$t \times y'(t) + y(t) = t^2$
e.	$y(t) = \sqrt{t+1}$	$y(0) = 1$	$y(t) \times y'(t) = \frac{1}{2}$
f.	$y(t) = \sqrt{1+t^2}$,	$y(0) = 1$,	$y(t) \times y'(t) = t$
g.	$y(t) = \sqrt{4+t^2}$	$y(0) = 2$	$y(t) \times y'(t) = t$

	Solution	Initial Condition	Differential Equation
h.	$y(t) = \frac{1}{t+1}$,	$y(0) = 1$,	$y'(t) + (y(t))^2 = 0$
i.	$y(t) = 0.5 + 0.5e^{-0.2\sin t}$	$y(0) = 1$,	$y'(t) + 0.2(\cos t)y(t) = 0.1\cos t$
j.	$y(t) = \tan t$,	$y(0) = 0$,	$y'(t) = 1 + (y(t))^2$
k.	$y(t) = 3$	$y(0) = 3$,	$y'(t) = (y(t) - 1) \times (y(t) - 3) \times (y(t) - 5)$
l.	$y(t) = 5$,	$y(0) = 5$,	$y'(t) = (y(t) - 1) \times (y(t) - 3) \times (y(t) - 5)$

Exercise 17.2.2 Which of the following possible solutions satisfies the initial condition and the differential equation.

	Possible Solution	Initial Condition	Differential Equation
a.	$y(t) = e^{2t} + 2e^t$	$y(0) = 2$	$y'(t) - y(t) = e^{2t}$
b.	$y(t) = e^{2t} + 2e^t$	$y(0) = 3$	$y'(t) - y(t) = e^{2t}$
c.	$y(t) = \sqrt{t+1}$	$y(0) = 1$	$y(t) \times y'(t) = 1$
d.	$y(t) = t^3$	$y(1) = 1$	$y'(t)/y(t) = 3t$
e.	$y(t) = \frac{1}{t}$	$y(0) = 1$	$y'(t)/y(t) = -y(t)$
f.	$y(t) = t^3$	$y(1) = 1$	$y'(t)/y(t) = 3t$
g.	$y(t) = te^t$	$y(0) = 0$	$y'(t) - y(t) = e^t$

Exercise 17.2.3 The special case of $y' = f(t,y)$ in which $f(t,y) = F(t)$ (f is independent of y) has a familiar solution from the Fundamental Theorem of Calculus I. Check by substitution that

$$y(t) = y_a + \int_a^t F(x)\,dx \qquad \text{solves} \qquad y(a) = y_a \qquad \text{and} \qquad y'(t) = F(t)$$

The differential equation

$$y(a) = y_a \qquad y'(t) = F(t)$$

has therefore been completely solved. Henceforth we will consider that f is dependent on y and possibly also on t.

17.3 Direction Fields.

Shown in Figures 17.2 and 17.3 are the graphs of solutions of certain differential equations. For each differential equation, the different solutions correspond to different points at which the curves cross the Y-axis, each such point representing a specific value of $y(0)$ - the initial condition. *You should get the idea that for any differential equation* **that we study**, *the plane is filled with graphs of functions that solve the differential equation and that no two of the graphs intersect.*

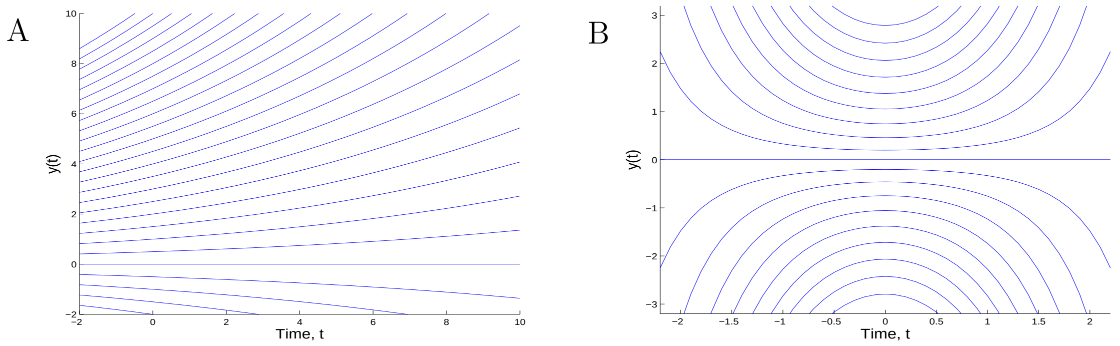


Figure 17.2: A. Solutions to $y'(t) = 0.1y$. B. Solutions to $y'(t) = t * y$

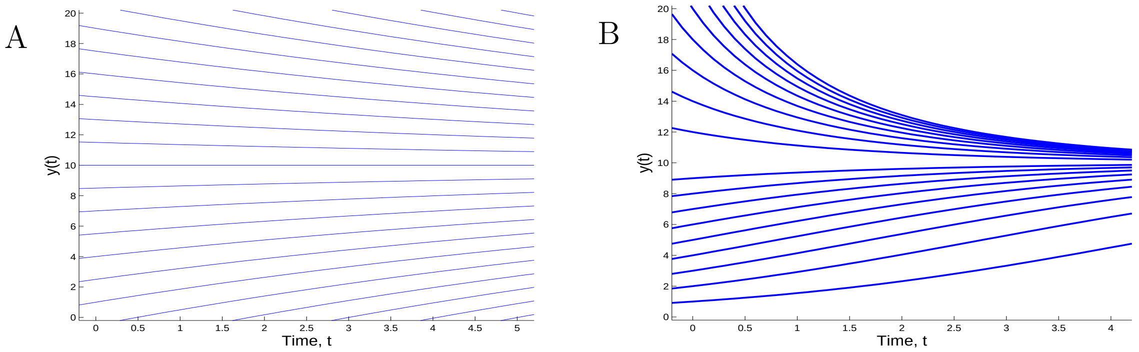


Figure 17.3: A. Solutions to $y'(t) = 1 - y/10$. B. Solutions to $y'(t) = 0.5 * y * (1 - y/10)$

In order to construct the previous graphs, we needed to know the solutions. However, we can 'almost' construct the previous graphs with out knowing any solutions.

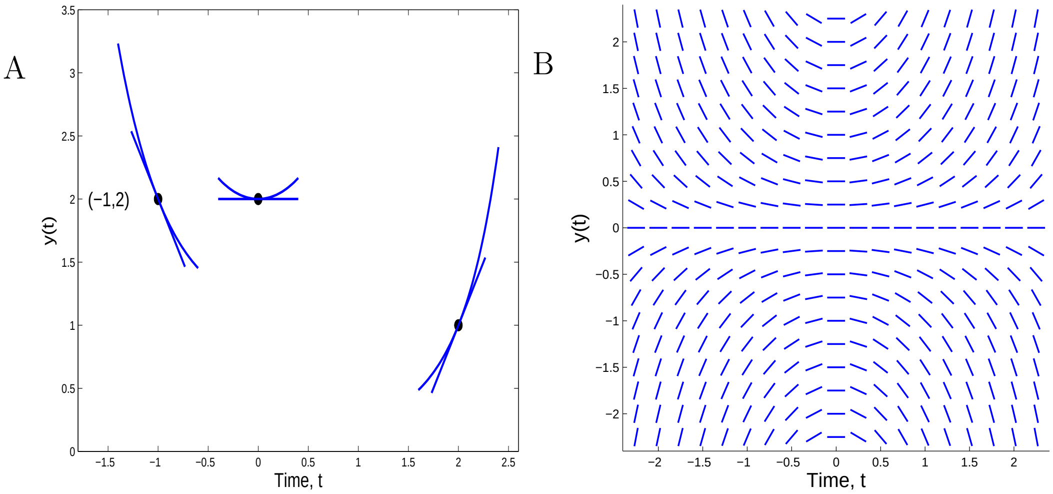


Figure 17.4: A. Solutions to $y'(t) = t \times y$ and tangents at selected points. B. Direction field for $y'(t) = t \times y$.

The differential equation

$$y'(t) = f(t, y)$$

specifies the **slope, $\mathbf{f(t, y)}$,** of the solution passing through, the point $(\mathbf{t}, \mathbf{y})$ in the plane. Knowing the point and the slope, we can construct a (short) tangent to the graph of the solution, *without having a formula for the solution.*

For the differential equation

$$y'(t) = t \times y$$

the slope of the solution at $(t, y) = (-1, 2)$ is

$$y' = (-1) \times 2 = -2$$

and is shown with some other slopes in Figure 17.4A. Slopes at many points appear in Figure 17.4B.

Explore 17.3.1 Find the three direction field tangents in Figure 17.4B that correspond to the three tangents shown in Figure 17.4A.

If for a single differential equation we draw enough of the short tangent lines, we obtain a picture that strongly suggests the shapes of the solution curves. The short tangents show the 'directions' of the solutions, and graphs of the short tangent lines are called **direction fields**.

A direction field for

$$y' = t - 2ty$$

is shown in Figure 17.5. Starting from the point A and following the direction tangents, it is fairly obvious that the curve shown approximates the graph of a solution.

Now look at the point B in the direction field for $y' = t - 2ty$, and follow the direction tangents for the solution through B. It should appear that the horizontal line through B is the graph of a solution to

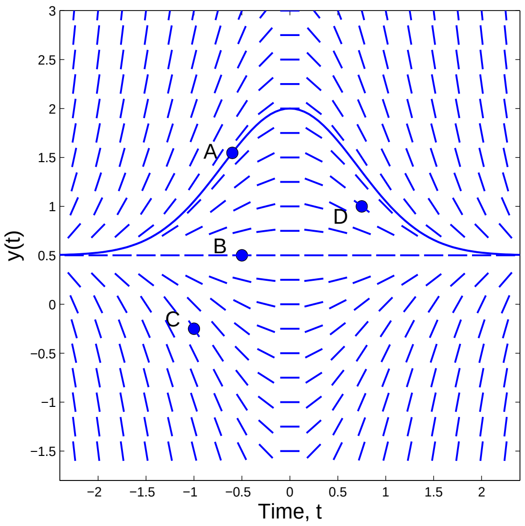


Figure 17.5: Direction field for $y' = t - 2ty$ and a solution through the point A.

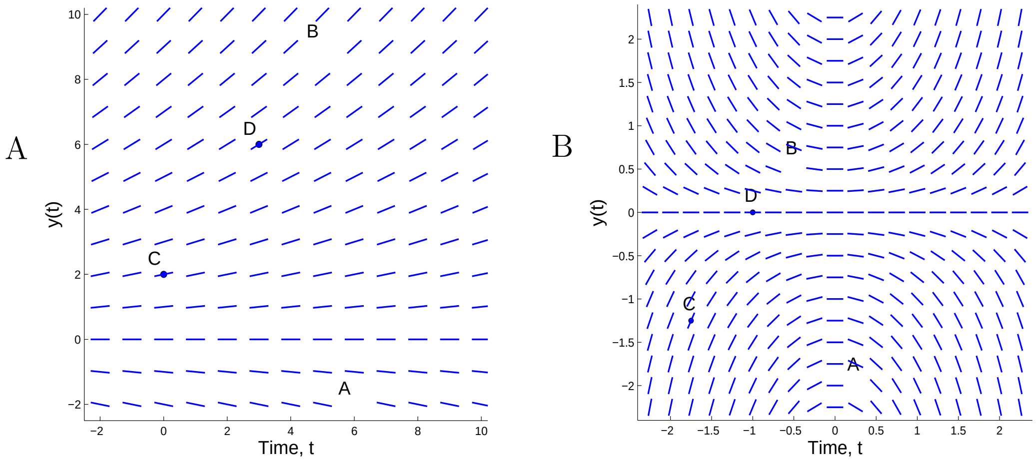


Figure 17.6: A. Direction field for $y'(t) = 0.1y$. B. Direction field for $y'(t) = t * y$

$y' = t - 2ty$. That line intersects the Y-axis at $(0, \frac{1}{2})$ and $y = \frac{1}{2}$ is an equation of a solution. We suggest that

$$y = \frac{1}{2} \qquad \text{is a solution to} \qquad y' = t - 2ty,$$

and check to see that it is:

$$\begin{array}{cccc}
\text{LHS: } y'(t) & & \text{RHS: } t - 2t \times y(t) & \\
y'(t) = \left[\frac{1}{2}\right]' & & t - 2t \times y(t) = t - 2t \times \left(\frac{1}{2}\right) & \\
0 & & t - 2t \times \frac{1}{2} & \\
0 & = & 0 & \textbf{It Checks!}
\end{array}$$

Direction fields for the four examples in Figures 17.2 and 17.3 are shown in Figures 17.6 and 17.7

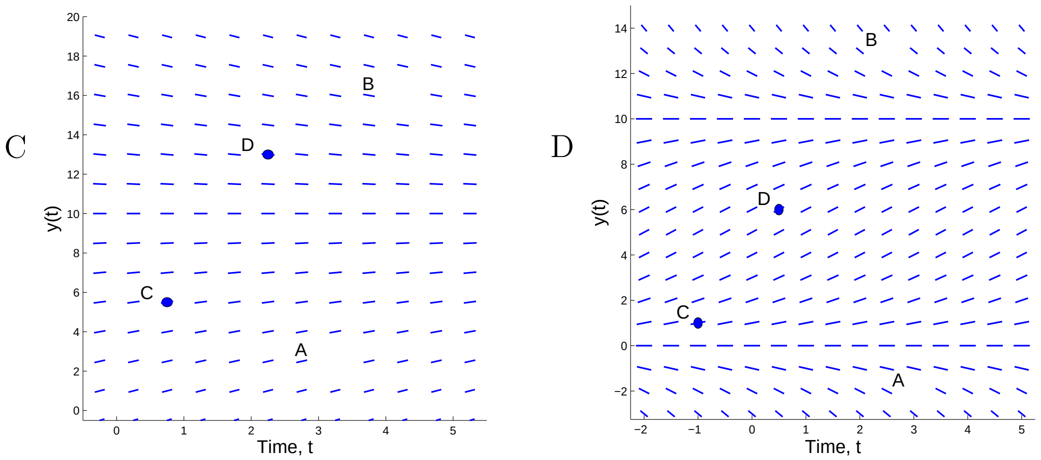


Figure 17.7: C. Direction field for $y'(t) = 1 - y/10$. D. Direction field for $y'(t) = 0.5 * y * (1 - y/10)$

Exercises for Section 17.3, Direction Fields.

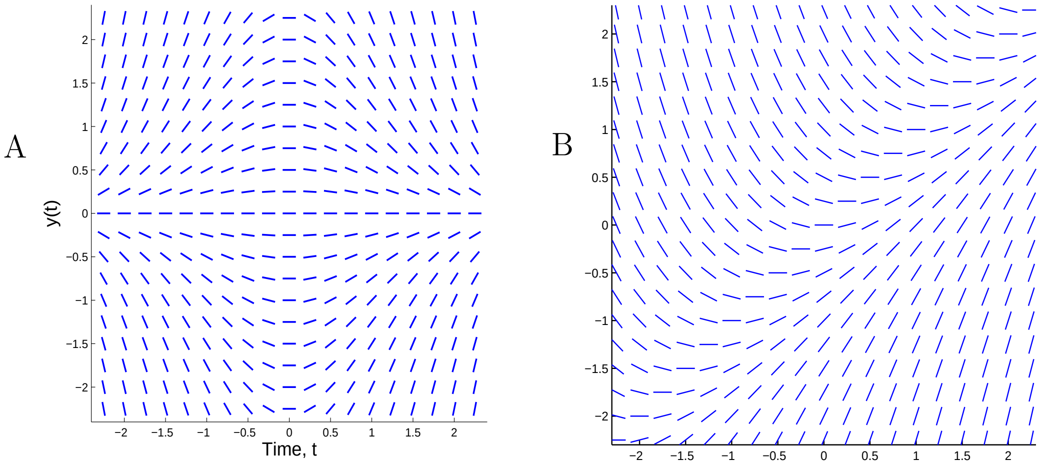
Exercise 17.3.1 For the direction field for the differential equation $y' = t - 2ty$ in Figure 17.5 draw solutions through the points C and D.

Exercise 17.3.2 For the four direction fields in Figures 17.6 and 17.7 tangents have been omitted at points marked A and B. Compute the slopes of those tangents for those points.

Exercise 17.3.3 For the four direction fields in Figure 17.6 and 17.7 draw an approximate solution that passes through the point C and another solution through the point D.

Exercise 17.3.4 Shown in Exercise Figure 17.3.4 are direction fields for two differential equations. For each direction field, draw (approximately) the graphs of three solutions.

Figure for Exercise 17.3.4 A. Direction field for $y\prime = -t \times y$. B. Direction field for $y' = t - y$.



Exercise 17.3.5 Draw direction fields in the quadrant, $t > 0$, $y > 0$, and three approximate solutions for the following differential equations. Note: d. is a special case that we said we would not consider.

$$\begin{array}{llll} \text{a.} \quad y'(t) = 1 & & \text{b.} \quad y'(t) = 0.5y \\ \text{c.} \quad y'(t) = t & & \text{d.} \quad y'(t) = 2\sqrt{y} \end{array}$$

Exercise 17.3.6 The direction field of

$$y' = (y-1) \times (y-3)$$

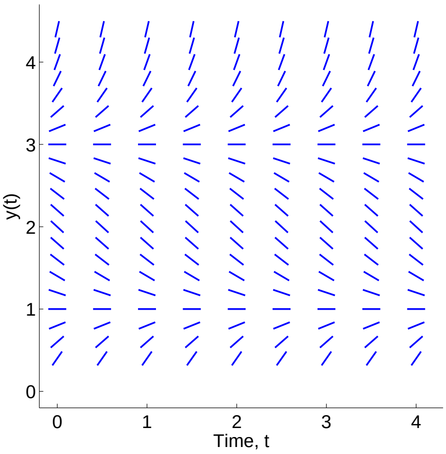
is shown in Exercise Figure 17.3.6

a. Draw exactly the graphs of two solutions of the differential equation.

b. Find a formula for the function that is one of the solutions to the equation.

c. Check that your formula satisfies the differential equation.

Figure for Exercise 17.3.6 Direction field for $y\prime = (y-1) \times (y-3)$.



Exercise 17.3.7 The following MATLAB program draws a direction field for a differential equation.

a. For what differential equation does the program draw a direction field?

b. How would you change the program to draw a direction field for the differential equation $y' = t + y$?

c. If you have access to MATLAB, enter the program and run it.

```
close all;clc;clear;
tt=[-2:0.25:2];
yy=[-2:0.25:2];
r=0.05;
axis([-2.2 2.2 -2.2 2.2])
```

```
hold
for i = 1:length(tt)
    for j = 1:length(yy)
        slope=tt(i)*yy(j);
        angle=atan(slope);
        cc=cos(angle);ss=sin(angle);
        plot([tt(i)-r*cc tt(i)+r*cc],[yy(j)-r*ss yy(j)+r*ss],...
                                              'linewidth',2)
    end
end
```

17.4 Phase planes and stability of constant solutions.

A differential equation of the form

$$y' = f(y) \qquad\qquad \text{(RHS is independent of } t\text{)}$$

is called an **autonomous** differential equation.

$$y' = y^2 \qquad \text{is autonomous} \qquad y' = t + y^2 \qquad \text{is not autonomous.}$$

A differential equation that models an isolated population in a constant environment is typically autonomous. If there is occasional migration into or out of the population or the environment fluctuates with time, the model differential equation will not be autonomous. A differential equation that models penicillin clearance following a single injection or even constant infusion is autonomous; if there are subsequent injections or variation in the infusion, the model differential equation is not autonomous.

Definition 17.4.1 For an autonomous differential equation

$$y' = f(y),$$

if $\overline{y}$ is a number such that $f(\overline{y}) = 0$, then $y(t) = \overline{y}$ is a solution to $y' = f(y)$, $\overline{y}$ is called an **equilibrium point** for $y' = f(y)$, and $y(t) = \overline{y}$ is called an *equilibrium solution.*

For example, the equation

$$y' = f(y) = (y-2) \times (y-3)$$

is autonomous and $f(2) = 0$ and $f(3) = 0$. Therefore $\overline{y} = 2$ and $\overline{y} = 3$ are equilibrium points. Also $y(t) = 2$ is an equilibrium solution to the differential equation:

$$\begin{array}{lcl} \text{LHS: } y'(t) & & \text{RHS: } (y(t)-2)\times(y(t)-3) \\ y'(t) = [2]' & & (y(t)-2)\times(y(t)-3) = (2-2)\times(2-3) \\ 0 & = & 0 \qquad \textbf{It Checks.} \end{array}$$

The RHS of Verhulst's logistic differential equation

$$p'(t) = r \times p(t) \times \left(1 - \frac{p(t)}{M}\right) \tag{17.10}$$

of the dynamics of a single population does not depend explicitly on t; the equation might be written

$$p' = r \times p \times \left(1 - \frac{p}{M}\right)$$

and is autonomous.

The direction field for the logistic differential equation is shown in Figure 17.8A. The graph of p' *vs* p, or the graph of f where

$$f(p) = r \times p \times \left(1 - \frac{p}{M}\right)$$

is shown in Figure 17.8B. This graph is called a **phase plane** graph. Phase plane graphs are easier to draw than are direction fields, but give similar information. In this case the phase plane graph is a parabola. The arrows point to the right where $f(p) > 0$ and point to the left where $f(p) < 0$.

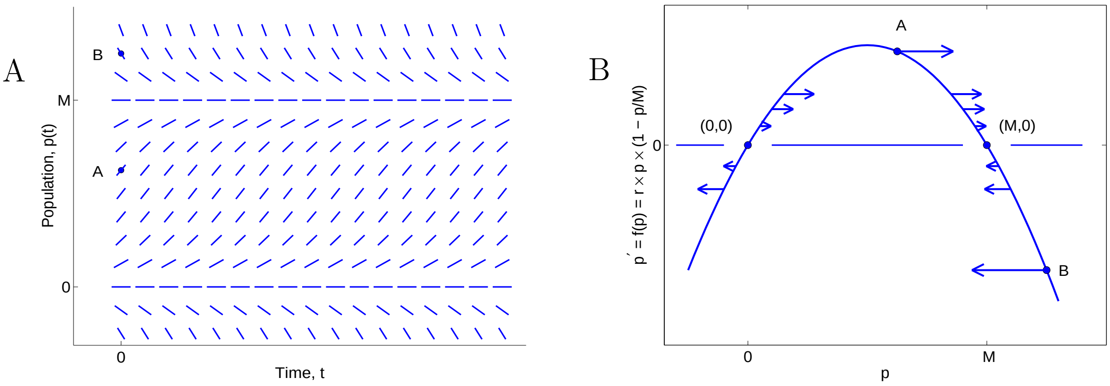


Figure 17.8: A. Direction field for the logistic equation $p' = r \times p \times (1 - p/M)$. B. Phase plane graph of $f(p) = r * p * (1 - p/M)$. The arrows point to the right where $f(p) > 0$ and point to the left where $f(p) < 0$.

From the direction field, we can immediately see two horizontal lines that are graphs of two solutions,

$$p = 0 \qquad \text{and} \qquad p = M$$

both of which are equilibrium solutions.

In the phase plane we can see the same equilibrium points. Simply observe that

$$f(0) = 0 \qquad \text{and} \qquad f(M) = 0$$

The meaning of the equilibrium solution, $p = 0$, is fairly obvious; if there are no rabbits at time $t = 0$ there will be no rabbits at any subsequent time (the model does not account for migration into the population, nor spontaneous generation of rabbits).

The equilibrium solution, $p = M$, is not quite so obvious, but makes sense. If the population size equals the the carrying capacity, the environment will support that size population and no more, and the population size will persist at that value.

Now examine the direction field in Figure 17.8A for the solution that passes through the point, $(0, A)$, for which

$$0 < p(0) = A < M \qquad \text{Initial population less than carrying capacity.}$$

The graph will rise from $(0, A)$ and will be asymptotic to the line $y = M$; the population increases to the maximum the environment will support.

The point A that marks $(A, f(A))$ is shown on the phase plane. We can see from that point that $f(A) > 0$, implying that p' is positive so that the population is increasing. Thus from $(A, f(A))$ we have drawn a horizontal arrow pointing to the right, or pointing toward increasing values of p. We do not see the shape of the solution from the phase plane, but we can see the qualitative character that the solution is increasing when the population size is A.

Next examine the direction field and the solution that passes through the point, $(0, B)$, for which

$$M < B = p(0) \qquad \text{Initial population exceeds the carrying capacity.}$$

The graph will decrease from $(0, B)$ and also will be asymptotic[5] to the line $y = M$.

The point, $(B, f(B))$, is shown on the phase plane. We can see from that point that $f(B) < 0$, implying that p' is negative so that the population is decreasing. Thus from $(B, f(B))$ we have drawn a horizontal arrow pointing to the left, or pointing toward decreasing values of p.

Stability of solutions.

Two important properties of the solutions $p = 0$ and $p = M$ of $p'(t) = r \times p(t) \times \left(1 - \frac{p(t)}{M}\right)$ are apparent from the direction field in Figure 17.8. For a solution that is near the line $p = M$, as time progresses the short tangent segments guide that solution toward that line. Every solution near $p = M$ is asymptotic to $p = M$ with increasing time, and the solution $p = M$ is said to be **asymptotically stable**. If, for example, the population is at the carrying capacity and random environmental effects cause the population to either increase or decrease from carrying capacity, the population size will move back toward the carrying capacity as time increases.

On the other hand, the arrows near the equilibrium solution $p = 0$ will be guided away from $p = 0$ as time progresses. There are solutions near the line $p = 0$ that move away from the line as time increases, and $p = 0$ is said to be a **nonstable** solution. If you introduce a single rabbit (perhaps two!) into an environment with out rabbits (Australia, for example), then the rabbit population will soon expand to carrying capacity (the asymptotically stable solution).

---

[5] According to the model, if the current population exceeds the carrying capacity, M, then the population decreases to M. It is probably more realistic to suppose that if the current population exceeds the maximum supportable population by very much, there will be a rather catastrophic immediate decrease to a level less than M (hopefully still positive), followed by a gradual increase to M.

Stability of constant solutions can also be determined from the phase plane. At points near the equilibrium points in Figure 17.8 a horizontal arrow is drawn pointing to the right at $(p, f(p))$ if $f(p) > 0$ and pointing to the left if $f(p) < 0$. The lengths of the arrows reflect the magnitude of $f(p)$. Near $(M, 0)$ the arrows all point toward M indicating the M is asymptotically stable, and near (0,0) the arrows all point away from 0 indicating that 0 is not stable.

Example 17.4.1 *An extreme example.* Shown in Figure 17.9 are the direction field and the phase plane graph of

$$y' = (y-1) \times (y-2) \times (y-3) \times (y-4)$$

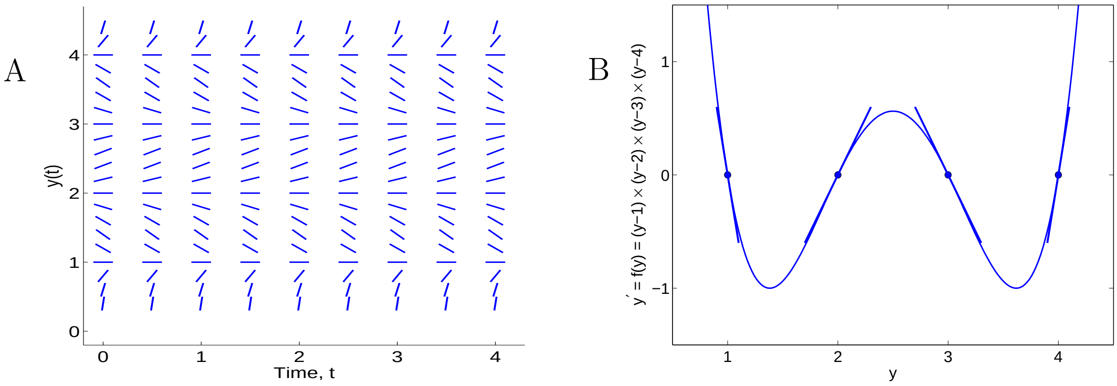


Figure 17.9: A. Direction field for the $y' = (y-1) \times (y-2) \times (y-3) \times (y-4)$. B. Phase plane graph of the same equation.

It is easy to solve $f(y) = (y-1)\,(y-2)\,(y-3)\,(y-4) = 0$ and see that 1, 2, 3 and 4 are equilibrium points, and equivalently that $y = 1$, $y = 2$, $y = 3$ and $y = 4$ are equilibrium solutions to the differential equation. Solutions, $y(t)$, with $y(0)$ close to 1 will be asymptotic to $y = 1$ and solutions with $y(0)$ close to 3 will be asymptotic to $y = 3$. Also, solutions starting close to 2 will not be asymptotic to $y = 2$ and solutions starting close to 4 will not be asymptotic to $y = 4$. ■

Explore 17.4.1 For the differential equation $y' = (y-1)\,(y-2)\,(y-3)\,(y-4)$, what qualitative character of the phase plane graph, Figure 17.9 B, of

$$\mathbf{f(y) = (y-1) \times (y-2) \times (y-3) \times (y-4)}$$

distinguishes the stable equilibrium points 1 and 3 from the non-stable equilibrium points 2 and 4? ■

Definition 17.4.2 Asymptotically stable equilibrium point. If $f(y)$ and $f'(y)$ are continuous, an equilibrium point $\overline{y}$ of

$$y' = f(y)$$

is an *asymptotically stable equilibrium point* means that there is an interval (a, b) containing $\overline{y}$, and every solution, $y(t)$, to $y' = f(y)$ for which $a < y(0) < b$ is asymptotic to the horizontal line $y = \overline{y}$.

We hope you guessed from Explore 17.4.1 the following theorem.

Theorem 17.4.1 If $f(y)$ and $f'(y)$ are continuous, an equilibrium point y_e of

$y' = f(y)$ is asymptotically stable if $f'(y_e)$ is negative.

Proof: **This proof is technical and your reading of it can be delayed — a long time.**

Suppose the hypothesis of the theorem and let $-m = f'(y_e) < 0$. By hypothesis, y_e is an equilibrium point so that $f(y_e) = 0$; therefore the function $\overline{y}(t) \equiv y_e$ is a solution to $y'(t) = f(y(t))$.

For convenience suppose that $y_e = 0$ (so that the equilibrium solution $\overline{y}(t) \equiv 0$). Then $F'(0) < 0$; let $-m = F'(0)$. See Figure 17.10 where the direction field of the very simple case, $f(y) = -y$, the equilibrium solution, $\overline{y}(t) = 0$, and a typical solution, $y = e^{-t}$, are drawn. We need to show that in the general case the solutions are similar to the "typical" solution. Observe that because $y' = f(y)$ is autonomous, the direction field slopes along any horizontal line are constant. A solution for which $y(0) = 1$, say, will be parallel to a solution for which $y(17) = 1$.

Because $f'(0) = -m$, there is $a > 0$ such that if $-a \le y \le a$, then $-2m \le f'(y) \le -m/2$. On the strip, $-a \le y \le a$, $0 \le t$, the hypothesis of the Existence and Uniqueness Theorem, 17.6.1, is satisfied and no two solutions intersect. Therefore, any function, $y(t)$, for which $y(0) = y_0 > 0$, will not intersect the equilibrium solution, $\overline{y(t)} = 0$ and will be positive for all t.

Suppose $0 < y_0 < a$ and $y(t)$ is the solution to $y(0) = y_0$, $y'(t) = f(y)$. The argument for $-a < y_0 < 0$ is similar. We will show that

$$e^{-2mt} \le y(t) \le e^{-(m/2)t}.$$

Suppose $t_0 > 0$ and $0 < y(t) \le a$ for $0 \le t \le t_0$. Note that because $a > y_0 = y(0)$ and $y'(0) = f(y_0) < -m/2$, there is such a number t_0. Then

$$\text{for} \qquad 0 \le t \le t_0, \quad y'(t) = f(y(t)) = f(y(t)) - f(0) = f'(c_t) \times y(t)$$

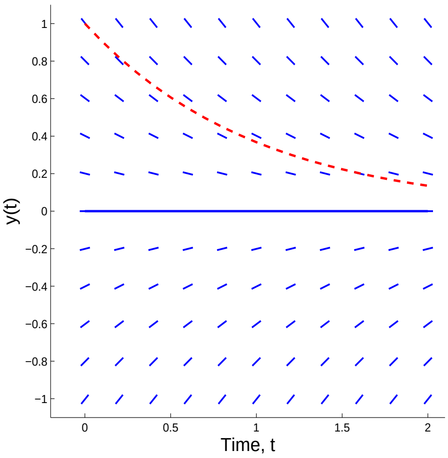


Figure 17.10: Direction field for $y' = -y$; the equilibrium solution, $\overline{y} = 0$, and a solution, $y = e^{-t}$.

where $0 < c_t < y(t) \leq a$. Because $0 < c_t < a$, $-2m \leq f'(c_t) \leq -m/2$. Then for $0 \leq t \leq t_0$

$$\begin{array}{rcccl} -2m & \leq & f'(c_t) & \leq & -m/2 \\ -2m \times y(t) & \leq & f'(c_t) \times y(t) & \leq & -m/2 \times y(t) \\ -2m \times y(t) & \leq & y'(t) & \leq & -m/2 \times y(t) \\ -2m & \leq & \dfrac{y'(t)}{y(t)} & \leq & -m/2 \\ \displaystyle\int_0^{t_0} -2m\,dt & \leq & \displaystyle\int_0^{t_0} \frac{y'(t)}{y(t)}\,dt & \leq & \displaystyle\int_0^{t_0} -m/2\,dt \\ -2m\,t_0 & \leq & \ln y(t_0) - \ln y_0 & \leq & -(m/2)\,t_0 \\ y_0e^{-2mt_0} & \leq & y(t) & \leq & y_0e^{-(m/2)t_0} \end{array}$$

We have shown that if $0 \leq y(t) \leq a$ on $0 \leq t \leq t_0$ then $y_0e^{-2mt_0} \leq y(t) \leq y_0e^{-(m/2)t_0}$. Suppose for some t_1, $a \leq y(t_1)$. Then, using the Completion Property of the numbers, Axiom 1 of Section 5.2, there is a least such number. Let t_0 denote that least number. Then $y(t) \leq a$ for $0 \leq t_0$,
$0 < y_0e^{-2mt_0} \leq y(t_0) \leq y_0e^{-(m/2)t_0} < a$, which is a contradiction. There is no such number t_1.
End of Proof.

In case $f'(\overline{y}) > 0$ then $\overline{y}$ is not stable. If $f'(\overline{y}) = 0$, $\overline{y}$ may be stable and it may be non-stable.

17.4.1 Parameter Reduction.

Example 17.4.2 Parameter Reduction. It is customary to divide the logistic differential equation

$$p'(t) = r \times p(t) \times \left(1 - \frac{p(t)}{M}\right)$$

by M to obtain

$$\frac{p'(t)}{M} = r \times \frac{p(t)}{M} \times \left(1 - \frac{p(t)}{M}\right)$$

Then let

$$u(t) = \frac{p(t)}{M} \qquad \text{and note that} \qquad u'(t) = \frac{p'(t)}{M}$$

to obtain

$$u'(t) = r \times u(t) \times (1 - u(t)) \qquad \text{Fraction Logistic.} \tag{17.11}$$

The function, u, is the fraction of the carrying capacity, M, used by the population. Because u is the ratio of p to M, both of which have units of population numbers, u is dimensionless.

Explore 17.4.2 What is the dimension of r in Equation 17.11? ■

If we then rescale time by $\tau = r \times t$ and let $v(\tau) = u(t)$, then

$$u'(t) = \frac{d}{dt}u(t) = \frac{d}{dt}v(\tau) = \frac{d}{d\tau}v(\tau)\frac{d\tau}{dt} = v'(\tau) \times r. \tag{17.12}$$

We can substitute $u(t) = v(\tau)$ and $u'(t) = v'(\tau) \times r$ into Equation 17.11 and obtain

$$v'(\tau) \times r = r\, v(\tau)(1 - v(\tau)), \qquad v'(\tau) = v(\tau)(1 - v(\tau)). \tag{17.13}$$

The qualitative characteristics of the original logistic equation, $p' = rp(1 - p/M)$ are the same as that of $v'(\tau) = v(\tau)(1 - v(\tau))$, which has no parameters. ■

Exercises for Section 17.4, Phase planes and stability of constant solutions to $y' = f(y)$.

Exercise 17.4.1 In discussion of release of chemical into a lake in Section 17.1.5 we obtained the differential equation

$$P'(t) = T - \frac{P(t)}{V} \times R$$

What is the equilibrium amount of chemical in the lake? Is the equilibrium stable?

Exercise 17.4.2 Find the equilibrium points and for each determine whether or not it is stable.

$$\begin{array}{llll} \text{a.} & y' = y - 1 & \text{b.} & y' = -y + 1 \\ \text{c.} & y' = y^2 - 1 & \text{d.} & y' = 1 - y^2 \\ \text{e.} & y' = e^{-y} - 1 & \text{f.} & y' = e^{y} - 1 \\ \text{g.} & y' = \sin y & \text{h.} & y' = -y + y^2 \\ \text{i.} & y' = -y^3 & \text{j.} & y' = y^3 \end{array}$$

For parts i. and j. draw the phase plane with arrows to determine the question of stability.

Exercise 17.4.3 Haldane's equation for nitrogen partial pressure, $N(t)$, in a given tissue of volume V in a scuba diver is

$$[V \times N(t)]' = \quad k \quad \times \left(0.8 + 0.8 \times \frac{d(t)}{10} - N(t)\right) \tag{17.14}$$

where $d(t)$ is measured in meters. See Exercise D.2.24, Decompression illness in deep water divers. Suppose a diver descends to a depth, 30 meters, and stays at that depth. Water pressure at 30 meters is approximately $30/10 = 3$ atmospheres. Then Haldane's equation is

$$N'(t) = \quad \frac{k}{V} \quad \times \left(0.8 + 0.8 \times \frac{30}{10} - N(t)\right)$$

a. What is the equilibrium nitrogen partial pressure for the tissue?

b. Is the equilibrium nitrogen partial pressure stable?

Exercise 17.4.4 *Continuous infusion of penicillin.* Suppose a patient recovering from surgery is to be administered penicillin intravenously at a constant rate of 5 grams per hour. The patient's kidneys will remove penicillin at a rate proportional to the serum penicillin concentration. Let $P(t)$ be the penicillin concentration t hours after infusion is begun. Then a simple model of penicillin pharmacokinetics is

$$\begin{array}{ccccc} \text{Net Rate of Increase} & & \text{Clearance} & & \text{Infusion} \\ & & & & \\ P'(t) & = & -K \times P(t) & + & 5 \\ & & & & \\ \frac{\text{gm}}{\text{hr}} & & \frac{1}{\text{hr}} \times \text{gm} & & \frac{\text{gm}}{\text{hr}} \end{array}$$

The proportionality constant, K, must have units $\frac{1}{\text{hr}}$ in order for the units on the equation to balance. We initially assume that $K = 2.5\frac{1}{\text{hr}}$ which is in the range of physiological reality. It is reasonable to assume that there was no penicillin in the patient at time $t = 0$, so that $P(0) = 0$.

a. Draw the phase plane for the differential equation

$$P(0) = 0 \qquad P'(t) = -2.5P(t) + 5$$

b. Find the equilibrium point of $P' = -2.5P + 5$.

c. Is the equilibrium point stable?

d. Show that the units of the equilibrium point are grams.

e. Suppose the patient's kidneys are impaired and only operating at 60% of normal. Then $K = 1.5$ instead of 2.5. What effect does this have on the equilibrium point.

Note: Because of the uncertainty of the kidney clearance rate, it is not common to administer penicillin as a continuous infusion.

Exercise 17.4.5 Draw the phase plane or the direction field for $y' = -y^2$ and decide whether 0 is a stable equilibrium point. What relevance is Theorem 17.4.1 to this?

Exercise 17.4.6 Suppose $y' = f(y)$ has three and only three equilibrium points, e_1, e_2, and e_3, where f and f' are continuous and $f'(e_1) \neq 0$, $f'(e_2) \neq 0$, and $f'(e_3) \neq 0$. Argue that one of e_1, e_2, and e_3 is stable. Hint: Draw a potential phase plane (graph of f) in which e_1 and e_3 are unstable.

Exercise 17.4.7 Consider a modification of the Lotka-Volterra equations for competition between two species in which $\alpha_{1,2} = 0$.

$$\begin{aligned} p_1'(t) &= r_1 \times p_1(t) \times \left(1 - \frac{p_1(t) + 0 \times p_2(t)}{M_1}\right) = r_1 \times p_1(t) \times \left(1 - \frac{p_1(t)}{M_1}\right) \\ p_2'(t) &= r_2 \times p_2(t) \times \left(1 - \frac{p_2(t) + \alpha_{2,1} p_1(t)}{M_2}\right) \end{aligned}$$

Thus population 1 is not affected by population 2 but population 2 is affected by population 1. Suppose that

$$0 < p_1(0) < M_1 \qquad \text{and} \qquad 0 < p_2(0) < M_2$$

a. Argue that $p_1(t) \to M_1$ as $t \to \infty$.

b. Argue that

$$\text{if} \qquad M_2 < \alpha_{2,1} M_1 \qquad \text{then} \qquad p_2(t) \to 0 \qquad \text{as} \qquad t \to \infty.$$

For ease, assume that there is a number t_0 and $p_1(t) = M_1$ for $t \geq t_0$. Also it is true that: If $z(t)$ satisfies $z'(t) < -kz(t)$ for $z(0) > 0$ then $z(t) \to 0$ as $t \to \infty$.

c. What happens to the second population if $M_2 > \alpha_{2,1} M_1$?

Exercise 17.4.8 a. Show that Ricker's equation,

$$p'(t) \quad = \quad \alpha\, p\, e^{-p/\beta} - \gamma\, p,$$

is equivalent to

$$v'(\tau) \quad = \quad v\, e^{-v} - \gamma_0\, v \tag{17.15}$$

with the substitutions, $u(t) = p(t)/\beta$, $\tau = \alpha\, t$, and $\gamma_0 = \gamma/\alpha$.
Note: Recall Equation 17.12 where $\tau = r \times t$ and $v(\tau) = u(t)$:

$$u'(t) = \frac{d}{dt} u(t) = \frac{d}{dt} v(\tau) = \frac{d}{d\tau} v(\tau) \frac{d\tau}{dt} = v'(\tau) \times r.$$

b. Show that the Beverton-Holt equation,

$$p'(t) \quad = \quad \frac{r \times p}{1 + p/\beta},$$

is equivalent to

$$v'(\tau) \quad = \quad \frac{v}{1+v} \tag{17.16}$$

with proper substitutions.

c. Show that the Gompertz equation,

$$p'(t) \quad = \quad -r\ \ln\frac{p}{\beta},$$

with proper substitutions, is equivalent to an equation with no parameters.

Exercise 17.4.9 Identify the stable and nonstable solutions of

$$u'(t) = u(t)\,(1 - u(t))$$

Exercise 17.4.10 Suppose a marine fish population when not subject to harvest is reasonably modeled by

$$u'(t) = r \times u(t) \times (1 - u(t))$$

with time measured in years. Suppose a harvest procedure is initiated, and that a fraction, h, of the *existing population* is harvested every year. The harvest is not a fixed amount each year, but depends on the number of fish available.

The growth rate will be the difference between the natural birth-death process and the harvest and may be modeled by

$$u'(t) = r \times u(t) \times (1 - u(t)) - h \times u(t) \tag{17.17}$$

a. Assume $h = r$ (the harvest rate equals the low density growth rate) Substitute $h = r$ in Equation 17.17, and simplify.

Show that

$$u(t) = \frac{1}{rt + 1/u_0} \qquad \text{where} \qquad u(0) = u_0$$

is a solution for this model. What will be the eventual annual fish harvest under this harvest strategy?

b. Assume $h = \frac{3}{4}r$ in Equation 17.17 and simplify.

Draw a direction field or phase plane for this model. What will be the eventual annual fish harvest under this harvest strategy?

c. Assume $h = \frac{1}{2}r$ in Equation 17.17, and simplify.

Draw a direction field for this model. What will be the eventual annual fish harvest under this harvest strategy?

d. Which of the three strategies will provide the largest long term harvest?

Exercise 17.4.11 Some population scientists have argued that population density can get so low that reproduction will be less than natural attrition and the total population will be lost. Named the Allee effect, after W. C. Allee who wrote extensively about it[6], this may be a basis for arguing, for example,

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[6] Allee, W.C. 1938 The Social Life of Animals. Norton, New York.

that marine fishing of a certain species (Atlantic cod on Georges Bank, for example [7]ee Paul Greenberg, "Four Fish, the Future of the Last Wild Food, The Penguin Press, 2010. The model is a lot more complicated than we present here.) should be suspended, despite the presence of a small residual population.

How should we modify the logistic differential equation, $u' = u\,(1-u)$, to incorporate such a threshold? Assume a fixed area and uniform density throughout the area and a threshold number, ϵ. If the population number is less than ϵ the population will decline; if the population number is more than ϵ the population will increase.

a. Modify the direction field for $u' = u\,(1-u)$ to account for the Allee effect. That is, a u, t plane, draw the line $u = 1$ and a threshold line $u = \epsilon$ where $\epsilon = 0.1$, say. Arrows below $u = \epsilon$ should point downward; arrows between $u = \epsilon$ and $u = 1$ should point upwards. Draw enough direction field arrows to indicate the paths of solutions for a threshold model.

b. Draw the the logistic phase plane graph and the phase plane graphs for the following three candidates of a threshold logistic differential equation where $\epsilon = 0.1$.

$$\begin{array}{lllllll} u' & = & f(u) & = & u \times (1-u) & & \text{Logistic} \\ u' & = & f_1(u) & = & u^{\frac{2}{3}} \times (u-\epsilon)^{\frac{1}{3}} \times (1-u) & & \text{Candidate 1} \\ u' & = & f_2(u) & = & u \times \frac{u-\epsilon}{u+\epsilon} \times (1-u) & & \text{Candidate 2} \\ u' & = & f_3(u) & = & u \times (u-\epsilon) \times (1-u) & & \text{Candidate 3} \end{array}$$

On first introduction, f_1 and f_2 look unbelievably complex. For $\epsilon \ll u$ $f_1(u) \doteq u(1-u)$ and $f_2(u) \doteq u(1-u)$, thus approximating the logistic $f(u) = u(1-u)$ when well above the threshold of disaster, ϵ. Argue that for each of f_1, f_2, and f_3, , $u = 0$ and $u = 1$ are stable solutions and and $u = \epsilon$ is an unstable solution.

We will find the f_1 has some technical problems that makes it unattractive but not absolutely impossible to use. f_2 and f_3 have an advantage in that we can analytically solve for u-inverse in the corresponding equations. A shortcoming of all three of f_1, f_2 and f_3 is that they are chosen only to match a pattern and do not derive from a fundamental hypothesis about the process of population dynamics near the threshold.

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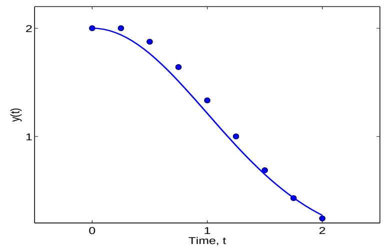
[7] S

17.5 Numerical approximations to solutions of differential equations.

Direction fields enable step by step construction of approximate solutions to

$$y(a) = y_a \qquad y'(t) = f(t, y)$$

$$a \le t \le b$$



Let $n > 0$ be an integer, and $h = \frac{b-a}{n}$.

$$\begin{aligned} \text{Let} \qquad t_k &= a + k * h \qquad & k = 0, 1, \cdots, n \\ \\ y_0 &= y_a \\ y_{k+1} &= y_k + h * f(t_k, y_k) \qquad & k = 0, \cdots, n-1 \end{aligned}$$

The points

$$(t_0, y_0) \qquad (t_1, y_1) \qquad (t_2, y_2) \qquad \cdots \qquad (t_n, y_n)$$

approximate the graph of y.

Example 17.5.1 We compute points to approximate the solution to

$$v(0) = 0.2 \qquad v'(t) = v(t) \times e^{-v(t)} - 0.1 * v(t) \qquad 0 \le t \le 10$$

This is Ricker's model of fish populations with parameter reduction, Equation 17.15. First we divide the time axis [0, 10] into intervals of length 2 and let

$$t_0 = 0 \qquad t_1 = 2 \qquad t_2 = 4 \qquad t_3 = 6 \qquad t_4 = 8 \qquad \text{and} \qquad t_5 = 10$$

Our objective is to compute v_0, v_1, $\cdots$, and v_5 so that the points (t_0, v_0), (t_1, v_1), $\cdots$, and (t_5, v_5) will lie close to the graph of the solution, $v(t)$. The method we use is called Euler's method (see Definition 17.5.1).

Step 0. let $v_0 = 0.2$. Then (t_0, v_0) *is* a point of the graph of the solution.

Step 1. From the differential equation, the slope of the solution at (0, 0.2) is

$$\begin{aligned} v'(0) &= v(0) \times e^{-v(0)} - 0.1 \times v(0) \\ &= 0.2e^{-0.2} - 0.1 * 0.2 \\ &= 0.1437 \end{aligned}$$

We construct the interval between $t_0 = 0$ and $t_1 = 2$ that starts at (0, 0.2) and has slope 0.1437. See Figure 17.11A. The right end point is at $t_1 = 2$ and the ordinate is

$$v_1 = 0.2 + 2 \times 0.1437 = 0.4874.$$

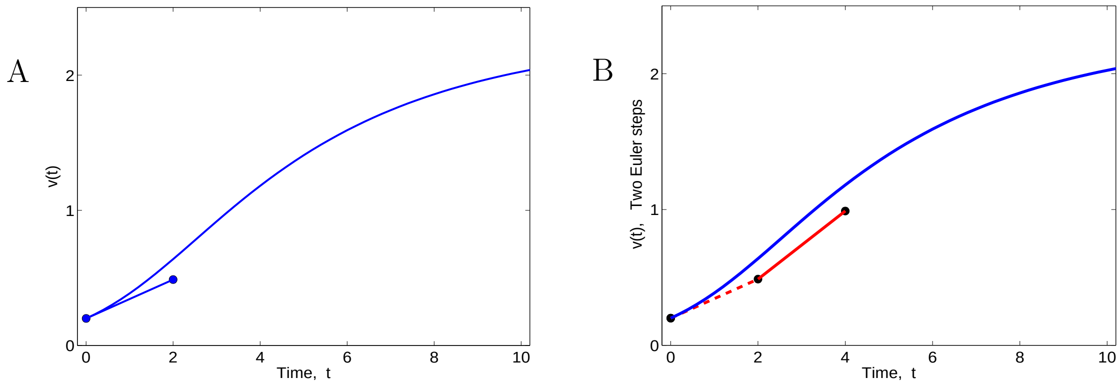


Figure 17.11: A. The first interval of Euler approximation to the solution of $v(0) = 0.2$, $v'(t) = v(t) \times e^{-v(t)} - 0.1 * v(t)$ on $[0, 10]$ using 5 intervals. B. The second interval.

Figure 17.11A. **Pattern:** Note that $0.1437 = v_0 \times e^{-v_0} - 0.1 * v_0$ so that

$$v_1 = v_0 + 2 \times (v_0 \times e^{-v_0} - 0.1 \times v_0)$$

Step 2. The slope of the direction field at $(t_1, v_1) = (2, 0.4874)$ is

$$v_1 \times e^{-v_1} - 0.1 \times v_1 = 0.4874 \times e^{-0.4874} - 0.1 * 0.4874 = 0.2506$$

We construct the interval between $t_1 = 2$ and $t_2 = 4$ that starts at (2, 0.4874) and has slope 0.2507. Figure 17.11B.

The right end point is at $t_2 = 4$ and

$$v_2 = 0.4876 + 2 \times 0.2506 = 0.9888$$

Pattern: Note that $0.2506 = v_1 \times e^{-v_1} - 0.1 \times v_1$ so that

$$v_2 = v_1 + 2 \times (v_1 \times e^{-v_1} - 0.1 \times v_1)$$

Step 3. We compute the slope of the direction field at (4,0.9888) and construct the interval with that slope between $t_2 = 4$ and $t_3 = 6$ that starts at (4, 0.9888). Following the previous two patterns, we use

$$v_3 = v_2 + 2 \times (v_2 \times e^{-v_2} - 0.1 v_2) = 0.9888 + 2 \times (0.9888 \times e^{-0.9888} - 0.1 \times 0.9888) = 1.5268$$

Step 4. We compute

$$v_4 = v_3 + 2(\times v_3 \times e^{-v_3} - 0.1 \times v_3) = 1.5268 + 2 \times (1.5268 e^{-1.5268} - 0.1 \times 1.5268) = 1.8848$$

where (t_4, v_4) is the right endpoint of the fourth interval tracking the solution.

Step 5. The right endpoint of the fifth interval is (t_5, v_5) where

$$v_5 = v_4 + 2 \times (v_4 \times e^{-v_4} - 0.1 \times v_4) = 1.8848 + 2 \times (1.8848 \times e^{-1.8848} - 0.1 \times 1.8848) = 2.0803$$

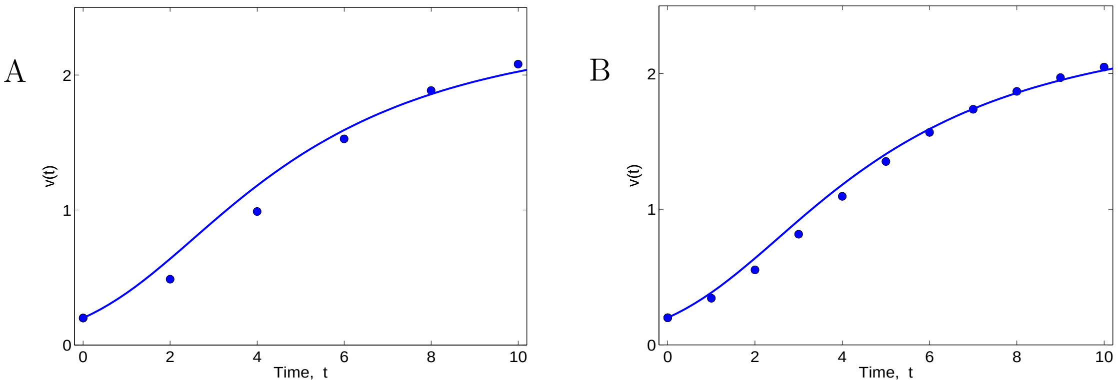


Figure 17.12: Euler approximations to the solution of $v(0) = 0.2$, $v'(t) = v(t) \times e^{-v(t)} - 0.1 * v(t)$ on $[0, 10]$, A. using 5 intervals in [0,10] and B. using 10 intervals in [0,10].

The graph of the solution and the points that we have computed are shown in Figure 17.12A. The points are perhaps close enough for some purposes, but we will find it easy to compute a closer fit.

The general pattern is that, for time interval size h,

$$v_{k+1} = v_k + h \times \text{slope}_k$$

and for $v' = v \times e^{-v} - 0.1 \times v$ the specific pattern is

$$v_{k+1} = v_k + h \times \Big(v_k \times e^{-v_k} - 0.1 \times v_k \Big)$$

For $h = 2$, these numbers are easily computed on your calculator with

Keystroke	Keystroke	Display
0.2	ENTER	0.20000
ANS + 2 × (ANS × e ^ (⁻ ANS) - 0.1 × ANS)	ENTER	0.4875
	ENTER	0.9888
	ENTER	1.5267
	ENTER	1.8847
	ENTER	2.0803

These numbers differ slightly from the previous numbers because the calculator is accurate to 11 or 12 digits, even if you do not want them.

Or you may write and execute the MATLAB program,

```
close all;clc;clear
h=2
v(1) = 0.2;
for k = 1:5
    v(k+1) = v(k)+(v(k)*exp(-v(k))-0.1*v(k))*h;
end
v.'
```

Table 17.1: Euler computations for $y(0) = 2$, $y'(t) = t - y(t)$ with interval size $h = 1$.

Step, k	t_k	y_k	$\text{slope}_k = t_k - y_k$	$y_{k+1} = y_k + h * \text{slope}_k$
0	0	2	0 - 2 = -2	$2 + 1 \times (-2) = 0$
1	1	0	1 - 0 = 1	$0 + 1 \times 1 = 1$
2	2	1	2 - 1 = 1	$1 + 1 \times 1 = 2$
3	3	2	3 - 2 = 1	$2 + 1 \times 1 = 3$
4	4	3		

We can easily improve the accuracy of our approximation by using a smaller time step size and a calculator.

For step size, $h = 1$, we use

0.1 ENTER ANS + 1 × (ANS × e ˆ (⁻ ANS) - 0.1 × ANS)

and press ENTER 10 times. The results are shown in Figure 17.12B.

Or you may change two of the MATLAB program lines to `h=1` and `k=1:10`.

Explore 17.5.1 Compute approximations to

$$v(0) = 0.2 \qquad v'(t) = v(t) \times e^{-v(t)} - 0.1 * v(t) \qquad 0 \leq t \leq 10$$

using 40 intervals of length 0.25.

Example 17.5.2 We find an approximate solution to

$$y(0) = 2 \qquad y'(t) = t - y$$

for $0 \leq t \leq 4$; the graph of the solution is shown in Figure 17.13A. Both t and y appear in the RHS in this problem.

The basic pattern is the same.

$$y_0 = y(0) = 2 \qquad y_{k+1} = y_k + h \times \text{slope}_k$$

The computations are organized in Table 17.1 for time-interval size $h = 1$.

Our approximation is shown in Figure 17.13A and is not close enough to the solution to satisfy us. The approximation computed using time-interval size $h = 0.25$ is shown in Figure 17.13B and it is more acceptable. The initial and final computations for $h = 0.25$ are shown in Table 17.2

Because the RHS of $y'(t) = t - y$ involves both t and y, these numbers are not computed on a calculator using only ANS, the previous answer key. A calculator program that will do the computations is included in Table 17.3.

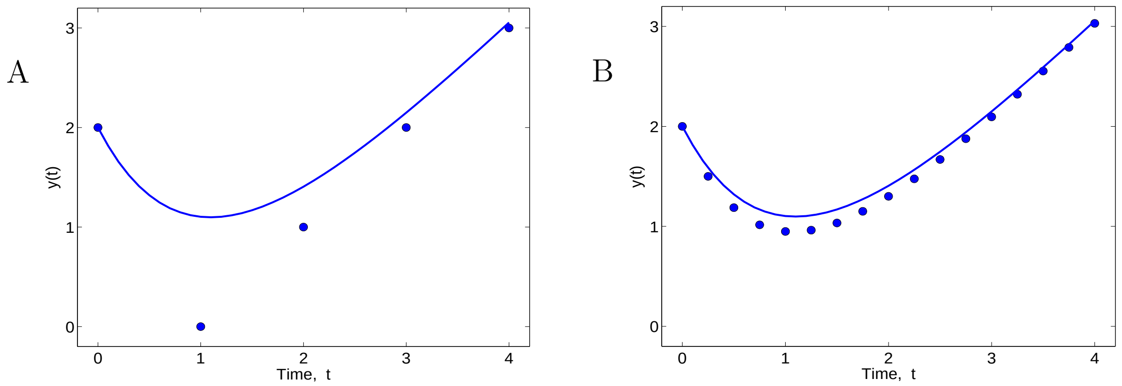


Figure 17.13: Euler approximations to the solution of $y(0) = 0.2$, $y'(t) = t - y(t)$ on $[0, 4]$ using 4 intervals in A and using 16 intervals in B.

Table 17.2: Euler computations for $y(0) = 2$, $y'(t) = t - y(t)$ on $0 \le t \le 4$ for $h = 0.25$.

Step, k	t_k	y_k	slope$_k = t_k - y_k$	$y_{k+1} = y_k + h * \text{slope}_k$
0	0.00	2.0000	-2.0000	1.5000
1	0.25	1.5000	-1.2500	1.1875
2	0.50	1.1875	-0.6875	1.0156
3	0.75	1.0156	-0.2656	0.9492
⋮	⋮	⋮	⋮	⋮
15	3.75	2.7901	0.9599	3.0301
16	4.00	3.0301		

Table 17.3: TI-86 calculator program, EU, to compute Euler approximation to $y(0) = 2$, $y'(t) = t - y(t)$ on $0 \le t \le 4$ for $h = 0.25$. Calculator program, TRAP, to compute trapezoid approximation to $y(0) = 2$, $y'(t) = -t * y(t)$ on $0 \le t \le 2$ with $h = 0.2$.

```
PROGRAM:EU
:Fix 3
:.25 -> H
:{17,2}->dim E
:0->E(1,1)
:2->E(1,2)
:For(K,1,16)
:H*K->E(K+1,1)
:SL=E(K,1)-E(K,2)
:E(K,2)+H*SL->E(K+1,2)
:End
:Pause  E
:Fix 9
```

```
PROGRAM:TRAP
:Fix 3
:0.2 -> H
:{11,2}->dim T
:0->E(1,1)
:2->E(1,2)
:For(K,1,10)
:H*K->E(K,1)
:SL1=-E(K,1)*E(K,2)
:SL2=-(E(K,1)+H)*(E(K,2)+H*SL1)
:SL = (SL1 + SL2)/2;
:E(K,2)+H*SL->E(K+1,2)
:End
:Pause E
:Fix 9
```

Table 17.4: MATLAB program, eu_tmy.m, to compute Euler approximation to $y(0) = 2$, $y'(t) = t - y(t)$ on $0 \le t \le 4$ for $h = 0.25$.

```
close all;clc;clear
h=0.25; t=[0:h:4];
xs=[0:0.05:4]; ys=2*exp((-xs.*xs/2));
plot(xs,ys,'linewidth',2); hold
v(1) = 2;
for i = 1:length(t)-1
    v(i+1) = v(i) - t(i)*v(i)*h;
end
plot(t,v,'o','MarkerSize',8,'markerfacecolor','g')
xlabel('Time, t','fontsize',16)
ylabel('y(0)=2, y^{\prime} = -ty, y(t)=2exp(-t^2/2)','fontsize',16)
```

Definition 17.5.1 Euler's Method. The scheme we have been using is called Euler's Method after Leonhard Euler (1707-1783) who introduced the method in 1768. The algorithm for solving

$$y(a) = y_a \qquad y'(t) = f(t, y) \qquad a \le t \le b$$

is

1. Choose n a number of intervals and let $h = \frac{b-a}{n}$.
2. Let $t_k = a + k * h$, for $k = 0,\ 1,\ 2,\ \cdots,\ n$ and let $y_0 = y_a$.
3. For $k = 0,\ 1,\ 2,\ \cdots,\ n-1$ let $y_{k+1} = y_k + h * f(t_k, y_k)$.

Euler's Method replaces a differential equation with a discrete difference equation. For example, for the logistic differential equation

$$y(a) = y_a \qquad y'(t) = r * y(t) * \left(1 - \frac{y(t)}{M}\right)$$

Euler's Method with interval size h yields

$$y_0 = y_a \qquad y_{k+1} = y_k + h * r * \left(1 - \frac{y_k}{M}\right)$$

This is the *discrete logistic equation* that we studied in Section 14.2 on page 144.

Satellites in orbit and space ships traveling in the solar system are subject to complex gravitational fields from the Earth, the Sun, the Moon, and other planets and their differential equations of motion are always solved numerically. Weather predictions are based on differential equations (called partial differential equations because their solution functions depend on three space variables and time). These equations are complex and involve a vast grid of space and time data points and can only be solved numerically.

17.5.1 The trapezoid and Runge-Kutta methods.

Euler's method is intuitive and instructional, but not very accurate. If you are traveling in a space ship to the Moon, you hope the folks at NASA Houston are using something better than Euler's Method to control your space ship, and there are numerous accurate methods for approximating solutions of differential equations. We present two additional methods that show one form of improvement.

Euler's Method estimates the slope of a solution at the left end point, t_k of the interval $[t_k, t_{k+1}]$ and assumes that slope is unchanged throughout the interval. This is similar to using rectangles to approximate an integral. For approximating an integral, we got better results using the trapezoid method which uses the average of the values of the integrand at the two end points of each interval in the partition. The trapezoid method for approximating a solution to a differential equation uses a similar procedure.

Definition 17.5.2 The **trapezoidal method** to approximate the solution to the differential equation

$$y(a) = y_a \qquad y'(t) = f(t, y)$$

is summarized by

1. Choose n a number of intervals and let $h = \frac{b-a}{n}$.
2. Let $t_k = a + k * h$ for $k = 0, 1, 2, \cdots, n$, and let $y_0 = y_a$.
3. For $k = 0, 1, 2, \cdots, n-1$, let
   (a) $\text{slope}_1 = f(t_k, y_k)$.
   (b) $\text{slope}_2 = f(t_k + h, y_k + h \times \text{slope}_1)$.
   (c) $\text{slope} = (\text{slope}_1 + \text{slope}_2)/2$.
   (d) $y_{k+1} = y_k + h \times$ slope.

A graphic of the trapezoid procedure is shown in Figure 17.14. $\text{slope}_1 = \text{sl}_1$ is the usual Euler's Method slope. $\text{slope}_2 = \text{sl}_2$ is the direction field slope at the point

$$f(t_k + h, y_k + h \times \text{sl}_1),$$

the point projected by Euler's method. Then the slope from y_k to y_{k+1} is the average of sl_1 and sl_2.

Example 17.5.3 We use the trapezoid method to approximate the solution to

$$y(0) = 2 \qquad y'(t) = t - y \qquad 0 \leq t \leq 4$$

using $n = 16$ intervals and compare the results with those obtained with Euler's method.

Step 0. $t_k = 0 + k * 0.25$, for $k = 0, 1, \cdots, 15$ and $y_0 = 2$.

Step 1. $\text{slope}_1 = t_0 - y_0 =$ 0 - 2 = -2

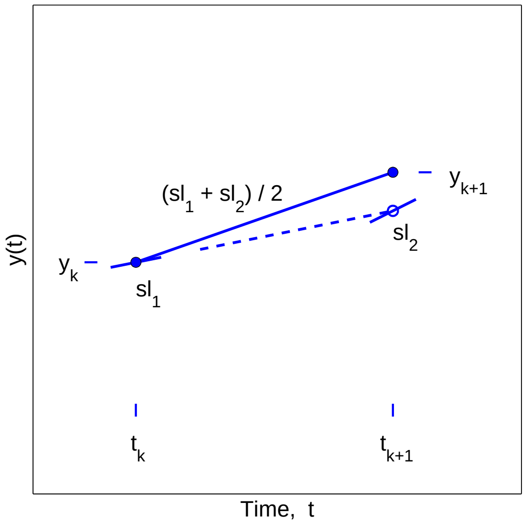


Figure 17.14: Graphic of the trapezoidal rule to compute y_{k+1} from y_k.

Euler's projected y_1 is $\hat{y}_1 = y_0 + h \times \text{slope}_1 = 2 + 0.25 \times (-2) = 1.5$.
Direction field slope at $(t_1, \hat{y}_1) = (0.25, 1.5)$ is $t_1 - \hat{y}_1 =$ 0.25 - 1.5 = -1.25 = slope$_2$.
slope = (slope$_1$ + slope$_2$)/2 = (-2 + -1.25)/2 = -1.625.
Trapezoid projected $y_1 = y_0 + h\times$ slope.

$$y_1 = 2.0 + 0.25 \times (-1.625) = 1.5938$$

The first two steps are drawn in Figure 17.15A.

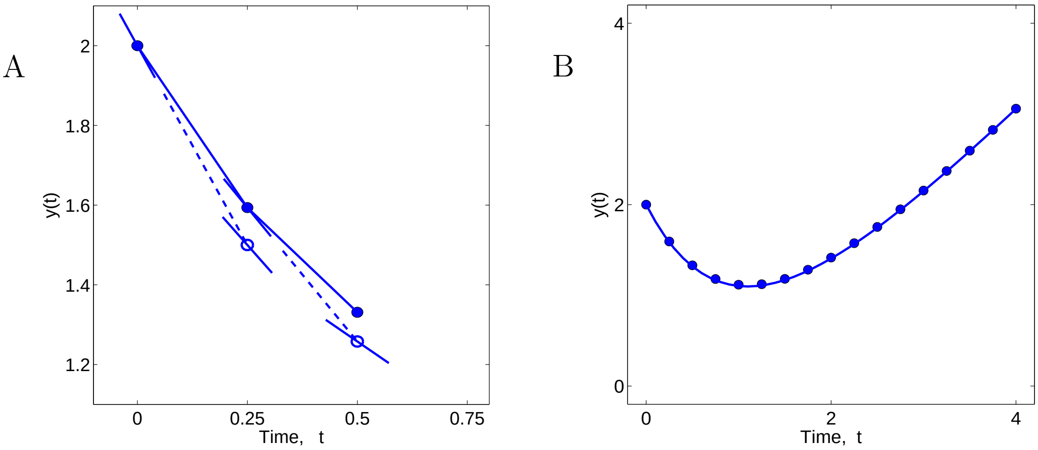


Figure 17.15: A. The first two steps of trapezoid approximations to the solution of $y(0) = 0.2$, $y'(t) = t - y(t)$ on $[0, 4]$ using 16 intervals. B. Graphs of both the trapezoid approximation and the solution. The two graphs are indistinguishable at this resolution.

Step 2. slope$_1 = t_1 - y_1 =$ 0.25 - 1.5938 = -1.3438
Euler's projected y_2 is $\hat{y}_2 = y_1 + h \times \text{slope}_1 = 1.5938 + 0.25 \times (-1.3438) = 1.2579$.
Direction field slope at $(t_2, \hat{y}_2) = (0.50, 1.2579)$ is $t_2 - \hat{y}_2 =$ 0.50 - 1.2579 = -0.7579 = slope$_2$.

Table 17.5: Comparison of Euler's method with 32 intervals and trapezoid approximation using 16 intervals to the solutions of $y(0) = 2$, $y'(t) = t - y(t)$.

Step, k			Time	Euler's	Trapezoid	Correct	Runge-
Euler	Trap	R-K	t	Method	Method	Solution	Kutta
0	0	0	0.00	2.0000	2.0000	2.0000	2.0000
2	1		0.25	1.5469	1.5938	1.5864	
4	2	1	0.50	1.2585	1.3311	1.3196	1.3203
6	3		0.75	1.0964	1.1805	1.1671	
8	4	2	1.00	1.0308	1.1176	1.1036	1.1045
16	8	4	2.00	1.3542	1.4163	1.4060	1.4067
24	12	6	3.00	2.1217	2.1551	2.1494	2.1497
32	16	8	4.00	3.0418	3.0578	3.0549	3.0551

slope $= (\text{slope}_1 + \text{slope}_2)/2$. Trapezoid projected $y_2 = y_1 + h\times$ slope

$$y_2 = 1.5938 + 0.25 \times \frac{(-1.3438) + (-0.7579)}{2} = 1.3311$$

The remaining 14 computations are similar and a graph of the trapezoid approximation solution and the actual solution is shown in Figure 17.15B. A table that compares the results of Euler's method and the trapezoid method is shown in Table 17.5.

It has to be acknowledged that the trapezoid method requires about twice as much arithmetic as that of Euler's method[8]. Therefore, we compare the accuracy of Euler's method using 32 intervals with the trapezoid method using 16 intervals.

It can be seen that with comparable arithmetic effort the trapezoid scheme gives closer approximations. The increased accuracy in this case may not seem impressive, but if you are trying to dock the space shuttle onto the space station, the improved accuracy may be appreciated.

Definition 17.5.3 The Runge-Kutta method. A popular and accurate extension of the trapezoid algorithm called the Runge-Kutta method is included for reference.

The Runge-Kutta algorithm to solve

$$y(a) = y_0, \qquad y'(t) = f(t,y) \qquad a \le t \le b$$

is

Choose a number n of intervals, let $h = (b-a)/n$ and $t_k = a + k \times h$, $k = 1, n$.
$y_0 = y(a)$ is given. For $k = 0, n-1$ compute:

$$s_1 = f(t_k, y_k)$$

---

[8] The usual way of evaluating the required work is to count the number of times $f(t,y)$ has to be evaluated; once for each Euler step and twice for each trapezoid step.

MATLAB program, trap_nty.m, to compute trapezoid approximation to $y(0) = 2$, $y'(t) = -t * y(t)$ on $0 \le t \le 2$ with $h = 0.2$.

```
close all;clear;clc
h = 0.25; Nsteps=4/h; t=[0:h:4];
ts=[0:0.1:4]; ys=3*exp(-ts)+ts-1; plot(ts,ys,'linewidth',2)
axis([-0.2 4.2 -0.2 4.2]); axis('square')
hold
y(1) = 2;
for k = 1:Nsteps
    slope1=t(k)-y(k);
    slope2=(t(k)+h) - (y(k)+slope1*h);
    slope = (slope1+slope2)/2;
    y(k+1) = y(k)+slope*h;
end
plot(t,y,'o','MarkerSize',8,'MarkerFaceColor','g')
set(gca,'xtick',[0  2  4],'fontsize',16)
set(gca,'ytick',[0 2 4],'fontsize',16)
xlabel('Time,  t','fontsize',16)
ylabel('y(t)','fontsize',16)
```

$$\begin{aligned} s_2 &= f(t_k + \frac{1}{2}h, y_k + \frac{1}{2}hs_1) \\ s_3 &= f(t_k + \frac{1}{2}h, y_k + \frac{1}{2}hs_2) \\ s_4 &= f(t_k + h, y_k + hs_3) \end{aligned}$$

$$y_{k+1} = y_k + \frac{h}{6}(s_1 + 2s_2 + 2s_3 + s_4)$$

The Runge-Kutta method requires four functional evaluations per step, so to compare it with the Euler method and trapezoid method illustrated in Table 17.5 only 8 steps with $h = 0.5$ are used. The Runge-Kutta method has even greater accuracy than that of the trapezoid method with no more computational effort.

Exercises for Section 17.5, Numerical approximations to solutions of differential equations.

Exercise 17.5.1 The function $y(t) = e^{0.5\,t}$ solves

$$y(0) = 1 \qquad y'(t) = 0.5 \times y(t) \qquad \text{for} \qquad 0 \le t \le 4$$

is shown in Exercise Figure 17.5.1A.

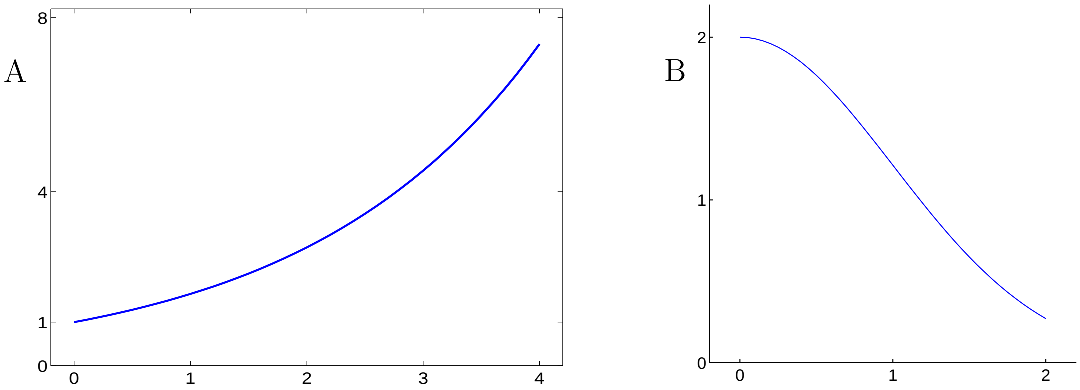
1. For time-interval size $h = 1$ compute Euler approximations to the solution, y, and compare them with the graph, $0 \le t \le 4$.

2. For time-interval size $h = 0.2$ compute

$$t_k = k \times h, \qquad k = 0, 1, \cdots, 20 \qquad y_0 = 1 \qquad y_{k+1} = y_k + h * 0.5 * y_k$$
$$k = 0, 1, \cdots, 19.$$

3. Plot the points, (t_5, y_5), (t_{10}, y_{10}), (t_{15}, y_{15}), and (t_{20}, y_{20}).

Figure for Exercise 17.5.1 A. Graph of $y = e^{0.5t}$ which is the solution to $y(0) = 1$, $y' = 0.5y$. B. Graph of $y(t) = 2e^{-t^2/2}$ which is the solution of $y(0) = 2$, $y'(t) = -t \times y$.



Exercise 17.5.2 Shown in Exercise Figure 17.5.1B is the graph of $y(t) = 2e^{-t^2/2}$ which is the solution of $y(0) = 2$, $y'(t) = -t \times y$. Use Euler's method and 10 intervals on $[0, 2]$ to approximate the solution to $y(0) = 2$, $y'(t) = -t \times y$.

Exercise 17.5.3 A differential equation with initial condition and its analytic solution are shown.

i. Show that the analytic solution satisfies the initial condition and the differential equation.

ii. Use Euler's method and the trapezoid methods to approximate the solution to the differential equation on the interval shown and using the step size shown. iii. Plot the solution and the Euler's and trapezoid approximations on a single t, y plane.

a. $y(0) = 1 \quad y'(t) = y^2 \quad y(t) = (1-t)^{-1} \quad 0 \le t \le 0.4 \quad h = 0.1$

b. $y(0) = 2 \quad y'(t) = -y^2 \quad y(t) = (t+0.5)^{-1} \quad 0 \le t \le 0.4 \quad h = 0.1$

c. $y(0) = 1 \quad y'(t) = t \times y \quad y(t) = e^{t^2/2} \quad 0 \le t \le 1 \quad h = 0.2$

d. $y(0) = 1 \quad y'(t) = \sqrt{y} \quad y = (t/2+1)^2 \quad 0 \le t \le 1 \quad h = 0.2$

Exercise 17.5.4 Use the Euler and trapezoid methods to compute the solutions to the following differential equations with initial conditions on the intervals shown and using the step sizes shown.

$$\begin{array}{lrclrclll}
\text{a.} & y(0) & = & 4 & y'(t) & = & t-\sqrt{y} & 0\le t\le 1 & h=0.2 \\
\text{b.} & y(0) & = & 0.5 & y'(t) & = & y/(1+y) & 0\le t\le 1 & h=0.1 \\
\text{c.} & y(0) & = & 0.5 & y'(t) & = & -\ln y & 0\le t\le 1 & h=0.1 \\
\text{d.} & y(0) & = & 0.15 & y'(t) & = & y(y-0.1)(1-y) & 0\le t\le 1 & h=0.4 \\
\text{e.} & y(0) & = & 0 & y'(t) & = & y(y-0.1)(1-y) & 0\le t\le 1 & h=0.4 \\
\text{f.} & y(0) & = & 0.05 & y'(t) & = & y(y-0.1)(1-y) & 0\le t\le 1 & h=0.4
\end{array}$$

17.6 Synopsis.

At this time you should see that close approximations to the solution of many differential equations,

$$y(a)=y_a \qquad y'(t)=f(t,y) \qquad a\le t\le b,$$

can be computed with simple arithmetic, perhaps a lot of it. It is even better than that, for we can use the same ideas to solve two differential equations involving two unknown functions, u and v,

$$\begin{array}{rclrcll}
u(a) & = & u_a & u'(t) & = & f(t,u,v) & \\
v(a) & = & v_a & v'(t) & = & g(t,u,v) &
\end{array} \qquad a\le t\le b$$

Recall the differential equations for penicillin clearance

$$\begin{array}{rcll}
a'(t) & = & (-r_{0,1}-r_{2,1})\,a(t)+r_{1,2}\,b(t) & a(0)=200 \\
b'(t) & = & r_{2,1}\,A_t-r_{1,2}\,B_t & b(0)=0
\end{array}$$

Euler's method applied to these equations (using step size 5 minutes) simply returns us to the difference equations we first used to study penicillin clearance. We might also use a trapezoidal rule that would improve our accuracy.

We have not shown that as the interval size, h, gets close to zero Euler's approximations get close to the actual solution to the equation. For 'well behaved' differential equations, they do, as do the trapezoidal approximations. The analysis, however, is beyond our goal.

We will soon show a few of the numerous important analytic techniques for finding exact solutions to differential equations. Analytic formulas for solutions provide a lot more insight and intuition about the solutions than do tables of numbers, even graphs of tables of numbers. And a formula can be more easily carried to the next stage of a problem than can a table of numbers.

Ambiguity. We have ignored a serious problem that can arise. The differential equation with initial condition

$$y(0)=0 \qquad y'(t)=2\sqrt{y(t)} \tag{17.18}$$

has two solutions:

$$y_1(t)=t^2 \qquad \text{and} \qquad y_2(t)=0$$

Both $y_1(t) = t^2$ and $y_2(t) = 0$ solve $y(0) = 0$, $y'(t) = 2\sqrt{y(t)}$:

$$\begin{array}{rcll@{\qquad\qquad}rcll}
y_1(t) & = & t^2 & & y_2(t) & = & 0 & \\[2ex]
y_1(0) & = & 0^2 = 0 & \textbf{Check} & y_2(0) & = & 0 & \textbf{Check.} \\[2ex]
y_1{}'(t) & = & 2t & & y_2'(t) & = & 0 & \\[2ex]
 & = & 2\sqrt{t^2} = 2\sqrt{y_1(t)} & & & = & 2\sqrt{0} = 2\sqrt{y_2(t)} & \\[2ex]
y_1 & & \textbf{Solves} & & y_2 & & \textbf{Solves} &
\end{array}$$

Euler's method to approximate a solution to

$$y(0) = 0 \qquad y'(t) = 2\sqrt{y(t)}$$

yields

$$y_0 = 0, \qquad y_1 = y_0 + 2\sqrt{y_0} = 0 + 2\sqrt{0} = 0, \qquad y_2 = 0, \qquad \cdots \quad y_n = 0$$

Euler's method gives no hint that $y(t) = t^2$ solves the equation. Almost surely, if your differential equation evolves from a real biological model, it will have unique solutions. Conversely, if your equation has multiple solutions, the model is probably not well formulated. It will be seen, however, that the model we derive for mold growth in Section 17.9.5 on page 374 comes perilously close to the example just presented.

Uniqueness. Mathematicians have derived conditions that will insure the uniqueness of solutions to

$$y(a) = y_a \qquad y'(t) = f(t, y) \qquad a \leq t \leq b$$

Recall that $f_2(t, y) = \frac{\partial}{\partial y} f(t, y)$ denotes the partial derivative of $f(t, y)$ with respect to the second variable (the first variable is held constant).

Theorem 17.6.1 Existence and Uniqueness of Solutions. *If $f(t, y)$ and $f_2(t, y) = \frac{\partial}{\partial y} f(t, y)$ are continuous on a rectangle*

$$a - d \leq t \leq a + d \qquad y_a - d \leq y \leq y_a + d \qquad d > 0,$$

then on an interval $a - e \leq t \leq a + e$, $0 < e < d$ there is a unique solution to

$$y(a) = y_a \qquad y'(t) = f(t, y) \qquad a \leq t \leq b$$

Under the hypothesis that f_2 is continuous on the rectangle, it follows that f_2 is bounded on the rectangle, and the proof of Theorem 17.6.1 hinges on this fact. In our example Equation 17.18

$$y(0) = 0 \qquad y'(t) = 2\sqrt{y(t)}$$

$$f(t, y) = 2\sqrt{y} \qquad \text{and} \qquad f_2(t, y) = \frac{1}{\sqrt{y}}$$

At the initial data point, $y(0) = 0$, $f_2(0, 0)$ is not even defined and $f_2(t, y) = \frac{1}{\sqrt{y}}$ is certainly not bounded in any rectangle containing the initial data point, $(0, 0)$. Equation 17.18 does not satisfy the hypothesis of Theorem 17.6.1.

Another type of exception may occur. The differential equation

$$y(0) = 0 \qquad y'(t) = 1 + y^2 \qquad 0 \leq t \leq 2$$

has a unique solution

$$y(t) = \tan t \qquad \text{Note:} \qquad y'(t) = \sec^2 t \qquad \sec^2 t = 1 + \tan^2 t$$

The problem is that the solution $y(t) = \tan t$ only extends on the interval $0 \leq t < \frac{\pi}{2} \doteq 1.57$. The original problem asked for a solution on the interval [0, 2].

$$\text{For} \qquad y'(t) = 1 + y^2 \qquad f(t, y) = 1 + y^2 \qquad \text{and} \qquad f_y(t, y) = 2y$$

and both f and f_2 are continuous for all (t, y). The hypothesis of Theorem 17.6.1 is satisfied, but the theorem only guarantees a solution on some interval $a - e \leq t \leq a + e$ and not throughout $[a, b]$.

Despite the two previous examples, you should expect differential equations that you derive from a biological model to have a unique solutions valid over the reasonable life of your system. You should expect to be able to compute a good approximation to the solution, perhaps using a more sophisticated system than either Euler's method or the trapezoid method.

Example 17.6.1 Although proof of Theorem 17.6.1 is beyond our scope, existence and uniqueness of solutions to a large number of differential equations follows rather easily from simple anti-derivative formulas and the Parallel Graph Theorem. For example, we know by simple substitution that the growth equation $(k > 0)$,

$$P(0) = P_0 > 0, \quad P'(t) = k\, P(t) \qquad \text{has a solution} \qquad P(t) = P_0\, e^{kt}.$$

Might there be another solution, say $Q(t)$? If so, then

$$Q(0) = P_0, \qquad Q'(t) = k\, Q(t).$$

Then

$$\begin{array}{rcll}
Q'(t) & = & k\,Q(t) & \text{Hypothesis} \\
e^{-k\,t}\,Q'(t) - ke^{-k\,t}\,Q(t) & = & 0 & \textbf{Blue Sky.} \\
\left[e^{-k\,t}\,Q(t)\right]' & = & [1]' & \text{Derivative formulas} \\
\int_0^t \left[e^{-k\tau}\,Q(\tau)\right]'\,d\tau & = & \int_0^t\,[1]'\,d\tau & \text{Identity in symbols} \\
e^{-k\,t}\,Q(t) + C_1 & = & C_2 & \text{Fundamental Theorem of Calculus II} \\
Q(t) & = & Ce^{k\,t} & C = C_2 - C_1 \\
P_0 & = & C\,e^{k\times 0} = C & Q(0) = P_0 \\
Q(t) & = & P_0\,e^{k\,t} &
\end{array}$$

Thus $Q(t) \equiv P(t)$ and there is one and only one solution to $P(t) = P_0 > 0$, $P'(t) = k\,P(t), \quad k > 0$.

The Blue Sky invoked above is part of the accumulated wisdom of 300 years of studying differential equations, and is described in the next section, Section 17.7. ■

Exercises for Section 17.6 Synopsis.

Exercise 17.6.1 Draw the direction field for $y'(t) = \sqrt{y(t)}$ and decide whether the equilibrium solution $y(t) = 0$ is stable.

Exercise 17.6.2 Consider the pair of differential equations

$$\begin{array}{rcl} u(0) & = & 1 \\ v(0) & = & 2 \end{array} \qquad \begin{array}{rcl} u'(t) & = & 0.5\times u(t) - 0.2\times u(t)\times v(t) \\ v'(t) & = & 0.1\times u(t)\times v(t) - 0.1\times v(t) \end{array} \qquad 0 \le t \le 1$$

This system is a predator prey system. We (including you!) will use Euler's method to approximate a solution on the time interval [0,1] with $n = 5$ subintervals. NOTE: SEE EXERCISE 17.6.3. IT MAY SAVE YOU SOME ARITHMETIC.

Step 0. $h = 0.2$. Let $t_0 = 0$, $u_0 = 1$, and $v_0 = 2$.

Step 1.

$$\begin{array}{rcl} \text{slope}_u & = & 0.5\times u_0 - 0.2\times u_0\times v_0 \\ u_1 & = & u_0 + h\times\text{slope}_u \end{array} \qquad \begin{array}{rcl} \text{slope}_u & = & 0.5\times 1 - 0.2\times 1\times 2 = 0.1 \\ u_1 & = & 1 + 0.2\times 0.1 = 1.02 \end{array}$$

$$\begin{array}{rcl} \text{slope}_v & = & 0.1\times u_0\times v_0 - 0.1\times v_0 \\ v_1 & = & v_0 + h\times\text{slope}_v \end{array} \qquad \begin{array}{rcl} \text{slope}_v & = & 0.1\times 1\times 2 - 0.1\times 2 = 0.0 \\ v_1 & = & 2 + 0.2\times 0 = 2 \end{array}$$

Continue the computations of (u_2, v_2), (u_3, v_3), (u_4, v_4), and (u_5, v_5).

Exercise 17.6.3 The following MATLAB program solves the previous problem, Exercise 17.6.2. In the second line, change `Lt=1;` to `Lt=50` and run it. Is the system stable?

```
close all;clc;clear
LT=1; h=0.2; t=[0:h:LT]; Nsteps=LT/h;
u(1)=1; v(1)=2;
for k=1:Nsteps
    slope_u(k) = 0.5*u(k) - 0.2*u(k)*v(k);
    slope_v(k) = 0.1*u(k)*v(k) - 0.1*v(k);
    u(k+1) = u(k) + h*slope_u(k);
    v(k+1) = v(k) + h*slope_v(k);
end
plot(u,v,'o','MarkerSize',10,'MarkerFaceColor','g')
hold
plot(u(1),v(1),'o','MarkerSize',10,'MarkerFaceColor','r')
[u.'  v.']
[slope_u.'  slope_v.']
```

Exercise 17.6.4 Recall that in Problem 17.4.11 on page 326 we suggested

$$u' = u^{\frac{2}{3}} \times (u-0.1)^{\frac{1}{3}} \times (1-u)$$

as a possible model for a logistic population with a threshold number below which the population would be lost. Show that the hypothesis of the Existence and Uniqueness of Solutions Theorem 17.6.1 is not satisfied for this equation.

In this case there are multiple solutions that peel off of the equlibrium solution, $u = 0.1$ as t increases; in a sense, it is super unstable. You should have found that the hypothesis of Theorem 17.6.1 is not satisfied near $u = 0$. Solutions with small positive initial values quickly join with $u = 0$; in a sense, $u = 0$ is super stable.

Exercise 17.6.5 Argue that

$$y = \tan t \quad \text{is the only solution to} \qquad y(0) = 0, \qquad y'(t) = 1 + y^2(t).$$

$$\text{Use the integral formula,} \quad \int \frac{y'(\tau)}{1+y^2(\tau)}\, d\tau = \arctan y(\tau) + C.$$

17.7 First order linear differential equations.

Analytic solutions to differential equations are shown in this and subsequent sections. Here we solve **First Order Linear Differential Equations** which are of the form

$$y(a) = y_a, \qquad y'(t) + p(t) \times y(t) = q(t) \tag{17.19}$$

where p and q are continuous functions defined on an interval, $[a, b]$. The problem of finding $y(t)$ reduces to

Theorem 17.7.1

Step 1. Define $u(t) = \int_a^t p(s)\,ds.$

Step 2. Define $v(t) = \int_a^t e^{u(s)} \times q(s)\,ds.$

Then the solution to Equation 17.19 is

$$y(t) = v(t)e^{-u(t)} + y_a e^{-u(t)} \tag{17.20}$$

Example 17.7.1 Consider a case in which $p(t) = 3$ and $q(t) = 5$ are constant. Solve

$$y(1) = 5 \qquad y'(t) + 3y(t) = 2, \qquad a = 1, \quad y_a = 5, \quad p(t) = 3, \qquad q(t) = 5$$

Define

$$\begin{aligned} u(t) &= \int_0^t 3\,dt = 3t \\ v(t) &= \int_0^t e^{3t} \times 2\,dt = \frac{2}{3}\left(e^{3t} - 1\right). \end{aligned}$$

Then

$$\begin{aligned} y(t) &= \frac{2}{3}\left(e^{3t} - 1\right) \times e^{-3t} + 5e^{-3t} \\ &= \frac{2}{3} + \frac{13}{3}e^{-3t} \end{aligned}$$

To check that $y(t)$ solves $y(0) = 5 \qquad y'(t) + 3y(t) = 5$ we compute

$$y(0) = \frac{2}{3} + \frac{13}{3}e^{-3\times 0} = \frac{2}{3} + \frac{13}{3} = 5. \qquad \textbf{Checks}.$$

Also

$$\begin{aligned} y'(t) + 3y(t) &= \left[\tfrac{2}{3} + \tfrac{13}{3}e^{-3t}\right]' + 3\left(\tfrac{2}{3} + \tfrac{13}{3}e^{-3t}\right) \\ &= 0 + \tfrac{13}{3}e^{-3t} \times (-3) + 2 + 13e^{-3t} \\ &= 2 \qquad \textbf{Checks}. \end{aligned}$$

Example 17.7.2 The formulas for $u(t)$ and $v(t)$ are explicit, but the integrals may not be computable in terms of familiar functions. In the equation

$$y(0) = 5 \qquad y'(t) - 2ty(t) = 1,$$

$$p(t) = -2t, \qquad u(t) = \int_0^t -2s\,ds = -t^2, \qquad \text{and} \qquad v(t) = \int_0^t e^{-s^2}\,ds.$$

There is no formula for $v(t)$ in familiar terms[9] for $v(t)$. It can be numerically approximated as you did in Chapter 11, The Integral, of Volume I, and is an important formula in statistics, but there is no expression for $\int_0^t e^{-s^2}\,ds$ in familiar terms. ■

That $y(t)$ in Equation 17.20 solves and is the only solution to the initial condition and equation of 17.19 is explored in Exercises 17.7.3 - 17.7.5

17.7.1 Rationale for Steps 1 and 2.

Steps 1 and 2 are presented as **Twin Lightning Bolts Out Of The Blue** to solve the equation

$$y'(t) + p(t) \times y(t) = q(t)$$

There is rationale leading to Steps 1 and 2 and the rationale is used in equations involving second and higher order derivatives.

First one should look at the special case of $q(t) = 0$

$$\text{LHS} \qquad y'(t) + p(t) \times y(t) = 0 \tag{17.21}$$

This is called a *homogeneous* linear equation and typically occurs when the system under study is isolated from external forces. q often represents environmental input.

$y = 0$ is a solution to the homogeneous equation. Because solutions are unique (Exercise 17.7.3), no other solution intersects the solution $y = 0$. Therefore every other solution is either always positive or always negative. For this discussion, we assume we are looking for a solution y_h (subscript h for 'homogeneous') that is always positive.

A procedure to solve Equation 17.21 is called *separation of variables*, meaning to write the equation in a form that has the terms involving y on one side and the terms involving t on the other side. We write (because $y_h > 0$)

$$\begin{aligned} y_h'(t) + p(t) \times y_h(t) &= 0 \\[2em] y_h'(t) &= -p(t) \times y_h(t) \\[2em] \frac{y_h'(t)}{y_h(t)} &= -p(t) \end{aligned}$$

---

[9] The *error function*, erf, is defined by $erf(x) = \frac{2}{\pi} \int_0^x e^{-t^2}\,dt$.

Now we search through the derivative formulas, and find

$$[\ln y(t)]' = \frac{y'(t)}{y(t)}.$$

Using the Fundamental Theorem of Calculus, (again, $y_h > 0$)

$$[\ln y_h(t)]' = -p(t) = -\left[\int_a^t p(s)\,ds\right]'$$

By the Parallel Graph Theorem, Theorem 12.3.2, of Chapter 12, Volume I,

$$\begin{aligned} \ln y_h(t) &= -\int_a^t p(s)\,ds + C_1 \\ y_h(t) &= e^{-\int_a^t p(s)\,ds + C_1} \\ y_h(t) &= C\,e^{-\int_a^t p(s)\,ds} \end{aligned}$$

Therefore, **Step 1.** Define $u(t) = \int_a^t p(s)\,ds$.
Given $u(t)$, for any number C

$$y_h(t) = Ce^{-u(t)} \qquad \text{solves the } \textit{homogeneous} \text{ equation} \qquad y'(t) + p(t)y(t) = 0$$

Think of $y_h(t) = Ce^{-u(t)}$ as a 'toe hold' for finding a solution $y(t)$ to

$$y'(t) + p(t) \times y(t) = q(t) \qquad\qquad \text{Nonhomogeneous equation.}$$

y might be some variation of y_h. Included with several trials, one asks whether C in $y_h = Ce^{-u(t)}$ might be a variable, $v(t)$ instead of a constant. That is, might a solution to the nonhomogeneous equation be of a form

$$y(t) = v(t) \times y_h(t)\ ? \tag{17.22}$$

You should guess that the answer is 'yes' is or we would not be writing about it.

Our goal is to find a function, v, so that

$$y(t) = v(t) \times e^{-u(t)} \qquad \text{solves} \qquad y'(t) + p(t) \times y(t) = q(t)$$

Remember that $[u(t)]' = p(t)$. By substitution,

$$\begin{aligned} \left[v(t) \times e^{-u(t)}\right]' + p(t) \times \left(v(t) \times e^{-u(t)}\right) &= q(t) \\ v'(t) \times e^{-u(t)} + v(t) \times \left[e^{-u(t)}\right]' + p(t) \times \left(v(t) \times e^{-u(t)}\right) &= q(t) \\ v'(t) \times e^{-u(t)} + v(t) \times e^{-u(t)}\,[-u(t)]' + p(t) \times \left(v(t) \times e^{-u(t)}\right) &= q(t) \end{aligned}$$

$$\begin{aligned}
v'(t) \times e^{-u(t)} + v(t) \times e^{-u(t)}\,(-p(t)) + p(t) \times \left(v(t) \times e^{-u(t)}\right) &= q(t) \\[2em]
v'(t) \times e^{-u(t)} &= p(t) \\[2em]
v'(t) &= e^{u(t)} \times p(t)
\end{aligned}$$

Thus we arrive at **Step 2.** Define $v(t) = \int_a^t e^{u(s)}\, p(s)\, ds$.

The constant, C, in $y_h = Ce^{-u(t)}$ is sometimes called a parameter, and replacing C by the variable $v(t)$ is called *variation of parameter(s).* Variation of parameters has wide use, but we will not employ it again. However, the method of separation of variables is used to find an analytic solution to the logistic equation and other autonomous equations in Section 17.8.

Example 17.7.3 Two integral formula examples. Steps 1 and 2 of Theorem 17.7.1 are easy to understand but require some integral formulas to compute the integrals. We will use the following two formulas to solve two problems:

$$\begin{aligned}
\int te^{at}\, dt &= \tfrac{1}{a}te^{at} - \tfrac{1}{a^2}e^{at} + C \\[1em]
\int e^{at} \sin bt\, dt &= \tfrac{a}{a^2+b^2}e^{at} \sin bt - \tfrac{b}{a^2+b^2}e^{at} \cos bt + C
\end{aligned} \tag{17.23}$$

Explore 17.7.1 You should differentiate the RHS of each of the formulas in Equations 17.23 to see that we have not made a mistake. ■

Problem 1. Solve $y(1) = 1, \quad y'(t) + \frac{1}{t}y(t) = e^t$.

$$\begin{aligned}
a &= 1, \quad y_1 = 1, \quad p(t) = \frac{1}{t}, \quad q(t) = e^t \\
u(t) &= \int_0^t p(s)\, ds = \int_0^t \frac{1}{s}\, ds = \ln s|_{s=1}^t = \ln t - 0 = \ln t \\
v(t) &= \int_0^t e^{u(s)}q(s)\, ds = \int_0^t e^{\ln s}e^s\, ds = \int_0^t se^s\, ds = se^s - e^s|_1^t = te^t - e^t \\
y &= v(t)e^{-u(t)} + y_a e^{-u(t)} = (te^t - e^t)e^{-\ln t} + 3e^{-\ln t} \\
&= e^t - \frac{1}{t}e^t + \frac{1}{t}
\end{aligned}$$

Problem 2. Solve $y(0) = 2, \quad y'(t) + 2y(t) = \sin 3t$.

$$\begin{aligned}
a &= 0, \quad y_0 = 2, \quad p(t) = 2, \quad q(t) = \sin 3t \\
u(t) &= \int_0^t p(s)\, ds = \int_0^t 2\, ds = 2s|_{s=0}^t = 2t - 0 = 2t \\
v(t) &= \int_0^t e^{u(s)}q(s)\, ds = \int_0^t e^{2s} \sin 3s\, ds \\
&= \frac{2}{4+9}e^{2s} \sin 3s - \frac{3}{4+9}e^{2s} \cos 3s\Big|_0^t \\
&= \frac{4}{13}e^{2t} \sin 3t - \frac{3}{13}e^{2t} \cos 3t - \left(\frac{2}{13}e^0 \sin 0 - \frac{3}{13}e^0 \cos 0\right)
\end{aligned}$$

$$\begin{aligned}
 &= \frac{4}{13}e^{2t}\sin 3t - \frac{3}{13}e^{2t}\cos 3t + \frac{3}{13} \\
y &= v(t)e^{-u(t)} + y_a e^{-u(t)} \\
 &= \left(\frac{2}{13}e^{2t}\sin 3t - \frac{3}{13}e^{2t}\cos 3t + \frac{3}{13}\right)e^{-2t} + 2e^{-2t} \\
 &= \frac{2}{13}\sin 3t - \frac{3}{13}\cos 3t + \frac{3}{13}e^{-2t} + 2e^{-2t}
\end{aligned}$$

■

Exercises for Section 17.7, First order linear differential equations.

Exercise 17.7.1 Use Steps 1 and 2 in Theorem 17.7.1 and the integral formulas shown there to compute the solutions, $y(t)$, to the following differential equations. After parts a. - d. the explicit dependence of y on t is suppressed in the differential equations. You will need Equations 17.23.

a. $y(0) = 2 \quad y'(t) + 0.2y(t) = 0.1$ b. $y(1) = 1 \quad y'(t) + \frac{1}{t}y(t) = 0.1$

c. $y(0) = 3 \quad y'(t) + 3y(t) = t$ d. $y(0) = 0 \quad y'(t) + y(t) = \sin t$

e. $y(1) = 1 \quad y' + \frac{1}{t}y = 1$ f. $y(0) = 3 \quad y' + y = e^{t}$

g. $y(0) = 7 \quad y' + 0.2y = e^{-0.2t}$ h. $y(0) = 5 \quad y' + 0.3y = e^{-0.2t}$

i. $y(0) = 2 \quad y' + (\sin t)\,y = \sin t$ j. $y(0) = 5 \quad y' + 3y = \sin 4t$

Exercise 17.7.2 Find the unique solutions to

a. $y(0) = 5 \quad y' + 2y = 0$ b. $y(0) = 0 \quad y' + 2y = 0$

c. $y(0) = 4 \quad y' + 3y = t$ d. $y(1) = 1 \quad y' + 3y = t$

e. $y(0) = 0 \quad y' + 0.2y = e^{-0.2t}$ f. $y(0) = 3 \quad y' + ty = t$

Exercise 17.7.3 The hypothesis of Theorem 17.6.1 may be modified to assume that $f(t,y)$ and $f_2(t,y)$ are continuous on a rectangle, $a \le t \le a + d$ and $y_a - d \le y \le y_a + d$ ((a, y_a) is on the left edge of the rectangle instead of in the middle) and then conclude that there is a solution to $y(a) = y_a$, $y' = f(t,y)$ on an interval $[a, a + \epsilon]$.

Suppose $p(t)$ and $q(t)$ are continuous on an interval $[a, a + d]$ Write the Equation 17.19

$$y(a) = y_a, \qquad y'(t) + p(t) \times y(t) = q(t)$$

as

$$y(a) = y_a, \qquad y'(t) = q(t) - p(t) \times y(t) = f(t,y).$$

Show that $f(t,y)$ and $f_2(t,y)$ are continuous on $a \le t \le a + d$, $y_a - d \le y_a + d$ so that existence and uniqueness of a solution to Equation 17.19 on some interval $[a, a + \epsilon]$ is assured.

Exercise 17.7.4 Suppose $p(t)$ and $q(t)$ of Equation 17.19 are defined and continuous on an interval $[a,b]$. Show that there is a solution $y(t)$ in Equation 17.20 defined on all of $[a,b]$.

Exercise 17.7.5 Show that

$$y(t) = v(t)e^{-u(t)} + y_a e^{-u(t)}, \quad \text{where} \quad u(t) = \int_a^t p(t)\,dt, \quad v(t) = \int_a^t e^{u(s)} \times q(s)\,ds$$

defined in Equation 17.20 satisfies the initial condition and the differential equation of 17.19

$$y(a) = y_a, \qquad y'(t) + p(t) \times y(t) = q(t).$$

a. The initial condition, $y(a) = y_a$ is a piece of cake. Compute $u(a)$ and $v(a)$ and $y(a)$.

b. Showing that $y(t)$ solves Equation 17.19 is messier. Complete:

$$\begin{aligned} y'(t) &= \left[v(t)e^{-u(t)} + y_a e^{-u(t)}\right]' \\ &= -v(t)e^{-u(t)}p(t) + q(t)e^{u(t)} \times e^{-u(t)} - y_a e^{-u(t)}p(t) \\ &= -y(t) \times p(t) + q(t) \end{aligned}$$

Exercise 17.7.6 For a SCUBA diver whose depth t minutes after the start of the dive is $d(t)$ meters, the water pressure will be $1 + d(t)/10$ atmospheres. Air is 79 percent nitrogen. Haldane's equation for nitrogen partial pressure, $N(t)$, in a given tissue is (from Equation 17.14 with $K = k/V$)

$$N(0) = 0.79, \qquad N'(t) = K \times \left(0.79 \times (1 + \frac{d(t)}{10}) - N(t)\right). \tag{17.24}$$

This is a first order linear differential equation. See Exercise D.2.24 and Haldane's Mathematical Model D.2.9.

Your goal is to solve Equation 17.24

a. Rewrite Equation 17.24 to put it into the form, $N'(t) + p(t)\,N(t) = q(t)$.

b. Show that $u(t) = \int_0^t p(t)\,dt = Kt$.

c. Show that $v(t) = \int_0^t e^{u(s)}q(s)\,ds = 0.79e^{Kt} - 0.79 + \frac{K}{10}\int_0^t e^{Ks}d(s)\,ds$.

d. Show that

$$N(t) = 0.79 + 0.79\frac{K}{10}e^{-Kt} \quad \int_0^t e^{K\,s}\,d(s)\,ds. \tag{17.25}$$

Exercise 17.7.7 There is 'conventional wisdom' among SCUBA divers that if you are going to make a dive that involves two depths, 'do the deep part first'. This problem and the next explores rationale for that wisdom. To be concrete, assume that $K = 0.07$ 1/min which corresponds to approximately 10 minute half-life for the compartment ($(\ln 2)/0.07 = 9.9$ min).

a. Assume a diver (d1) descends immediately to 10 meters and stays there for 15 minutes, then descends to 30 meters and stays there for 10 minutes. Let

$$d_1(t) = \begin{cases} 10 & \text{for} \quad 0 \le t \le 15 \\ 30 & \text{for} \quad 15 < t \le 25 \end{cases}$$

Compute

$$N_1(25) = 0.79 + 0.007e^{-0.07\times 25}\int_0^{25} e^{0.07s} \times d_1(s)\,ds + 0.79e^{-0.07\times 25}.$$

You should recognize that

$$\int_0^{25} e^{0.07s} \times d_1(s)\,ds = \int_0^{15} e^{0.07s} \times 10\,ds + \int_{15}^{25} e^{0.07s} \times 30\,ds$$

Answer: $N_1(25) = 2.24$ Atmospheres.

b. Assume a diver (d2) descends immediately to 30 meters and stays there for 10 minutes, then ascends to 10 meters and stays there for 15 minutes. Let

$$d_2(t) = \begin{cases} 30 & \text{for} \quad 0 \le t \le 10 \\ 10 & \text{for} \quad 10 < t \le 25 \end{cases}$$

Compute

$$N_2(25) = 0.79 + 0.007e^{-0.07\times 25}\int_0^{25} e^{0.07x} \times d_2(x)\,dx + 0.79e^{-0.07\times 25}$$

Answer: $N_2(25) = 1.72$ Atmospheres.

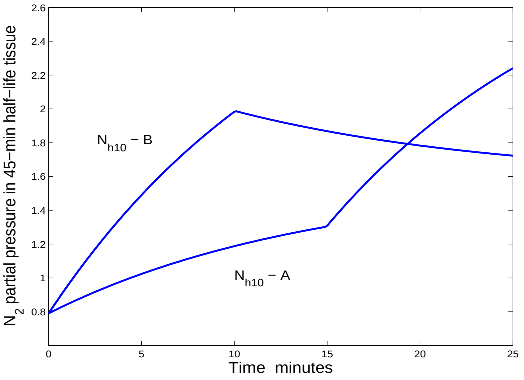
Thus by doing the deep part of the dive first, the end state of N_2 partial pressure in this compartment is at least 20% less than if she had made the shallow part of the dive first.

Exercise 17.7.8 It is instructive to see the graphs of $N_{d1}(t)$ and $N_{d2}(t)$ of the previous problem. You can plot these graphs on your calculator. We plotted N_2 on the TI-86 by

$$\begin{aligned} y1 &= \exp(0.07x) \\ y2 &= (x < 10) * \text{fnInt}(y1 * 30, x, 0, x) \\ &+ (10 \le x)(434.5 + \text{fnInt}(y1 * 10, x, 10, x) \\ y3 &= 0.79 + 0.79 * 0.007\exp(-0.07x) * y2 \end{aligned}$$

Also see Exercise Figure 17.7.8 Use the graphs to explain how nitrogen absorption differs with the two dive profiles.

Figure for Exercise 17.7.8 Nitrogen profiles in a 10-minute half-life compartment of a diver with two dive plans. One (A) is to dive to 10 meters for 15 minutes, then to 30 meters for 10 minutes. The other (B) profile is for a dive to 30 meters for 10 minutes followed by 10 meters for 15 minutes.



Exercise 17.7.9 A diver is to survey the life on the bottom of a near-shore marine environment. She is to follow a transect that declines 30° and will travel 40 meters at 1 meter per minute along the 30° slope. You will need the formula:

$$\int te^{at}\,dt = \frac{1}{a}te^{at} = \frac{1}{a^2}e^{at} + C.$$

a. She starts at the shore and travels 40 meters along the 30° slope. At one meter per minute she will be at depth

$$d(t) = \frac{1}{2}t \qquad t \text{ minutes into the dive.}$$

and the water pressure in atmospheres will be $1 + d(t)/10$. What will be the partial pressure of N_2 in tissue with $K = 0.07$ at the end of her dive?

b. She swims out to a point above the end point of the previous dive, descends and travels back toward shore. At one meter per minute she will be at depth

$$d(t) = 20 - \frac{1}{2}t \qquad t \text{ minutes into the dive.}$$

What will be the partial pressure of N_2 in tissue with $K = 0.07$ at the end of her dive?

Exercise 17.7.10 J. S. Haldane did the initial work on decompression illness in divers and used the model of the previous exercises. He assumed the body would have tissues with half-lives of 5, 10, 20, 40 and 75 minutes and set a criteria that the partial pressure of nitrogen in a diver's tissue should not be more than 2 times the partial pressure of nitrogen in her blood. (Actually, he began this work in 1905 and it was *his* blood, not *her* blood.) Compute curves analogous to those in Exercise Figure 17.7.8 for a compartment of 5 minute half-life ($K = (\ln 2)/5 = 0.139$) and a compartment of 40 ($K = (\ln 2)/40 = 0.0173$) minute half-life.

17.8 Separation of variables.

Separation of Variables. If in the differential equation, $y' = f(t, y)$, the RHS factors into

$$f(t, y) = g(t) \times h(y)$$

then $y' = f(t, y)$ can be written

$$y'(t) = g(t) \times h(y) \qquad \frac{1}{h(y)}y'(t) = g(t) \tag{17.26}$$

and *the variables t and y are separable.*

The solution to 17.26 involves two *antiderivative* problems; the Leibnitz notation is very helpful.

SV Step 1, LHS Find $H(y)$ such that

$$\frac{d\,H(y)}{dy} = \frac{1}{h(y)} \qquad \text{so that by the Chain Rule, Theorem 6.2.1,}$$

$$\frac{d\,H(y(t))}{dt} = \frac{d}{dy}\,H(y) \times \frac{d\,y(t)}{dt} = \frac{1}{h(y)}\,y'(t)$$

SV Step 2, RHS Find $G(t)$ such that $\frac{d\,G(t)}{dt} = g(t)$.

From Equation 17.26

$$\begin{aligned} \frac{1}{h(y)}y'(t) &= g(t) \\ \frac{d\,H(y(t))}{dt} &= \frac{d\,G(t)}{dt} \end{aligned}$$

Then by the Parallel Graph Theorem, Theorem 12.3.2, there is a constant, C, so that

$$H(y(t)) = G(t) + C \tag{17.27}$$

is an implicit expression of $y(t)$.

If a data point, $y(a) = y_a$ is given, then C can be computed from

$$H(y_a) = G(a) + C$$

To see that a function, $y(t)$, implicitly defined by Equation 17.27, solves Equation 17.26 we differentiate 17.27 with respect to t using the Chain Rule.

$$\begin{aligned} \frac{d}{dt}\,[H(y(t))] &= \frac{d}{dt}\,[G(t) + C] \\ \frac{d}{dy}\,H(y) \times \frac{d\,y(t)}{dt} &= \frac{d}{dt}\,G(t) \\ \frac{1}{h(y)}y'(t) &= g(t) \\ y'(t) &= g(t) \times h(y) \end{aligned}$$

Example 17.8.1 Consider

$$y'(t) = 2t \times y$$

Then

$$\frac{1}{y}y' = 2t$$

SV Step 1, LHS Find $H(y)$ such that

$$\frac{d\,H(y)}{dy} = \frac{1}{y}. \qquad \text{Choose} \quad H(y) = \ln y.$$

SV Step 2, RHS Find $G(t)$ such that $\frac{d}{dt}G(t) = 2t$. Choose $G(t) = t^2$.

Then

$$\frac{d}{dt}\ln y(t) = \frac{1}{y(t)}\frac{d\,y(t)}{dt} = \frac{1}{y}\,y' = 2\,t = \frac{d}{dt}\,G(t) = \frac{d}{dt}\,t^2.$$

By the Parallel Graph Theorem there is a number, C_1, such that

$$\ln y(t) = t^2 + C_1$$

This implicit expression for y can be solved explicitly.

$$y(t) = e^{t^2+C_1} = e^{t^2} \times e^{C_1} = Ce^{t^2} \tag{17.28}$$

If we are also given that, for example, $y(0) = 3$ then we write

$$y(0) = 3 = Ce^{0^2} = C \qquad\qquad y(t) = 3e^{t^2}$$

Explore 17.8.1 Alternatively, we might be given that $y(0) = -3$. Return to Equation 17.28 and solve for the solution. ■

■

Example 17.8.2 Of the following six equations, the variables can be separated in only two.

$$\begin{aligned} y'(t) &= t + y & \qquad y' &= e^{t+y} \\ y'(t) &= \ln(t+y) & y' &= e^{t\times y} \\ y'(t) &= \ln(t\times y) & y' &= \ln\left(t^y\right) \end{aligned}$$

The two equations in which variables are separable are shown below.

$$\begin{aligned} y' &= e^{t+y} = e^t \times e^y & \qquad y' &= \ln t^y = y \times \ln t \\ e^{-y} \times y' &= e^t & \tfrac{1}{y}y' &= \ln t \\ [-e^{-y}]' &= [e^t]' & [\ln y]' &= [t\ln t - t]' \\ -e^{-y} &= e^t + C & \ln y &= t\ln t - t + C_1 \\ y &= -\ln\left(-e^t - C\right) & y &= C \times t^t \times e^{-t} \end{aligned}$$

■

Example 17.8.3 *The variables can be separated in every autonomous differential equation*

$$y' = f(y) \qquad \frac{1}{f(y)}y' = 1$$

To find an implicit solution to any autonomous differential equation only the problem

SV Step 1 Find $F(y)$ such that $\quad F'(y) = \frac{1}{f(y)} \quad$ requires attention.

SV Step 2 is easy: Find $G(t)$ such that $G'(t) = 1$ Answer: $G(t) = t$

Example 17.8.4 To solve the autonomous equation, $y' = -y^2$ we write

$$\begin{aligned} y' &= -y^2 \\ \frac{1}{y^2}y' &= -1 \qquad \frac{d\,H(y)}{dy} = \frac{1}{y^2}, \quad \text{choose} \quad H(y) = -\frac{1}{y} \\ \frac{d}{dt}\left[-\frac{1}{y(t)}\right] &= \frac{d}{dt}\,[-t] \\ -\frac{1}{y(t)} &= -t + C \\ y(t) &= \frac{1}{t-C} \end{aligned}$$

If also an initial condition is given, for example $y(0) = 0.5$, we write

$$y(0) = \frac{1}{0-C}, \qquad 0.5 = \frac{1}{-C}, \qquad C = -2, \qquad y(t) = \frac{1}{t+2}.$$

■

17.8.1 The logistic equation.

By far the most important differential equation that we solve by separation of variables is the logistic equation

$$p' = r \times p \times \left(1 - \frac{p}{M}\right)$$

We treat the Case: $0 < p(0) = p_0 < M$. From the direction field for the logistic equation in Figure 17.8 we have seen that $0 < p(0) < M$ implies that for all time, t, $0 < p(t) < M$.

First we separate variables

$$\frac{1}{p \times (1 - p/M)} \times p' = r$$

Then we use an algebraic identity (see the following section, Partial Fractions)

$$\frac{1}{p \times (1 - p/M)} = \frac{1}{p} + \frac{1}{M-p} \qquad (17.29)$$

to write

$$\left(\frac{1}{p} + \frac{1}{M-p}\right) \times p' = r$$

Two derivative formulas

$$[\ln p]' = \frac{1}{p} \qquad \text{and} \qquad [-\ln(M-p)]' = \frac{1}{M-p}$$

provide a solution to

SV Step 1 Find $H(p)$ such that

$$H'(p) = \frac{1}{p} + \frac{1}{M-p} \qquad \text{Answer:} \qquad H(p) = \ln p - \ln(M-p)$$

Note: The assumption that $0 < p < M$ assures that both $\ln p$ and $\ln(M-p)$ are defined.

SV Step 2. Find $G(t)$ such that $G'(t) = r$. (Whew! Easy) $G(t) = rt$.

Then we write

$$\begin{aligned} \ln p - \ln(M-p) &= rt + C_1 \qquad (17.30) \\ \ln \frac{p}{M-p} &= rt + C_1 \\ \frac{p}{M-p} &= e^{rt+C_1} = e^{rt} \times e^{C_1} \\ \frac{p}{M-p} &= Ce^{rt} \end{aligned}$$

The initial condition, $p(0) = p_0$, yields

$$\frac{p_0}{M-p_0} = Ce^{r\times 0} = C$$

Therefore

$$\frac{p}{M-p} = \frac{p_0}{M-p_0}e^{rt}$$

and solving for p yields

$$p(t) = \frac{Mp_0}{p_0 + (M-p_0)e^{-rt}} \qquad (17.31)$$

Explore 17.8.2 Check the algebraic step

$$\frac{1}{p \times (1-p/M)} = \frac{1}{p} + \frac{1}{M-p}$$

by adding the two fractions on the right and showing that the sum can be written as the fraction on the left. ■

17.8.2 The Method of Partial Fractions.

$$\text{The ratio} \qquad \frac{P(x)}{Q(x)} \qquad \text{where} \qquad P(x) \quad \text{and} \quad Q(x)$$

are polynomials and the **degree of $P(x)$ is less than the degree of $Q(x)$** can be partitioned into the sum of fractions with simple denominators.

The pattern is illustrated by the four examples:

$$\frac{px+q}{(x-a)\times(x-b)} = \frac{A}{x-a}+\frac{B}{x-b} \qquad \text{Example a.}$$

$$\frac{px^2+qx+r}{(x-a)\times(x-b)\times(x-c)} = \frac{A}{x-a}+\frac{B}{x-b}+\frac{C}{x-c} \qquad \text{Example b.}$$

$$\frac{px^2+qx+r}{(x-a)\times(x-b)^2} = \frac{A}{x-a}+\frac{B}{x-b}+\frac{C}{(x-b)^2} \qquad \text{Example c.}$$

$$\frac{px^2+qx+r}{(x-a)\times(x^2+bx+c)} = \frac{A}{x-a}+\frac{Bx+C}{x^2+bx+c} \qquad \text{Example d.}$$

The method is used to replace a complex integral such as

$$\int \frac{px^2+qx+r}{(x-a)\times(x-b)\times(x-c)}\,dx$$

by (see Example b.) the sum of three relatively easy integrals

$$\int \frac{A}{x-a}\,dx + \int \frac{B}{x-b}\,dx + \int \frac{C}{x-c}\,dx$$

Example 17.8.5

$$\frac{3x+4}{(x+3)\times(x-2)} = \frac{A}{x+3}+\frac{B}{x-2}$$

To find A and B, multiply by $(x+3)\times(x-2)$ and get

$$3x+4 = A\times(x-2)+B\times(x+3)$$

Then substitute

$$\begin{array}{llcll} x=-3 & 3(-3)+4 & = & A\times(-3-2)+B\times(-3+3) & \\ & -5 & = & A\times(-5) & A=1 \\ & & & & \\ x=2 & 3(2)+4 & = & A\times(2-2)+B\times(2+3) & \\ & 10 & = & B\times(5) & B=2 \end{array}$$

$$\text{Thus} \qquad \frac{3x+4}{(x+3)\times(x-2)} = \frac{1}{x+3}+\frac{2}{x-2}$$

The method is used to compute an integral:

$$\int \frac{3x+4}{(x+3)\times(x-2)}\,dx = \int \frac{1}{x+3}\,dx + \int \frac{2}{x-2}\,dx = \ln|x+3| + 2\ln|x-2| + C$$

In substituting values of x (we used $x = -3$ and $x = 2$ above) any two values of x are acceptable. The chosen values, -3 and 2, each annihilate one of the terms and leave only one term with one of the unknown parameters A and B. It is curious that the most convenient values of x, -3 and 2, are exactly the values of x for which the original fraction

$$\frac{3x+4}{(x+3)\times(x-2)}$$

is meaningless.

Example 17.8.6

$$\frac{3x^2-5x+1}{(x+1)\times(x-2)^2} = \frac{A}{x+1} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$$

To find A, B and C, multiply by $(x+1)\times(x-2)^2$ and get

$$3x^2-5x+1 = A\times(x-2)^2 + B\times(x-2)\times(x+1) + C\times(x+1)$$

Then substitute

$$\begin{array}{lrcl}
x=-1 & 3(-1)^2-4(-1)+1 & = & A\times(-1-2)^2 + B\times(-1-2)(-1+1) + C\times(-1+1) \\
 & 9 & = & A\times(9) \\
 & A & = & 1 \\
 & & & \\
x=2 & 3(2)^2-5\times2+1 & = & A\times(2-2)^2 + B\times(2-2)(2+1) + C\times(2+1) \\
 & 3 & = & C\times(3) \\
 & C & = & 1 \\
 & & & \\
x=0 & 3(0)^2-5\times0+1 & = & A\times(0-2)^2 + B\times(0-2)(0+1) + C\times(0+1) \\
 & 1 & = & 4A-2B+C \\
 & 1 & = & 4\times1-2B+1 \\
 & B & = & 2
\end{array}$$

$$\text{Thus} \qquad \frac{3x^2-5x+1}{(x+1)\times(x-2)^2} = \frac{1}{x+1} + \frac{2}{x-2} + \frac{1}{(x-2)^2}$$

The method is used to compute an integral:

$$\begin{aligned}
\int \frac{3x^2-5x+1}{(x+1)\times(x-2)^2}\,dx &= \int \frac{1}{x+1}\,dx + \int \frac{2}{x-2}\,dx + \int \frac{2}{(x-2)^2}\,dx \\
&= \ln|x+1| + 2\ln|x-2| - 2(x-2)^{-1} + C
\end{aligned}$$

In this case, there were only two values of x that annihilated other terms, -1 and 2, and a third value of x was required because there were three unknown parameters, A, B, and C. We chose $x = 0$ as a convenient third value to use.

Example 17.8.7 In Problem 17.4.11 on page 326 three differential equations were suggested as possible population models illustrating the Allee effect, in which there is a threshold population number, ϵ, with the property that if the density were less than ϵ the population would be so thinly distributed that reproductive success would not counter natural loss and the population will become extinct. Two of these equations can be solved by the method of separation of variables using partial fraction expansion. (We assume that time is scaled so that $r = 1$.)

$$\begin{aligned} u' &= u \times \frac{u-\epsilon}{u+\epsilon} \times (1-u) & \text{Candidate 2} \\ u' &= u \times (u-\epsilon) \times (1-u) & \text{Candidate 3} \end{aligned}$$

For Candidate 3 the method of partial fractions converts

$$u' = u \times (u-\epsilon) \times (1-u) \qquad \frac{u'}{u \times (u-\epsilon) \times (1-u)} = 1$$

into

$$\left(\frac{-1/\epsilon}{u} + \frac{1/(\epsilon\,(1-\epsilon))}{u-\epsilon} + \frac{1/(1-\epsilon)}{1-u} \right) u' = 1$$

which can be integrated to obtain

$$-\frac{1}{\epsilon}\ln|u(t)| + \frac{1}{\epsilon\,(1-\epsilon)}\ln|u(t)-\epsilon| - \frac{1}{1-\epsilon}\ln|1-u(t)| = t + C \tag{17.32}$$

It seems unlikely that anyone will solve Equation 17.32 for $u(t)$. We can, however, look at t as a function of u, the inverse of the solution of interest. And we can reflect that graph about the line $u = t$ to get the graph of interest.

An example of Equation 17.32 is shown in Figure 17.16. For the purpose of the graph, we have let $\epsilon = 0.1$. This is high; the threshold number is likely much less than 10% of the capacity of the environment. The value of C is determined by u_0 and we used both $u_0 = 0.12$ ($C = 1.024$) and $u_0 = 0.08$ ($C = 0.05664$), above and below the threshold. Some values of t are negative in Figure 17.16 and illustrates the curve if time runs backward from 0. The graphs in Figure 17.16A are of the equation

$$t + C = -10\ln u + \frac{10}{0.9}\ln|u-0.1| - \frac{1}{0.9}\ln(1-u)$$

where

$$\begin{aligned} C &= -10\ln 0.12 + \frac{10}{0.9}\ln|0.12-0.1| - \frac{1}{0.9}\ln(1-0.12) = -22.12 \\ C &= -10\ln 0.08 + \frac{10}{0.9}\ln|0.08-0.1| - \frac{1}{0.9}\ln(1-0.08) = -18.12 \end{aligned}$$

The analysis of Candidate 2 is left for you in Exercise 17.8.12. ■

Exercises for Section 17.8, Separation of variables.

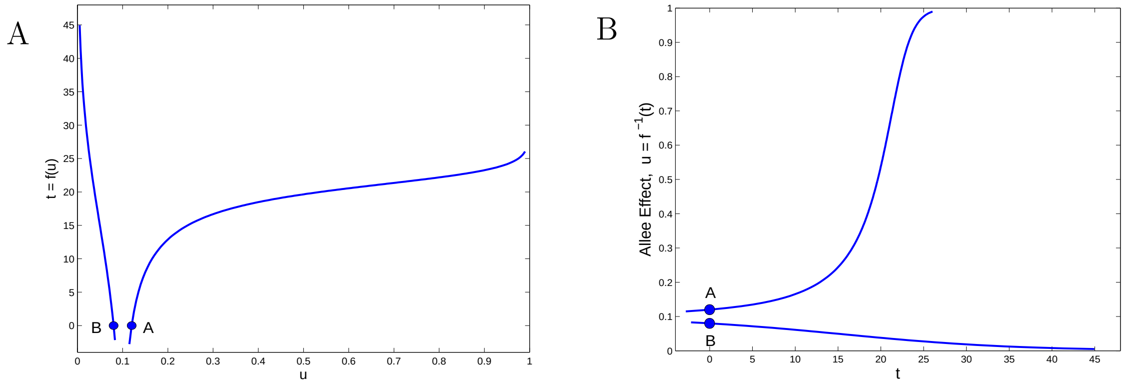


Figure 17.16: A. The graph of t versus u for Equation 17.32 using $\epsilon = 0.1$, and $u_0 = 0.12$ for the branch marked A and $u_0 = 0.8$ for the branch marked B. B. The reflection of the graph in A. with negative t values trimmed.

Exercise 17.8.1 Find r and M so that the solution

$$p(t) = \frac{Mp_0}{p_0 + (M - p_0)e^{-rt}} \qquad \text{of the logistic equation} \qquad p' = r \times p \times \left(1 - \frac{p}{M}\right)$$

approximates the data

pH 6.25		
Time (min)	Index t	Population Density
0	0	0.022
16	1	0.036
32	2	0.060
48	3	0.101
64	4	0.169
80	5	0.266
96	6	0.360
112	7	0.510
128	8	0.704
144	9	0.827
160	10	0.928

A solution of the discrete logistic equation was fit to the data in Section 14.6.1 and the low density growth rate and carrying capacity should be similar in both solutions.

Exercise 17.8.2 Show that the variables are not separable in the equation $y'(t) = t + y$. That is, there are not two functions, $g(t)$ and $h(y)$, which

$$\text{for all } t \text{ and } y \qquad t + y = g(t) \times h(y)$$

A procedure is to assume two such functions, $g(t)$ and $h(y)$ exist and then show that the following equations are incompatible.

$$\begin{array}{ccc} t = 0 \quad y = 0 & t = 0 \quad y = 1 & t = 1 \quad y = 0 \\ g(0) \times h(0) = 0 + 0 = 0 & g(0) \times h(1) = 0 + 1 = 1 & g(1) \times h(0) = 1 + 0 = 1 \end{array}$$

Show that $g(0) \times h(0) = 0$, $g(0) \times h(1) = 1$ and $g(1) \times h(0) = 1$ are incompatible.

Exercise 17.8.3 Show that the variables are not separable in the equation

$$\text{a.} \quad y'(t) = \ln(t \times y) \qquad \text{b.} \quad y'(t) = \ln(t + y)$$

Exercise 17.8.4 Find an implicit or explicit expression for $y(t)$ for each equation. Then use the given data point to evaluate the constant C of integration.

The following derivative formulas will be helpful.

$$[\ln(y-1)]' = \frac{1}{y-1} \qquad \left[\ln(t^2+1)\right]' = \frac{2t}{t^2+1} \qquad [\ln(1-y)]' = -\frac{1}{y-1}$$

$$\begin{array}{llllllll}
\text{a.} & y' = \frac{t}{y} & & y(0) = 2 & \text{b.} & y' = \frac{y}{t} & & y(1) = 1 \\
\text{c.} & y' = \frac{t-1}{y} & & y(2) = 1 & \text{d.} & y' = \frac{t}{y(t^2+1)} & & y(0) = 1 \\
\text{e.} & y' = y - 1 & & y(0) = 1.5 & \text{f.} & y' = 1 - y & & y(0) = 0.5 \\
\text{g.} & y' = y^2 - 1 & & y(0) = 2 & \text{h.} & y' = e^{t-y} & & y(0) = 1.5
\end{array}$$

Exercise 17.8.5 Prove the validity of Equation 17.29:

$$\frac{1}{p \times (1 - p/M)} = \frac{1}{p} + \frac{1}{M - p}$$

Exercise 17.8.6 Solve for p in

$$\frac{p}{M-p} = \frac{p_0}{M-p_0} e^{rt}$$

to obtain Equation 17.31,

$$p(t) = \frac{M p_0}{p_0 + (M - p_0)e^{-rt}} \qquad ''$$

It will be useful to first solve for p in

$$\frac{p}{M-P} = K \qquad \left(K \quad \text{replaces} \quad \frac{p_0}{M-p_0} e^{rt} \right)$$

You should get

$$p = M \times \frac{K}{1+K}$$

Then substitute

$$K = \frac{p_0}{M-p_0} e^{rt}$$

and simplify. As a final step, divide numerator and denominator by e^{rt}.

Exercise 17.8.7 Show that for

$$p(t) = \frac{M p_0}{p_0 + (M - p_0)e^{-rt}}$$

$$\text{a.} \ p(0) = p_0 \qquad \text{b.} \ \lim_{t \to \infty} p(t) = M$$

Exercise 17.8.8 Let $M = 10$ and $r = 0.1$ and plot the graphs of

$$p(t) = \frac{Mp_0}{p_0 + (M - p_0)e^{-rt}}$$

for $0 \le t \le 80$ and

$$\text{a. } p_0 = 1 \qquad \text{b. } p_0 = 12 \qquad \text{c. } p_0 = 10$$

Exercise 17.8.9 Suppose population is described by Equation 17.31, $P(t) = p_0 M/(p_0 + (M - p_0)\, e^{-rt})$ and $0 < p_0 < M$.

1. Compute $P'(t)$.
2. Compute $P''(t)$.
3. Let t_{steep} be the value of t for which $P''(t) = 0$. Compute $e^{-rt_{steep}}$.
4. Compute $P(t_{steep})$.
5. Interpret t_{steep} as the time at which the population size is increasing the fastest.

Exercise 17.8.10 Suppose a population growth is is described by Equation 14.32

$$p' = r \times p \times \left(1 - \frac{p}{M}\right).$$

For what value of p is the population growing the fastest? That is, for what value of p is $r \times p \times \left(1 - \frac{p}{M}\right)$ the largest?

Exercise 17.8.11 Parameter reduced population models are shown below. In the Ricker equation, find a condition on γ_0 that will insure that there is a value of v for which the population is growing the fastest. In the Beverton-Holt equation show that there is no value of v for which the population is growing the fastest. In the Gompertz equation, find the value of v for which the population is growing the fastest.

$$\begin{array}{lrcll} \text{a.} & v' & = & ve^{-v} - \gamma_0 v & \text{Ricker} \\ \text{b.} & v' & = & \dfrac{v}{1+v} & \text{Beverton-Holt} \\ \text{c.} & v' & = & -v\ln(v) & \text{Gompertz} \end{array}$$

The three previous problems have important implications for wildlife management, at least conceptually. Suppose you are managing a wildlife population, salmon, for example, as a renewable resource, and wish to annually harvest as many salmon as possible. If you harvest too severely, the next years spawn will be low, and four years later the harvest will be limited. Your optimum strategy is to maintain the population at the level where the growth is the greatest.

Exercise 17.8.12 The second candidate suggested in Exercise 17.4.11 to model the Allee effect of an extinction threshold for a population is

$$u' = u \times \frac{u - \epsilon}{u + \epsilon} \times (1 - u)$$

a. Show that
$$\left(\frac{-1}{u}+\frac{2/(1-\epsilon)}{u-\epsilon}+\frac{(1+\epsilon)/(1-\epsilon)}{1-u}\right)u'=1$$

HINT: Solve for A, B, and C in

$$\begin{aligned}\frac{u+\epsilon}{u(u-\epsilon)(1-u)} &= \frac{A}{u}+\frac{B}{u-\epsilon}+\frac{C}{1-u}\\ \\ u+\epsilon &= A(u-\epsilon)(1-u)+Bu(1-u)+Cu(u-\epsilon)\end{aligned}$$

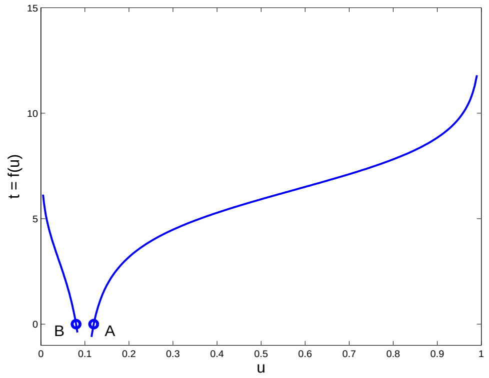
by letting $u=0$, $u=\epsilon$, and $u=1$.

b. Show that
$$-\ln|u|+\frac{2}{1-\epsilon}\ln|u-\epsilon|-\frac{1+\epsilon}{1-\epsilon}\ln|1-u|=t+C \tag{17.33}$$

c. Let $\epsilon=0.1$, and compute C for $u_0=0.08$, and draw the graph of t versus u. Then draw the graph of u versus t. You may wish to use the following MATLAB program, which draws the graph that follows the program.

```
close all;clc;clear
eps=0.1;   u=[0.115:0.005:0.99];  u0=0.12;
C = -log(u0) + (2/(1-eps))*log(abs(u0-eps))-...
                                ((1+eps)/(1-eps))*log(1-u0)
t = -log(u) + (2/(1-eps))*log(abs(u-eps))-...
                                ((1+eps)/(1-eps))*log(1-u) - C;
plot(u,t,'linewidth',2);  axis([0 1 -1 15]);  hold
u=[0.005:0.002:0.0846];  u0 = 0.08;
C = -log(u0) + (2/(1-eps))*log(abs(u0-eps))-...
                                ((1+eps)/(1-eps))*log(1-u0)
t = -log(u) + (2/(1-eps))*log(abs(u-eps))-...
                                ((1+eps)/(1-eps))*log(1-u) - C;
plot(u,t,'linewidth',2)

plot(0.08,0.0,'o','MarkerSize',8,'linewidth',3)
text(0.03,-0.3,'B','fontsize',16)
plot(0.12,0.0,'o','MarkerSize',8,'linewidth',3)
text(0.145,-0.3,'A','fontsize',16)
xlabel('u','fontsize',16)
ylabel('t = f(u)','fontsize',16)
```



Let $\epsilon = 0.1$, and compute C for $u_0 = 0.12$, and draw the graph of t versus u. Then draw the graph of u versus t.

Exercise 17.8.13 Show that if $\epsilon = 0$, Equation 17.33 becomes the implicit solution to the logistic equation, Equation 17.30.

17.9 Examples and exercises for first order ode's.

Generally, first order differential equations apply to problems of the type

> The rate of change of [a quantity, $y(t)$] is proportional to [a quantity described in terms of $y(t)$ and possibly also t].

For example,

Mathematical Model 17.9.1 The rate of change of oxygen in the air sac of an egg is proportional to the difference between the external oxygen concentration and the oxygen concentration inside the sac.

Let $\Omega(t)$ be the amount of oxygen (measured as mg of O_2) in the air sac at time t, and let V be the volume of the sac, and $[O_2]$ be the ambient partial pressure of oxygen. Then the statement can be written as an equation

$$\frac{d}{dt}\Omega(t) = K \times \left([O_2] - \frac{\Omega(t)}{V} RT\right)$$

where R is the gas constant and T is temperature. If the metabolic activity of the chicken embryo and the nonembryonic membranes inside the egg is also considered, one might have

> The rate at which the embryo absorbs oxygen from the air sac of an egg is a function, $G(t)$ of the incubation time.

Then the equation may be modified to

$$\frac{d}{dt}\Omega(t) = K \times \left([O_2] - \frac{\Omega(t)}{V} RT\right) - G(t) \qquad (17.34)$$

Exercise 17.9.1 Shown below are data and a graph of O_2 absorption as a function of day of incubation[10].

---

[10] Alexis L. Romanoff, Biochemistry of the Avian Embryo, John Wiley, 1967

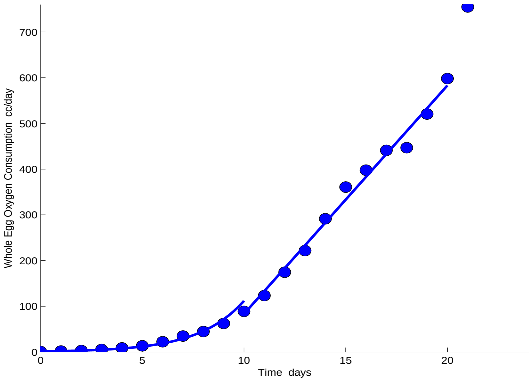
a. What is $G(t)$ (approximately) on $[0, 10]$? Assume the density of oxygen is 1.43 mg/cc.

b. By what method might the resulting Equation 17.34 be solved.

c. Write a modification of Equation 17.34 appropriate for the time interval, [10,20] days.

d. You might assume $\Omega(0) = 1$. What would be your procedure to solve Equation 17.34 over the interval [0,20]?

Whole Egg O_2 Consumption			
Age days	cc/day	Age days	cc/day cc/day
0	0.96	11	123.36
1	1.92	12	174.48
2	3.12	13	221.76
3	5.52	14	291.37
4	8.88	15	360.72
5	13.44	16	397.68
6	22.32	17	441.12
7	34.80	18	446.80
8	44.64	19	520.50
9	62.40	20	598.08
10	88.80	21	754.80



The curve from $t = 0$ to $t = 10$ is $y = 1.28e^{0.45t}$. The line from $t = 10$ to $t = 20$ is $y = 50t - 416$.

Exercise 17.9.2 Write an equation that describes the temperature of an egg after it is uncovered (the adult bird leaves the nest to feed). Assume that the rate of change of the temperature of the egg is proportional to difference between the air temperature and the egg temperature.

17.9.1 Light decay with depth in water.

It is an observed fact and may have been your experience that light intensity decreases as one descends from the surface of a lake or ocean. This problem addresses the question as to what is the light intensity, $I(x)$, at a distance, x meters, below the surface of a lake or ocean, assuming that the light intensity penetrating the surface is a known quantity, I_0.

We will use an hypothesis about light transmission in water (see Figure 17.17).

Hypothesis L: The amount of light absorbed by a (horizontal) layer of water is proportional to the thickness of the layer and to the amount of light entering the layer. Hypothesis L implies

Implication 1. That the light absorbed by a 10 cm layer of water is twice the light absorbed by a 5cm layer of water and

Implication 2. A layer that absorbs 10% of a dim light will absorb 10% of a bright light.

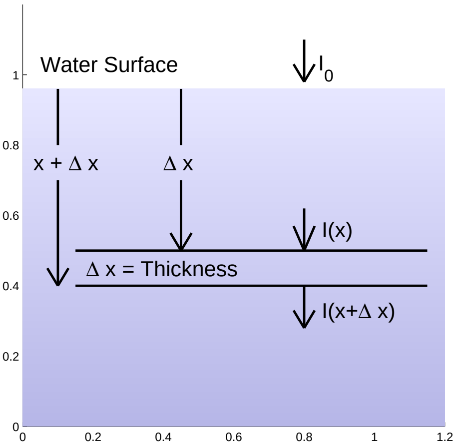


Figure 17.17: Illustration of light absorbance by a layer of water.

Implication 1. is approximately true for thin layers and for low levels of turbidity. Implication 2. is fairly robust.

The double proportionality of Hypothesis L is handled by assuming that the amount of light absorbed is proportional to the *product* of the thickness of the layer and the intensity of the light incident to the layer. That is, there is a number, K, such that if $I(x)$ is the light intensity at depth x and $I(x+\Delta)$ is the light intensity at depth $x+\Delta$, then

$$I(x+\Delta) - I(x) \doteq -K \times \Delta \times I(x) \tag{17.35}$$

The product, $K \times \Delta \times I(x)$, has the advantage that for fixed incident light intensity, $I(x)$, the light absorbed, $I(x+\Delta) - I(x)$, is proportional to the thickness, Δ, and for fixed thickness Δ, the light absorbed is proportional to the incident light, $I(x)$.

Equation 17.35 can be rearranged to

$$\begin{aligned} \frac{I(x+\Delta) - I(x)}{\Delta} &\doteq \frac{-K\Delta I(x)}{\Delta} \\ \frac{I(x+\Delta) - I(x)}{\Delta} &\doteq -KI(x) \end{aligned}$$

The approximation ($\doteq$) gets better as the layer thickness, Δ, gets close to zero. Therefore

$$\begin{aligned} \lim_{\Delta \to 0} \frac{I(x+\Delta) - I(x)}{\Delta} &= -KI(x) \\ I'(x) &= -KI(x) \end{aligned} \tag{17.36}$$

Exercise 17.9.3 Assume that 1000 w/m^2 of light is striking the surface of a lake and that 40% of that light is reflected back into the atmosphere. Solve the initial value problem

$$\begin{aligned} I'(x) &= -KI(x) \\ I(0) = 600 & \end{aligned}$$

Suppose the light intensity at 10 meters is 500 w/m^2. Find the value of K.

17.9.2 Spectrophotometer scales.

Equation D.7 has an important application in interpretation of readings of light absorbance as measured in spectrophotometers. The spectrophotometer has two scales; the top scale is marked fraction (or percent) transmission, the bottom scale is marked 'absorbance.'

In $I'(x) = -KI(x)$, the constant, K, reflects the turbidity of the water. With high turbidity, light intensity decreases rapidly with depth, so that K is large. For clear water, light intensity decreases slowly with depth, and K is small. In the spectrophotometer, the thickness of the layer of liquid is constant, being determined by the glass tube holding the sample. The variable is the turbidity of the liquid in the tube and is assumed proportional to the density of bacteria, dye, or other substance you wish to quantify. The general solution to

$$I'(x) = -KI(x) \qquad \text{is} \qquad I(x) = I(0)e^{-Kx}$$

$I(0)$ is determined by the light source in the spectrophotometer. Assume that the thickness of the liquid in the tube is τ. Then the constant quantities in $I(x) = I(0)e^{-Kx}$ are $x = \tau$ and $I(0)$. The quantity measured by a photocell in the spectrophotometer and displayed in the top scale is $I(\tau)$. We write

$$\begin{aligned}
I(\tau) &= I(0)e^{-K\tau} \\
\ln I(\tau) &= \ln I(0) + \ln\left(e^{-K\tau}\right) \\
\ln I(\tau) &= \ln I(0) + -K\tau \\
K = -\frac{1}{\tau}\left(\ln I(\tau) - \ln I(0)\right) & \\
K = -\frac{1}{\tau}\ln\frac{I(\tau)}{I(0)} & \qquad (17.37)
\end{aligned}$$

Therefore, the turbidity of the liquid sample is proportional to the logarithm of the fraction of light transmitted through the sample, and the bottom scale of the spectrophotometer is the negative of the logarithm of the top scale.

17.9.3 Carbon dating of ancient organic material.

Plants obtain carbon from atmospheric carbon dioxide; animals obtain carbon from plants. The fraction of ^{14}C carbon in the total carbon of living animals and plants is the same as that of the atmosphere. At present that fraction is approximately 1 part ^{14}C per 10^{12} total carbon atoms. Upon death, no new carbon is absorbed and the ^{14}C decays back to nitrogen

$$^{14}_{6}C \quad \longrightarrow \quad ^{14}_{7}N + \beta^- + \overline{\nu} + energy \qquad (17.38)$$

where β^- is an electron emitted from one of the neutrons of the $^{14}_{6}C$, changing it to a proton, and $\overline{\nu}$ is a neutrino that accompanies a β^- emission. Measure of the β^- emission is a measure of the ^{14}C content of the carbon under study. One gram of carbon from presently living tissue contains about 5×10^{10} ^{14}C atoms and emits about 15.3 β^- particles per minute. In carbon from ancient plants and animals, some of the ^{14}C has decayed and the rate of β^- emission is smaller.

Mathematical Model of ^{14}C decay. The rate at which ^{14}C decreases is proportional to the amount of ^{14}C present. The reaction is describe by

$$\frac{d}{dt}\,^{14}C = -K \times ^{14}C \tag{17.39}$$

For carbon dating assume that time is measured with zero at the year 1950 and negative times before that. For example, $t = -3000$ corresponds to 1050 BC (or BCE, before the common era). Let $E(t)$ be the level of β^- emission per minute from one gram of carbon in living tissue at time t. Scientists initially assumed that $E(t)$ is equal to the present value, $E(0) = 15.3$, for all time, but as will be seen in Figure 17.18, $E(t)$ is not constant. Let $E_{t_0}(t)$ be β^- emission per minute from one gram of carbon at time t in the remains of an organism that died at a time $t_0 \leq t$. Note that $E_{t_0}(0) = E(t_0)$ and $E_{t_0}(t_0)$ is the β^- emission that we would measure today from a fossil of an animal that lived at time t_0.

The solution to Equation 17.39 may be written:

$$^{14}C(t) = ^{14}C(t_0)e^{-K\,(t-t_0)}$$

Because β^- emission is directly proportional to ^{14}C content, we write that

$$E_{t_0}(t) = E(t_0)e^{-K\,(t-t_0)}. \tag{17.40}$$

Laboratory measurements have shown that in about 5730 years, one-half of the ^{14}C in any sample will decay to nitrogen. Therefore, for any t_0 and $t = t_0 + 5730$

$$\begin{aligned} E_{t_0}(t_0 + 5730) &= E_{t_0}(t_0)e^{-K\,5730} \\ &= 0.5 \times E_{t_0}(t_0). \end{aligned}$$

Thus

$$\begin{aligned} e^{-K\,5730} &= 0.5 \\ K &= \frac{\ln 2}{5730}. \end{aligned}$$

Equation 17.40 may now be written

$$E_{t_0}(t) = E_{t_0}(t_0)e^{-\frac{\ln 2}{5730}\,(t-t_0)} \tag{17.41}$$

Example 17.9.1 *Problem* Suppose the one gram of carbon from deer bone recently found among American Indian artifacts is emits 7 β^- particles per minute. How old is the bone?

Solution. Let t_0 be the time at which the deer died. **Assume** that $E(t_0) = E(0) = 15.3$. Then

$$\begin{aligned} E_{t_0}(t)\big|_{t=0} = E_{t_0}(0) &= E(t_0)e^{-\frac{\ln 2}{5730}\,(0-t_0)} \\ 7 &= \mathbf{1}5.3e^{-\frac{\ln 2}{5730}\,(-t_0)} \\ \ln(\frac{7}{15.3}) &= \frac{\ln 2}{5730}\,t_0 \end{aligned}$$

$$t_0 = -6464$$

Thus the bone is 6464 years old. ■

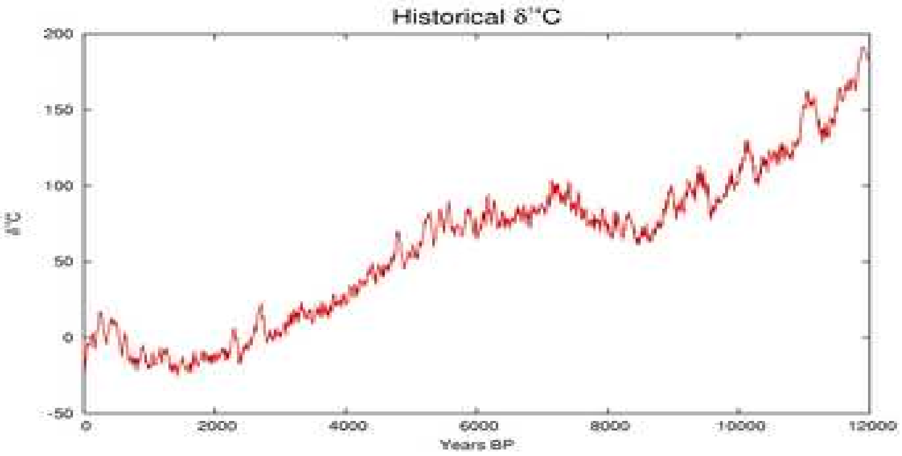


Figure 17.18: Ambient $\delta^{14}C$ levels for the most recent 12000 years. Produced by Leland McInnes using publicly available sources. http://en.wikipedia.org/wiki/File:Carbon-14-calibration.png

Exercise 17.9.4 Suppose the β^- emission per gram of carbon in a recently discovered bone is $(1/2^n)\,E(0) = (1/2^n)\,15.3$ where n is an integer. **Assume** that $E(t) = E(0) = 15.3$ for all t. How old is the bone?

Note: Accurate measurement of $(1/2^n)\,15.3$ β^- emission is limited to $n \leq 10$.

Date adjustment for varying ambient ^{14}C content.

We found the bone of Example 17.9.1 to be approximately 6464 years old, so that the humans occupied the camp in which which the bone was found approximately 6,464 years ago. We assumed that the β^- emission of carbon 6,646 years ago was the same as the present β^- emission. Techniques for measuring the ^{14}C radiation and estimating the age of organic matter were developed by Willard F. Libby in the 1950's and have been checked by measuring the β^- emission of carbon from artifacts of known age; for two examples, carbon of heart wood of a giant sequoia tree shown by tree rings to be 2928± 50 years old, and by measuring the β^- emission of 4900 year old wood from a 1st dynasty Egyptian tomb.

Careful analysis of these woods have shown that the ambient β^- emission of carbon has varied over the years, and age estimates must be modified accordingly. A graph of ambient β^- emission levels for 9000 years is shown in Figure 17.18[11].

The vertical units in Figure 17.18 are measured as

$$\delta(t) = \left(\frac{E(t) - E(0)}{E(0)}\right) \times 1000, \qquad E(t) = E(0)\left(1 + \frac{\delta(t)}{1000}\right)$$

From the graph, for 6,464 years ago, (4514 B.C.), $\delta(6464) \doteq 80$, and

$$E(-6464) \quad = \quad E(0)\left(1 + \frac{80}{1000}\right)$$

---

[11] **I was not able to locate a source to obtain copyright of this figure.** From M. Bruns, et al, The atmospheric ^{14}C level in the 7th millenium BC, Proceedings of the First International Symposium on ^{14}C and Archeology, (1981), Groningen, Netherlands. An alternate graph is inserted.

If $E(t_0)$ is $1.08E(0) = 1.08 \times 15.3 = 16.52$ and the β^- emission of the sample is 7, as assumed above, then the age $(-t_0)$ of the bone would be found by

$$\begin{aligned} 7 &= 16.52e^{-\frac{\ln 2}{5730}*(-t_0)} \\ t_0 &= -7098 \end{aligned}$$

Thus a substantial adjustment must be made in the age of the sample, from 6464 years old to 7098 years old. However, this may not be the final answer, for you may see that 7098 years ago, the assumption that the ambient $\delta(t)$ is 80 is not valid. One then can make another adjustment, and hope that the procedure converges to a single value.

Exercise 17.9.5 Read the $\delta^{14}C$ level for 7098 years ago (5148 BC) from Figure 17.18 and convert that to $E(7098)$. Use that level to compute the next estimate of the age of a bone sample that has present day β^- emission = 7 particles per gram of carbon.

In their article, Bruns et al remark that "The overall trend of the cosmogenic ^{14}C follows remarkably closely a sine-curve with a period of 11,300 years and a peak to peak amplitude of 10.2%." They only considered the previous 9000 years. in Figure 17.18.

Exercise 17.9.6 By inspection of the graph for the last 9000 years, explain why

$$E(t) = E(0)\left(1.035 - 0.050 \sin\left(\frac{2 * \pi}{11,200}(t + 4200)\right)\right) \tag{17.42}$$

where t is time before the present, gives a reasonable estimate of β^- emission of tissue living t years before the present. How does the graph work for 9000-12000 years ago?

Unique dates from ^{14}C analysis. Graphs of Equation 17.42 (solid) and an exponential decay curve (dashed) are shown in Figure 17.19. The curves intersect at (-4900,14.3) and (-1600,13.4), and the value of the decay curve at $t = 0$ is 13.0. If the decay curve were that of ^{14}C decay, with a half life of 5730 years, a wood sample with a present day ^{14}C radiation level could either be 4900 years old or 1600 years old. Fortunately, the decay curve illustrated has a half-life of 35,000 years and is not representative of ^{14}C decay.

Exercise 17.9.7 Use the Extended Mean Value Theorem 14.5.2 to show that the graphs of

$$E(t) = 15.3\left(1.035 - 0.050 \sin\left(\frac{2 * \pi}{11200}(t + 4200)\right)\right)$$

and

$$E_{t_0}(t) = E(t_0)\, e^{-\frac{\ln 2}{5730}(t-t_0)}$$

can not intersect at two points for $-10000 \le t_0 \le 0$ and $t_0 \le t \le 0$. Argue as follows. They obviously intersect at $(t_0, E(t_0))$. Argue that:

a. $E'(t) \ge -0.000429$ for $t_0 \le t$.

b. $E'_{t_0}(t) \le -0.00045$ for $t_0 \le t$.

c. Suppose the graphs intersect at another point, $(t_1, E(t_1))$ with $t_0 \le t_1$. Then at some time τ between t_0 and t_1 the slopes of the two graphs are equal. This leads to a contradiction.

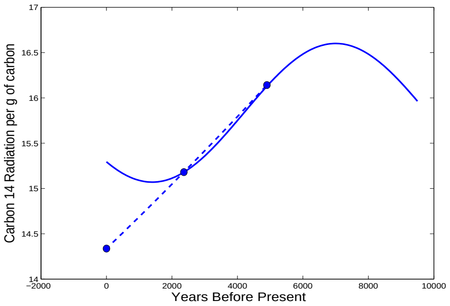


Figure 17.19: Graphs of a sine curve approximating ^{14}C radiation (Equation 17.42) and a decay curve (–) $D_{t_0}(t) = E(t_0)\, e^{-\frac{\ln 2}{5\cdot 5730}(t-t_0)}$ with $t_0 = -5000$. $D_{t_0}(t)$ decays one fifth as fast as $E_{t_0}(t) = E(t_0)\, e^{-\frac{\ln 2}{5730}(t-t_0)}$

17.9.4 Chemical Kinetics

Chemical kinetics is the study of the rates of chemical reactions. The rate of a reaction is expressed in terms of the rate of disappearance of the reactants or the rate of appearance of the products. In the reaction

$$A \longrightarrow B \tag{17.43}$$

The reaction rate is

$$\text{Rate} = -\frac{dA}{dt} = \frac{dB}{dt} \tag{17.44}$$

We consider only simple reactions in which the reaction rate is a function of the concentrations of the reactants[12].

A reaction of the form

$$\frac{dA}{dt} = -K \times A \tag{17.45}$$

such as ^{14}C decay is called a first order reaction. It also is a first order differential equation, but not for the same reason.

A reaction that occurs by the collision and combination of two molecules A and B,

$$A + B \longrightarrow AB$$

has a reaction rate that is proportional to the concentrations of A and B and a rate equation

$$-\frac{d[A]}{dt} = k[A][B] \tag{17.46}$$

In the event that the concentrations of A and B are equal, Equation 17.46 becomes

$$\frac{d[A]}{dt} = -k[A][A], \tag{17.47}$$

---

[12] Refer to Linus Pauling, *General Chemistry* 1970, W. H. Freeman, New York, 1988, Dover, New York. Pauling, p552. "The factors that determine the rate of reaction are manifold. The rate depends not only upon the composition of the reacting substances, but also upon their physical form, the intimacy of their mixture, the temperature and pressure, the concentrations of the reactants, special physical circumstances such as irradiation with visible light, ultraviolet light, x-rays, neutrons, or other waves or particles, and the presence of other substances that affect the reaction but are not changed by it."

a first order differential equation that describes a second order reaction.

Equal quantities of gaseous hydrogen and iodine are mixed resulting in the reaction

$$H_2 + I_2 \longrightarrow 2HI$$

Because the initial amounts of H_2 and I_2 are equal and they combine equally, their concentrations will be equal. Let $y(t)$ be $[H_2] = [I_2]$. Then from equation 17.46,

$$\frac{dy}{dt} = -k\,y(t) \times y(t) \tag{17.48}$$

Explore 17.9.1 Solve Equation 17.48 and show that

$$y(t) = \frac{y_0}{k\,y_0\,t + 1} \qquad \text{where} \qquad y_0 = y(0)$$

The general reaction

$$mA + nB \longrightarrow A_m B_n$$

has a rate equation

$$\frac{d[A]}{dt} = -k[A]^m[B]^n$$

and some special cases are of interest.

In the event that the initial concentration of B greatly exceeds that of A, then the concentration of B will be relatively constant during the reaction. With A being the rate limiting substance, the reaction will be an order m reaction. Let $y(t) = [A]$ and assume $[B]$ is constant. Then Equation 17.9.4 becomes

$$\frac{dy(t)}{dt} = -K\,[B]^n\,(y(t))^m = -\hat{K}\,(y(t))^m \tag{17.49}$$

For

$$m = 1, \qquad y(t) = y_0\,e^{-\hat{K}t} \qquad \text{and for} \qquad m = 2, \qquad y(t) = \frac{y_0}{y_0\,\hat{K}\,t + 1}.$$

Exercise 17.9.8 Show that for $m = 3$ the solution to

$$\frac{dy(t)}{dt} = -\hat{K}\,(y(t))^m \qquad \text{is} \qquad y(t) = \frac{y_0}{\sqrt{2\,y_0^2\,\hat{K}\,t + 1}}$$

We now can see that for $m = 1$, 2, or 3

$$\left.\begin{aligned} m = 1 &\implies & \ln y(t) &= \hat{K}t + \ln y_0 \\ m = 2 &\implies & \tfrac{1}{y(t)} &= \hat{K}t + \frac{1}{y_0} \\ m = 3 &\implies & \frac{1}{(y(t))^2} &= 2\hat{K}t + \frac{1}{y_0^2} \end{aligned}\right\} \tag{17.50}$$

If data from a reaction are given then one of $\ln y(t)$, $\frac{1}{y(t)}$, or $\frac{1}{(y(t))^2}$ may be linear in t, depending on whether each reaction step requires $m = 1$, 2, or 3 molecules of A. If neither is linear in t, then it would be assumed that the reaction step requires more than 3 molecules of A, or that the concentration of B is changing enough to affect the data.

Example 17.9.2 Data from Reaction 1 of Exercise Table 17.9.9 are plotted in Figure 17.20 and it is clear that the data are from a second order reaction, and $m = 2$. The reaction is thus

$$2A + nB \longrightarrow A_2B_n$$

for some n (found below). ■

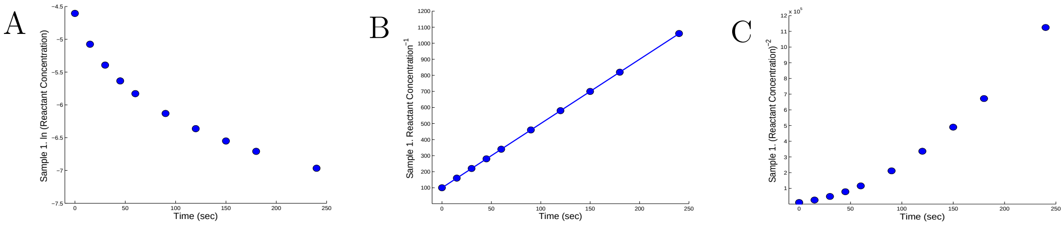


Figure 17.20: Graphs of Reaction 1 data. A. ln Concentration vs t. B. 1/Concentration vs t. C. 1/(Concentration)^2 vs t. B. is linear and the line y = 99.96 + 4 t fits the data. Therefore, the reaction is second order, $\hat{K} = 4$, and $\frac{1}{y(t)} = \hat{K}t + \frac{1}{y_0}$, and $y(t) = 1/(4t + 99.96)$.

Exercise 17.9.9 Sample concentrations of a reaction $mA + nB \longrightarrow A_mB_n$ with initial concentrations $[A]_0 = 0.01\,M$ and $[B]_0 = 0.2\,M$ are shown in Exercise Table 17.9.9. For samples 2 and 3, what are the values of m?

Table for Exercise 17.9.9 Sample concentrations of a substance A in a chemical reaction $mA + nB \longrightarrow A_m\,B_n$ with initial concentrations $[A]_0 = 0.01M$ and $[B]_0 = 0.2M$.

Time (sec)	Reaction 1 Concentration of A (mol)	Reaction 2 Concentration of A (mol)	Reaction 3 Concentration of A (mol)
0	0.010000	0.010000	0.010000
15	0.006250	0.003430	0.008607
30	0.004545	0.002500	0.007408
45	0.003571	0.002063	0.006376
60	0.002941	0.001796	0.005488
90	0.002174	0.001474	0.004066
120	0.001724	0.001280	0.003012
150	0.001429	0.001147	0.002231
180	0.001220	0.001048	0.001653
240	0.000943	0.000909	0.000907

Once the correct linear relation has been determined by comparing the data with Equations 17.50 so that m is known, another experiment will reveal the value of n. Observe that $[B]^n$ is a factor of $\hat{K}$ in the slopes of each of the equations in Equations 17.50. Therefore, if the experiment is repeated with twice the initial concentration $[B]$, the slope of the line first observed will be altered by 2^n.

Example 17.9.2 Continued. For the Reaction 1 with $[B] = 0.2$ mol, we found that the reaction was second order, $m = 2$ and $\hat{K} = 4$. The data for Reaction 1 with $[B] = 0.4$ mol is plotted in Figure 17.21 as for a second order reaction, 1/Concentration(A) vs time, and it is found that $y = 104 + 31.9t$ fits the data. Now $\hat{K}_{[B]=0.4} = 32$ which is 8 times $\hat{K}_{[B]=0.2}$. Therefore, $2^n = 8$, $n = 3$, and the reaction is third order in B. The reaction is thus

$$2A + 3B \longrightarrow A_2B_3 \qquad \blacksquare$$

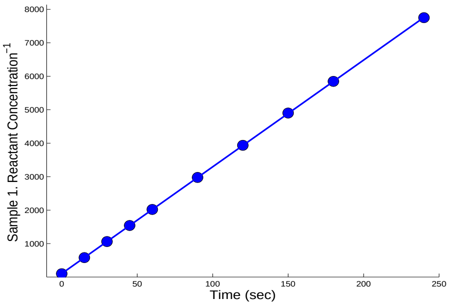


Figure 17.21: Graph of the data for Reaction 1 with $[B] = 0.4$, plotted as for a second order reaction (m=2) in A, 1/Concentration(A) vs time. The line fitting the data has equation $y = 104 + 31.9t$, so $\hat{K}_{[B]=0.2} = 4$ (from above) and $\hat{K}_{[B]=0.4}$ is $32 = 8 \times \hat{K}_{[B]=0.2}$. Thus $2^n = 8$, $n = 3$, and the reaction is third order in B.

Exercise 17.9.10 The experiments of the previous problem are repeated with initial concentrations $[A]_0 = 0.01M$ and $[B]_0 = 0.4M$, and the concentrations of $[A]$ are shown in Exercise Table 17.9.10. What are the values of n for reactions 2 and 3?

Table for Exercise 17.9.10 Sample concentrations of a substance A in a chemical reaction $mA + nB \longrightarrow A_mB_n$ with initial concentrations $[A]_0 = 0.01M$ and $[B]_0 = 0.4M$.

Time (sec)	Reaction 1 Concentration of A (mol)	Reaction 2 Concentration of A (mol)	Reaction 3 Concentration of A (mol)
0	0.010000	0.010000	0.010000
15	0.001724	0.001796	0.007408
30	0.000943	0.001280	0.005488
45	0.000649	0.001048	0.004066
60	0.000495	0.000909	0.003012
90	0.000336	0.000743	0.001653
120	0.000254	0.000644	0.000907
150	0.000204	0.000576	0.000498
180	0.000171	0.000526	0.000273
240	0.000129	0.000456	0.000082

Table 17.6: Area of mold colonies on days 0 to 9.

Day	0	1	2	3	4	5	6	7	8	9
Area mm^2	4	8	24	46	84	126	176	248	326	420

17.9.5 Mold Growth

You saw in Volume I, Chapter 2, that the area of mold growing on a solution of tea and water can be well described by a quadratic function. You may have attributed this to the fact that the growth is confined to the perimeter of the mold colony – the cells on the interior of the colony were not expanding. The pictures of the growth are shown on the page **??** of this volume.

A fairly accurate measure of the areas is shown in the Table 17.6. Each grid square in the photographs is 4 mm^2. If you have not done so, you should enter the data in statistical lists for your calculator and fit both an exponential function and a second degree polynomial to the data. You will find that the polynomial fits the data better, even though the exponential function shows a correlation of 0.97 with the data.

Let $A(t)$ be the area of the mold colony at time t and $P(t)$ be the length of the perimeter. Because cell division appears to be taking place on the perimeter of the mold, a reasonable mathematical model is

Mathematical Model 17.9.2 Mathematical Model. The rate of change of the area of the mold colony is proportional to the length of the perimeter.

We will also assume that the mold colony is essentially circular, of radius R.

Assumption. $A(t) = \pi R^2$. $P(t) = 2\pi R$.

Exercise 17.9.11 Use the Mathematical Model 17.9.2 and Assumption to show that

$$A'(t) = k \times 2 \times \sqrt{\pi}\sqrt{A(t)} \tag{17.51}$$

Exercise 17.9.12 Equation 17.51 can be written

$$\frac{A'(t)}{\sqrt{A(t)}} = k \times 2 \times \sqrt{\pi}$$

a. Show that

$$\left[2\sqrt{A(t)}\right]' = \frac{A'(t)}{\sqrt{A(t)}} \qquad \text{and} \qquad \left[k \times 2 \times \sqrt{\pi} \times t\right]' = k \times 2 \times \sqrt{\pi}$$

b. Conclude that there is a number C such that

$$\sqrt{A(t)} \quad = \quad k \times \sqrt{\pi} \times t + C$$

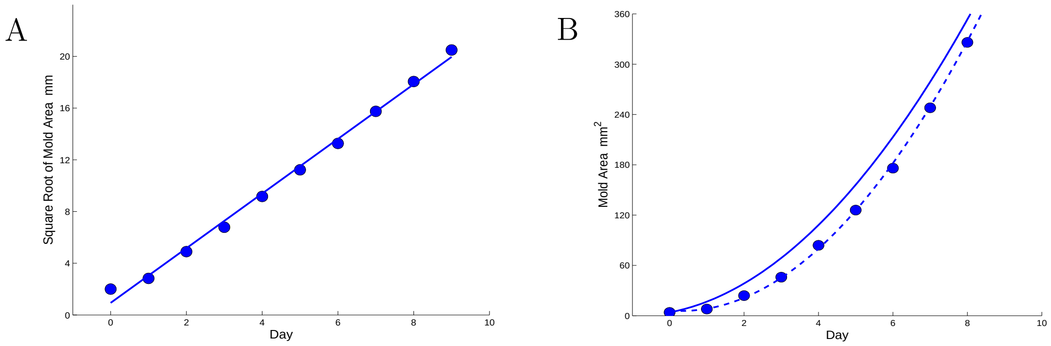
c. **Initial condition.** Assume that $A(0) = 4$, as observed in the pictures. Use $A(0) = 4$ and evaluate C.

d. The graph of $\sqrt{A(t)}$ *vs* t is shown in Exercise Figure 17.9.12. Use the graph to find an estimate of $k \times \sqrt{\pi}$.

e. From what theorem can you conclude that

$$A(t) \doteq (2.1\,t + 2)^2 \qquad (17.52)$$

Figure for Exercise 17.9.12 A. Graph of $\sqrt{A(t)}$ versus t for the data of Table 17.6. The equation of the line is $y = 0.94 + 2.1\,x$. B. Graphs of the mold data, $y = (2.1t + 2)^2$ (solid line), and the regression parabola, $P_2(t) = 5.4\,t^2 - 3.3\,t + 6.2$ (dashed line).



It is to be noted that

$$A(t) \doteq (2.1\,t + 2)^2$$

does not fit the data as well as does the regression curve,

$$P_2(t) = 5.4t^2 - 3.3t + 6.2$$

(as computed with a TI-86 and the data in the table.) The advantage of the solution $A(t) \doteq (2.1\,t + 2)^2$ is that it is based on a well defined model of the growth.

17.10 Chapter 17 Suggested solutions.

Explore 17.1.1.

$$\begin{array}{rcl} & & \text{Ricker} \\ U(p(t)) & = & \dfrac{Ae^{-p(t)/\beta} - 1}{A - 1} \\ \text{With} & & p(t) \ll \beta, \quad p(t)/\beta \doteq 0, \quad e^{-p(t)/\beta} \doteq 1. \\ \text{and} & & \dfrac{Ae^{-p(t)/\beta} - 1}{A - 1} \doteq \dfrac{A \times 1 - 1}{A - 1} = 1 \\ \text{Thus} & & p'(t) = r\,p(t)\,U(p(t)) \doteq r\,p(t). \end{array}$$

$$\begin{array}{rcl} & & \text{Gompertz} \\ U(p(t)) & = & -\ln(p(t)/\beta) \\ \text{With} & & p(t) \ll \beta, \quad p(t)/\beta \doteq 0, \quad \text{and} \quad -\ln(p(t)/\beta) \doteq -\ln 0 = \inf . \\ \text{Thus} & & p'(t) = r\,p(t)\,U(p(t)) \gg r\,p(t). \end{array}$$

Exercise 17.1.3

$$p'(t) = -rp(t)\ln(p(t)/\beta), \quad t \geq 0, \qquad p(0) = p_0$$

$$m(p_0) = p'(0) = -rp(0)\ln(p(0)/\beta) = -rp_0\ln(p_0/\beta)$$

By L'Hospital's Rule,

$$\lim_{p_0 \to 0} m(p_0) = \lim_{p_0 \to 0} -r\frac{\ln(p_0/\beta)}{1/p_0} = \lim_{p_0 \to 0} -r\frac{1/p_0}{-1/(p_0)^2} = \lim_{p_0 \to 0} rp_0 = 0.$$

Exercise 17.1.5. The carbon dioxide pressure in air is insignificant compared to the partial pressure of carbon dioxide in the Coca-Cola and we write

> **Mathematical Model**. The rate at which carbon dioxide leaves a Coca-Cola is proportional to the amount of carbon dioxide in the Coca-Cola.

Let $CO_2(t)$ denote the carbon dioxide partial pressure in a Coca-Cola at time t, where the bottle is opened at time $t = 0$.

$$\begin{array}{rcl} CO_2'(t) & = & -K\,CO_2(t) \\ CO_2(t) & = & CO_2(0)e^{-Kt} \end{array}$$

Exercise 17.1.6.

$$\textbf{Ricker:} \quad p_1' = r_1p_1\frac{A_1e^{-(p_1+\alpha1,2p_2)/\beta_1}}{A_1-1}, \qquad p_2' = r_2p_2\frac{A_2e^{-(\alpha_{2,1}p_1+p_2)/\beta_2}}{A_2-1}.$$

Exercise 17.1.9. We have

3. Without the prey, the predator decreases at a rate proportional to the size of the predator population (the predator has no alternate food source).

4. The predator growth rate is increased proportional to the rate of predation.

From 3, predator decrease without prey is $-cv(t)$. The rate of predation in the Prey Equation is $b \times u(t) \times v(t)$ and its effect on the predator population may be written as $d \times u(t) \times v(t)$. Thus

$$\textbf{Predator} \qquad v'(t) = -cv(t) + d \times u(t) \times v(t).$$

Exercise 17.1.11. If people become susceptible six months after recovery, we modify the definition of $R(t)$ to mean those who have recovered *and are not susceptible.* Then the SIR equations become

$$\begin{array}{lrcl} \textbf{Susceptible} & S'(t) & = & -\beta \times S(t) \times I(t) + R'(t-180) \\ \textbf{Infectious} & I'(t) & = & \beta \times S(t) \times I(t) - \gamma \times I(t) \\ \textbf{Recovered} & R'(t) & = & \gamma \times I(t) - R'(t-180) \end{array}$$

Exercise 17.1.15. You might write $P(0) = 200$,

$$P'(t) = -\frac{0.23}{5}P(t) = -0.046P(t), \quad P(t) = P(0)e^{-0.046\,t} = 200e^{-0.046\,t}$$

For which $P(10) = 126.3$. Write

$$P(0) = 200, \qquad P'(t) = -kP(t), \qquad P(t) = P(0)e^{-kt} = 200e^{-kt}$$

Because $P(5) = 118$mg,

$$118 = 200e^{-k\,5}, \qquad k = \frac{-\ln 0.59}{10} \doteq 0.05276 \qquad P(t) = 200e * -0.05276\,t$$

Explore 17.2.1.

$$\text{a. } y = \frac{1}{2}e^{\frac{t^2}{2}} \qquad \text{c. } y = 6e^{\frac{t^2}{2}}$$

$$\begin{array}{lcccl} \text{a. } y = \frac{1}{2}e^{\frac{t^2}{2}} & \text{LHS: } y'(t) & & \text{RHS: } t \times y(t) & \\ & y'(t) = \left[\frac{1}{2}e^{\frac{t^2}{2}}\right]' & & t \times y(t) = t \times \frac{1}{2}e^{\frac{t^2}{2}} & \\ & \frac{1}{2}e^{\frac{t^2}{2}} \times \left[\frac{t^2}{2}\right]' & & t \times \frac{1}{2}e^{\frac{t^2}{2}} & \\ & \frac{1}{2}e^{\frac{t^2}{2}} \times t & = & \frac{1}{2}t \times e^{\frac{t^2}{2}} & \textbf{It Checks!} \\ \text{c. } y = 6e^{\frac{t^2}{2}} & \text{LHS: } y'(t) & & \text{RHS: } t \times y(t) & \\ & y'(t) = \left[6e^{\frac{t^2}{2}}\right]' & & t \times y(t) = t \times 6e^{\frac{t^2}{2}} & \\ & 6e^{\frac{t^2}{2}} \times \left[\frac{t^2}{2}\right]' & & t \times 6e^{\frac{t^2}{2}} & \\ & 6e^{\frac{t^2}{2}} \times t & = & 6t \times e^{\frac{t^2}{2}} & \textbf{It Checks!} \end{array}$$

f. $y = e^{t^2}$ LHS: $y'(t)$ RHS: $t \times y(t)$

$$y'(t) = \left[e^{t^2}\right]' \qquad t \times y(t) = t \times e^{t^2}$$

$$e^{t^2} \times [t^2]' \qquad t \times e^{t^2}$$

$$e^{t^2} \times 2t \quad \neq \quad t \times e^{t^2}$$ **Does Not Check!**

Exercise 17.2.1.

a. $y(t) = e^{2t} + e^t, \qquad y(0) = 2, \qquad y'(t) - y(t) = e^{2t}$
$y(0) = e^{2\cdot 0} + e^0 = 1 + 1 = 2$ **Check.**
$y'(t) - y(t) = [e^{2t} + e^t]' - (e^{2t} + e^t) = 2e^{2t} + e^t - (e^{2t} + e^t) = e^{2t}$ **Check.**

c. $y(t) = te^t, \quad y(0) = 0, \quad y'(t) - y(t) = e^t$
$y(0) = 0e^0 = 0$ **Check.**
$y'(t) - y(t) = [te^t]' - te^t = te^t + e^t - te^t = e^t$ **Check.**

e. $y(t) = \sqrt{t+1}, \quad y(0) = 1, \quad y(t) \times y'(t) = \frac{1}{2}$
$y(0) = \sqrt{1+0} = 1$ **Check.**
$y(t) \times y'(t) = \sqrt{t+1}\left[\sqrt{t+1}\right]' = \sqrt{t+1}\frac{1}{2\sqrt{t+1}} = \frac{1}{2}$ **Check.**

g. $y(t) = \sqrt{4+t^2}, \quad y(0) = 2, \quad y(t) \times y'(t) = t$
$y(0) = \sqrt{4+0^2} = 2$ **Check.**
$y(t) \times y'(t) = \sqrt{4+t^2}\left[\sqrt{4+t^2}\right]' = \sqrt{4+t^2}\left(\frac{2t}{2\sqrt{4+t^2}}\right) = t$ **Check.**

i. $y(t) = 0.5 + 0.5e^{-0.2\sin t}, \quad y(0) = 1, \quad y'(t) + 0.2(\cos t)y(t) = 0.1\cos t$
$y(0) = 0.5 + 0.5e^{-0.2\sin 0} = 0.5 + 0.5 \times 1 = 1$ **Check.**
$y'(t) + 0.2(\cos t)y(t) = \left[0.5 + 0.5e^{-0.2\sin t}\right]' + 0.2\cos t\left(0.5 + 0.5e^{-0.2\sin t}\right)$
$= 0 - 0.1e^{-0.2\sin t}\cos t + 0.1\cos t + 0.1(\cos t)e^{-0.2\sin t} = 0.1\cos t$ **Check.**

k. $y(t) = 3, \quad y(0) = 3, \quad y'(t) = (y(t)-1) \times (y(t)-3) \times (y(t)-5)$
$y(0) = y(t)|_{t=0} = 3$ **Check.**
$y'(t) = [3]' = 0$
$(y(t)-1) \times (y(t)-3) \times (y(t)-5) = (3-1) \times (3-3) \times (3-5) = 0$**Check.**

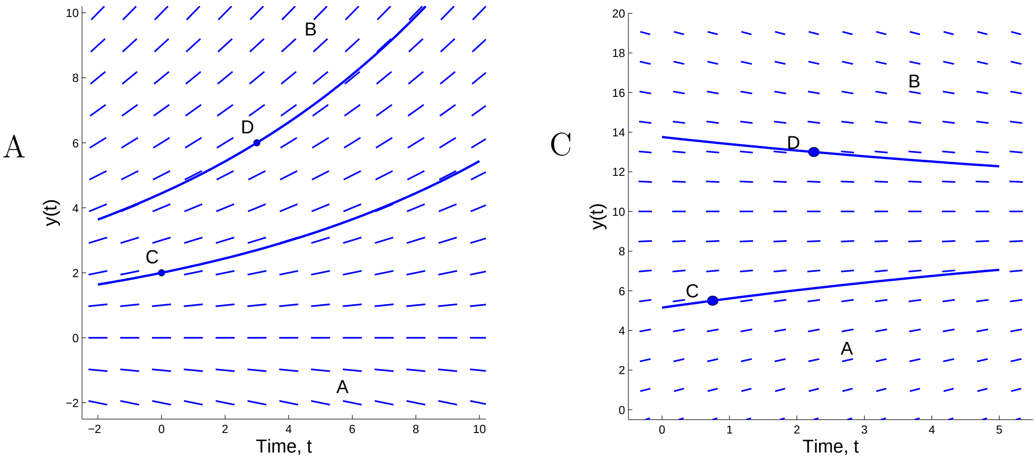


Figure 17.22: **Exercise 17.3.3.** A. Direction fields and solution curves for A. $y'(t) = 0.1y$, and C. $y'(t) = 1 - y/10$.

Exercise 17.2.2.

a. $y(t) = e^{2t} + 2e^{t}, \quad y(0) = 2, \quad y'(t) - y(t) = e^{2t}$
$y(0) = 3$ **Does Not Check**

c. $y(t) = \sqrt{t+1}, \quad y(0) = 1, \quad y(t) \times y'(t) = 1$
$y(0) = \sqrt{0+1} = 1$ **Checks**
$y(t) \times y'(t)\sqrt{t+1}\frac{1}{2\sqrt{t+1}} = \frac{1}{2}$ **Does Not Check**

e. $y(t) = \frac{1}{t}, \quad y(0) = 1, \quad y'(t)/y(t) = -y(t)$
$y(0) = \frac{1}{0}$, Undefined **Does Not Check**

Exercise 17.3.2.

A. $y'(t) = 0.1y$. A:(6,-2), slope=-0.2. B:(5,9), slope=0.9.

C. $y'(t) = 1 - y/10$. A:(13/4,5/2), slope=3/4. B:(17/4,16), slope=-0.6.

Exercise 17.3.3. See Figure 17.22.

Exercise 17.3.4. See Figure 17.23.

Exercise 17.3.5. See Figure 17.24.

Exercise 17.3.6. b. $y = 1$ is one solution.

Exercise 17.3.7. b. Change "slope=tt(i)*yy(j);" to "slope=tt(i)+yy(j);".

Explore 17.4.1. 1 and 3 are stable equilibria.

Exercise 17.4.1.TV/R is a stable equilibrium.

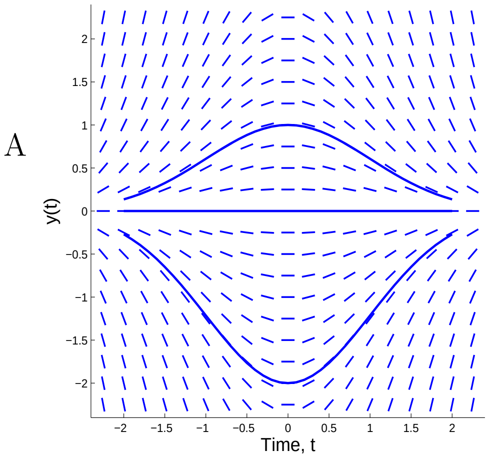


Figure 17.23: **Exercise 17.3.4.** Direction field and solution curves A. for $y\prime = -t \times y$.

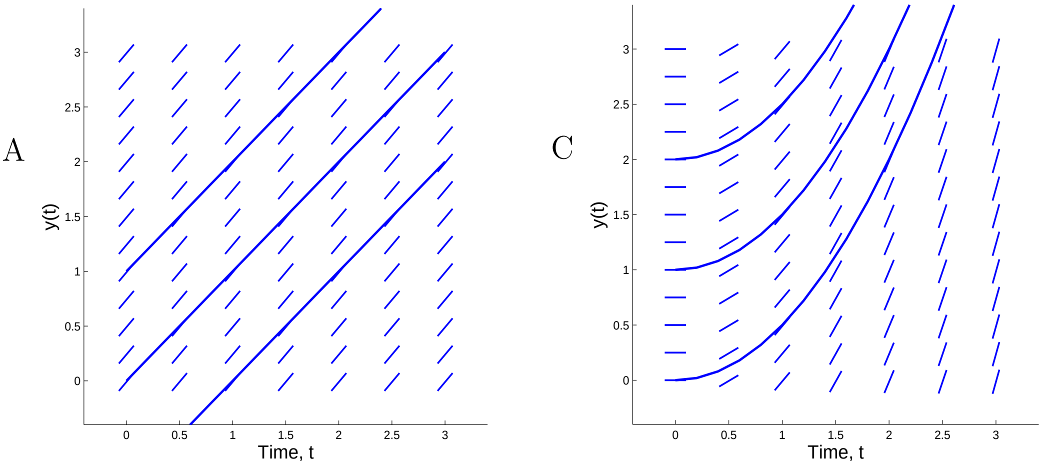


Figure 17.24: **Exercise 17.3.5.** Direction field and solution curves A. for $y\prime = 1$, and C. for $y\prime = 1$.

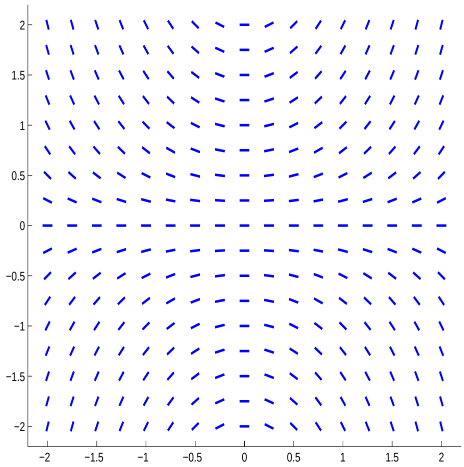


Figure 17.25: **Exercise 17.3.7.** Direction field for $y' = t \times y$.

Exercise 17.4.2.

a.	$y' = y - 1$	$f(y) = y - 1$	$f(1) = 0,$	$f'(1) = 1$	Not Stable
c.	$y' = y^2 - 1$	$f(y) = y^2 - 1$	$f(1) = 0,$	$f'(1) = 2$	Not Stable
			$f(-1) = 0$	$f'(-1) = -2$	Stable
e.	$y' = e^{-y} - 1$	$f(y) = e^{-y} - 1$	$f(0) = 0,$	$f'(0) = -1$	Stable
g.	$y' = \sin y$	$f(y) = \sin y$	$f(2k\pi) = 0,$	$f'(2k\pi) = 1$	Not Stable
			$f((2k+1)\pi+) = 0$	$f'((2k+1)\pi) = -1$	Stable
i.	$y' = -y^3$	$f(y) = -y^3$	$f(0) = 0,$	$f'(0) = 0$	??

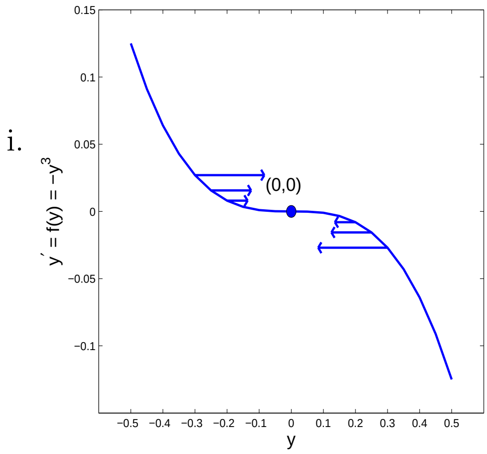


Figure 17.26: **Exercise 17.4.2.** Phase planes for i. $y' = -y^3$. Because the arrows in i. point toward 0, 0 is a stable point of $y' = -y^3$.

Exercise 17.4.3. $N = 3.2$ Atm.

Exercise 17.4.4. a. See Figure 17.27.

b. The equilibrium point, 2.5, is stable; all arrows point toward it. Furthermore, $F'(2.5) = -2.5 < 0$.

c. The units of the equilibrium point are the units of P.

$$-2\frac{1}{\text{hr}} \times P\text{gm}5\frac{\text{gm}}{\text{hr}} = 0 \qquad P\text{gm} = 2.5\frac{\text{gm}}{\text{hr}} \div \frac{1}{\text{hr}} = 2.5gm$$

d. The equation now is $P' = -1.5P + 5$ and the equilibrium point is now $5/1.5 = 3\ 1/3$ and the patient has more than 50 percent more penicillin at equilibrium.

Exercise 17.4.7.

a. The equation for p_1 is Verhulst's logistic equation and the stability of $p_1 = M_1$ is illustrated in Figure 17.8.

b. Suppose $M_2 < \alpha_{2,1}M_1$. Assume that $p_1(t) = M_1$ for $t \geq t_0$. then $M_2 < \alpha_{2,1}p_1(t)$ for $t_0 \leq t$. Let $\delta = \alpha_{2,1}M_1/M_2$. Observe that $\delta > 1$. Then in the equation for $p_2'(t)$

$$\begin{aligned} p_2'(t) &= r_2 \times p_2(t) \times \left(1 - \frac{\alpha_{2,1}p_1(t)}{M_2} - \frac{p_2(t)}{M_2}\right) \\ p_2'(t) &< rp(t)(1-\delta) = -kp_2(t) \qquad \text{for} \quad t_0 < t \end{aligned}$$

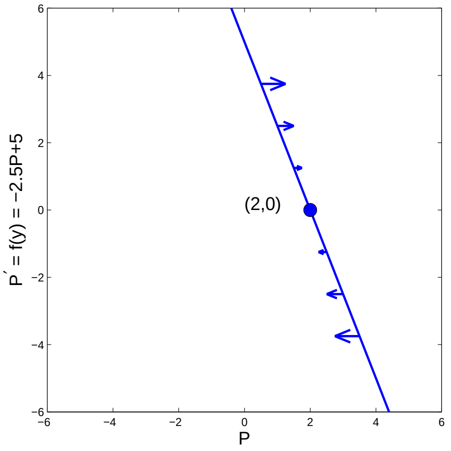


Figure 17.27: **Exercise 17.4.4.** Phase plane for $P'(t) = -2.5P(t)+5$. $f(P) = -1.5P+5$ The equilibrium point, 2.5, is stable.

It follows that $p_2(t) \to 0$.

c. $p_2 \to M_2 - \alpha_2 M_1 > 0$. Again we assume that $p_1(t) = M_1$ for $t \geq t_0$. If $M_2 > \alpha_{2,1} M_1$, then

$$\begin{aligned} p_2'(t) &= r_2 \times p_2(t) \times \left(1 - \frac{p_2(t) + \alpha_{2,1} p_1(t)}{M_2}\right) \\ p_2'(t) &= r_2 \times p_2(t) \times \left(1 - \frac{p_2(t) + \alpha_{2,1} M_1}{M_2}\right) \\ &= r\frac{M_2 - \alpha_{2,1} M_1}{M_1} P_2 \left(1 - \frac{p_2}{M_2 - \alpha_{2,1} M_1}\right) \end{aligned}$$

This is Verhulst's logistic equation and $p_2 \to M_2 - \alpha_2 M_1 > 0$.

Exercise 17.4.8. a. Recall Equation 17.12 for $\tau = r \times t$ and $v(\tau) = u(t)$:

$$u'(t) = \frac{d}{dt}u(t) = \frac{d}{dt}v(\tau) = \frac{d}{d\tau}v(\tau)\frac{d\tau}{dt} = v'(\tau) \times r.$$

Then Ricker's Equation:

$$\begin{aligned} p'(t) &= \alpha\, p\, e^{-p/\beta} - \gamma\, p \\ \frac{p'(t)}{\beta} &= \alpha\, \frac{p}{\beta}\, e^{-p/\beta} - \gamma\, \frac{p}{\beta} \\ u'(t) &= \alpha\, u\, e^{-u} - \gamma\, u \end{aligned}$$

where $u(t) = p(t)/\beta$. Now let $\tau = \alpha t$ and $v(\tau) = u(t)$. Then

$$u'(t) = v'(\tau)\alpha$$

Then

$$u'(t) = \alpha\, u\, e^{-u} - \gamma\, u$$

$$\begin{aligned} v'(\tau)\alpha &= \alpha\, v(\tau)\, e^{-v(\tau)} - \gamma\, v(\tau) \\ v'(\tau) &= v(\tau)\, e^{-v(\tau)} - \gamma_0\, v(\tau) \end{aligned}$$

where $\gamma_0 = \gamma/\alpha$.

Exercise 17.4.10.

b. See Figure 17.28. With $h = \frac{3}{4}r$

$$u' = ru(1-u) - \frac{3}{4}ru = \frac{1}{4}ru\left(1 - \frac{u}{1/4}\right).$$

Then $u = 1/4$ is a stable solution and eventual harvest will be $\frac{3}{4}r\frac{1}{4} = \frac{3}{16}r$.

d. $h = \frac{1}{2}r$ will yield the largest long term harvest.

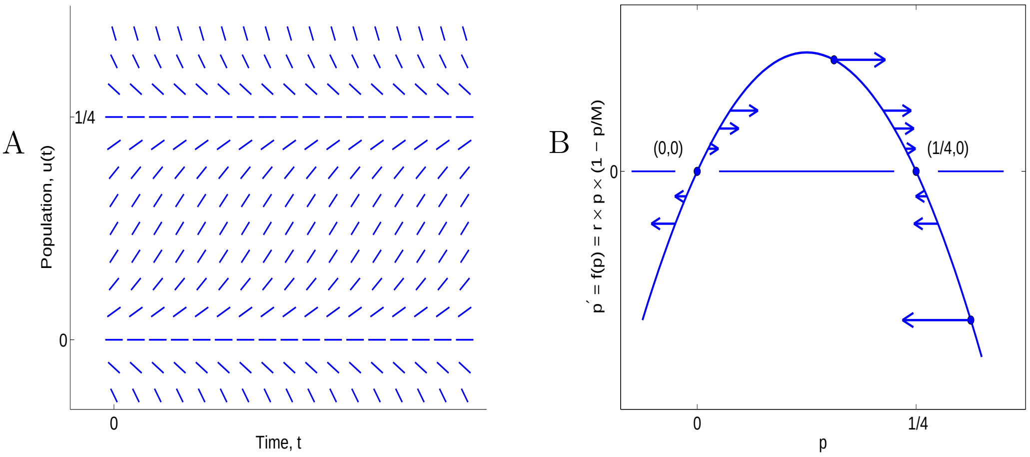


Figure 17.28: **Exercise 17.4.10.** A. Direction Field and B. phase plane for $u' = \frac{1}{4}r\left(1 - \frac{u}{1/4}\right)$. $u = 1/4$ is a stable solution. Eventually the harvest will be $\frac{3}{16}r$

Exercise 17.4.11. See Figure 17.29.

Exercise 17.5.1. See Figure 17.30.

Exercise 17.5.3.

$$\begin{array}{lrcll} \text{a.} & y(t) &=& (1-t)^{-1} \qquad y(0)=1 \qquad y'(t)=y^2, & \\ & y(0) &=& (1-0)^{-1} = 1, & \textbf{Check.} \\ & y'(t) &=& (-1)(1-t)^{-2}(-1) = \left((1-t)^{-1}\right)^2 = y^2, & \textbf{Check.} \end{array}$$

$$\begin{array}{lrcll} \text{c.} & y(t) &=& e^{t^2/2} \qquad y(0)=1 \qquad y'(t)=t\times y & \\ & y(0) &=& e^{0^2/2} = e^0 = 1 & \textbf{Check.} \\ & y'(t) &=& te^{t^2/2} = ty(t) & \textbf{Check.} \end{array}$$

Exercise 17.5.4.

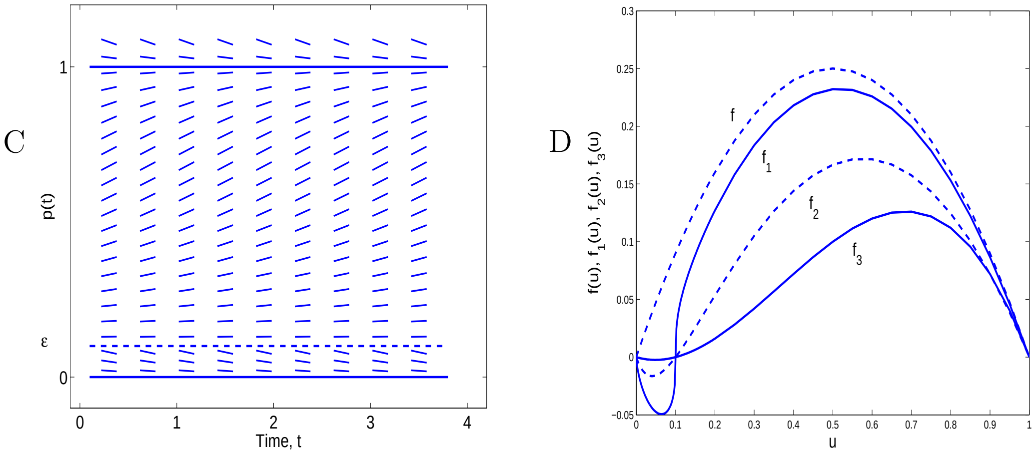


Figure 17.29: **Exercise 17.4.11.** A. Direction Field for logistic population curve modified to have a low density threshold, ϵ, below which population decreases and B. phase plane for logistic curve, f, and three candidate modifications of the logistic model that account for a threshold at $\epsilon = 0.1$. $f_1 = u^{2/3}(u - 0.1)^{1/3}(1 - u)$, $f_2 = u(u - \epsilon)/(u + \epsilon)(1 - u)$, and $f_3 = u(u - \epsilon)(1 - u)$. The tangents of each modification at (0,0) and (1,0) have negative slopes showing that 0 and 1 are stable, and the tangents at $(\epsilon, 0)$ have positive slopes showing that ϵ is not stable.

Table 17.7: Euler solutions to $u(0) = 1, u' = 0.3u - 0.2u\,v,\qquad v(0) = 2v' = 0.1uv - 0.1v$

Time	0	0.2	0.4	0.6	0.8	1.0
$u(t)$	1	$1 + 0.1 \cdot 0.2 = 1.02$	1.404	0.0612	1.0823	1.1037
$v(t)$	2	$2 + 0.0 \cdot 0.2 = 2.0$	2.0008	2.0024	2.0049	2.0082
$u'(t)$	0	0.1020	0.1039	0.1056	0.1072	
$v'(t)$	0.1	0.0	0.0040	0.0081	0.0165	

Exercise 17.6.1. A direction field for $y' = \sqrt{y}$ is shown in Figure 17.36. The tangents along $y = 0$ are horizontal and $y = 0$ is a solution. All of the tangents above the t-axis slope upward and it appears that another solution could 'leave' the t-axis. Indeed, $y = t^2/4$ also solves the $y' = \sqrt{y}$ and intersects the t-axis at (0,0).

Exercise 17.6.2. Euler solutions to $u(0) = 1,\quad u' = 0.3u - 0.2u\,v$, $v(0) = 2,\quad v' = 0.1uv - 0.1u$ are shown in Table 17.7.

Exercise 17.6.4.

$$\begin{aligned} u' &= u^{\frac{2}{3}} \times (u - 0.1)^{\frac{1}{3}} \times (1 - u) \\ \\ f(t, u) &= u^{\frac{2}{3}} \times (u - 0.1)^{\frac{1}{3}} \times (1 - u) \end{aligned}$$

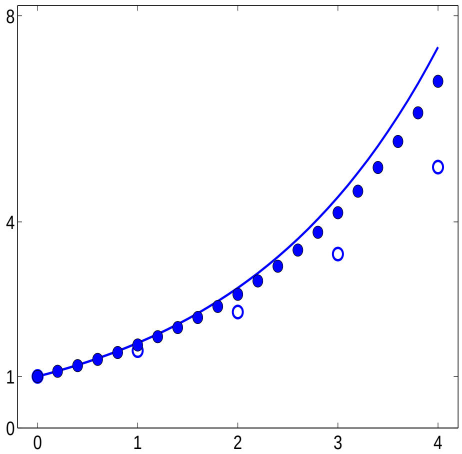


Figure 17.30: **Exercise 17.5.1.** Graphs of the solution to $y(0) = 1 \qquad y'(t) = 0.5y(t) \qquad 0 \le t \le 4$ and approximations using Euler's method and 4 intervals of length 1 (open circles) and using 20 intervals of length 0.2 (filled circles).

Exercise 17.5.3a.

$y(0) = 1, \quad y' = y^2, \quad y = (1-t)^{-1}$

t	Euler	Trapezoid	Solution
0	1.0000	1.0000	1.0000
0.1	1.1000	1.1105	1.1111
0.2	1.2210	1.2483	1.2500
0.3	1.3701	1.4248	1.4286
0.4	1.5578	1.6587	1.6667

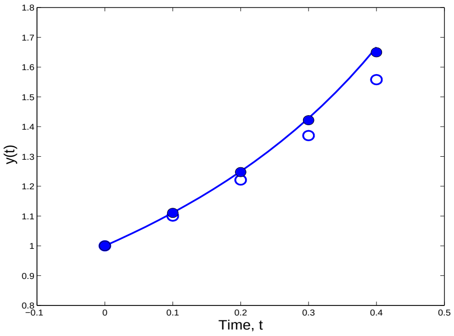


Figure 17.31: **Exercise 17.5.3a.** Euler (open circles) and trapezoid (filled circles) approximations and exact solution ($y = (1-t)^{-1}$) to $y(0) = 1$, $y' = y^2$.

Exercise 17.5.3c.

$y(0) = 1, \quad y' = ty \quad , \quad y = e^{t^2/2}$

t	Euler	Trapezoid	Solution
0	1.0000	1.0000	1.0000
0.2	1.0000	1.0200	1.0202
0.4	1.0400	1.0828	1.0833
0.6	1.1232	1.1963	1.1972
0.8	1.2580	1.3753	1.3771
1.0	1.4593	1.6448	1.6487

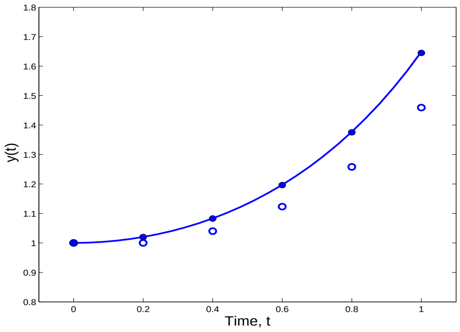


Figure 17.32: **Exercise 17.5.3c.** Euler (open circles) and trapezoid (filled circles) approximations and exact solution ($y = e^{t^2/2}$) to $y(0) = 1$, $y' = ty$.

Exercise 17.5.4a.

$y(0) = 1, \quad y'(t) = t - \sqrt{y(t)}$

t	Euler	Trapezoid
0	1.0000	1.0000
0.2	0.8000	0.8306
0.4	0.6611	0.7165
0.6	0.5785	0.6526
0.8	0.5464	0.6337
1.0	0.5585	0.6544

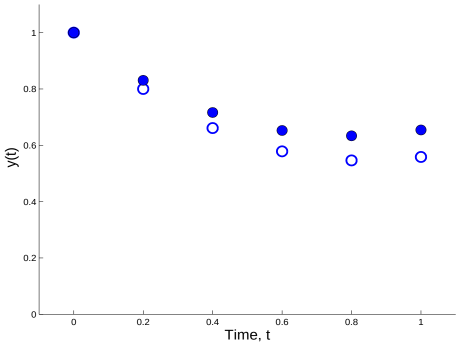


Figure 17.33: **Exercise 17.5.4a.** Euler (open circles) and trapezoid (filled circles) approximations to the solution of $y(0) = 4, \quad y'(t) = t - \sqrt{y(t)}$.

Exercise 17.5.4c.

$y(0) = 0.5, \quad y'(t) = -\ln y$

t	Euler	Trapezoid
0	0.5000	0.5000
0.2000	0.6256	0.6154
0.4000	0.7122	0.6987
0.6000	0.7754	0.7611
0.8000	0.8231	0.8092
1.0000	0.8597	0.8467

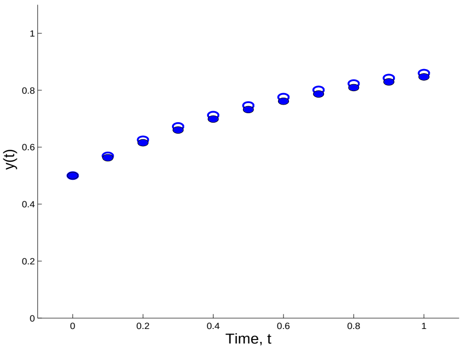


Figure 17.34: **Exercise 17.5.4c.** Euler (open circles) and trapezoid (filled circles) approximations to the solution of $y(0) = 0.5, \quad y'(t) = -\ln y$. Some data are not printed.

Exercise 17.5.4e.

$y(0) = 0.1, \quad y'(t) = y(y - 0.1)(1 - y)$

t	Euler	Trapezoid
0	0.1	0.1
0.4000	0.1	0.1
0.8000	0.1	0.1
1.2000	0.1	0.1
1.6000	0.1	0.1
2.0000	0.1	0.1

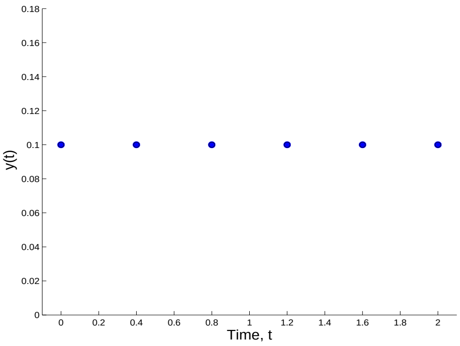


Figure 17.35: **Exercise 17.5.4e.** Euler (open circles) and trapezoid (filled circles) approximations to the solution of $y(0) = 0.1, \quad y'(t) = y(y - 0.1)(1 - y)$.

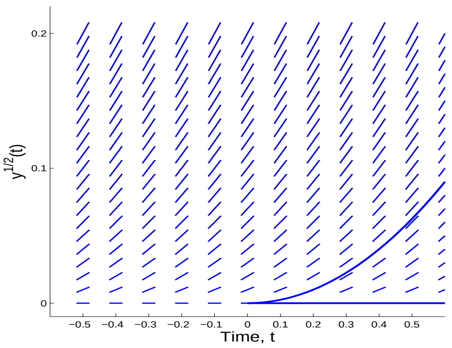


Figure 17.36: **Exercise 17.6.1.** Direction field for $y' = \sqrt{y}$. Only positive values of y are accepted. Two solutions, $y = 0$ and $y = t^2/4$, are shown.

$$\begin{aligned} f_2(t,u) &= \frac{2}{3}u^{-\frac{1}{3}} \times (u-0.1)^{\frac{1}{3}} \times (1-u) \\ &+ u^{\frac{2}{3}}\frac{1}{3} \times (u-0.1)^{-\frac{2}{3}} \times (1-u) \\ &+ u^{\frac{2}{3}} \times (u-0.1)^{\frac{1}{3}} \times (-1) \end{aligned}$$

Exercise 17.7.1.

a.
$$\begin{aligned} a &= 0, \quad y_0 = 2, \quad p = 0.2, \quad q = 0.1 \\ u &= \int_0^t 0.2\,ds = 0.2t, \quad v = \int_0^t e^{0.2s}0.1\,ds = \frac{1}{2}(e^{0.2t}-1) \\ y &= v(t)e^{-u(t)} + y_a e^{-u(t)} = \frac{1}{2}(e^{0.2t}-1)e^{-0.2t} + 2\cdot e^{-0.2t} \\ &= \frac{1}{2} + \frac{3}{2}e^{-0.2t} \end{aligned}$$

c.
$$\begin{aligned} a &= 0, \quad y_0 = 1, \quad p = 3, \quad q = t \\ u &= \int_0^t 3\,ds = 3t, \quad v = \int_0^t e^{3s}s\,ds = \frac{1}{3}se^{3s} - \frac{1}{9}e^{3s}\Big|_0^3 = \frac{1}{3}te^{3t} - \frac{1}{9}e^{3t} + \frac{1}{9}. \\ y &= v(t)e^{-u(t)} + y_a e^{-u(t)} = \left(\frac{1}{3}te^{3t} - \frac{1}{9}e^{3t} + \frac{1}{9}\right)e^{-3t} + 3\cdot e^{-3t} \\ &= \frac{1}{3}t - \frac{1}{9} + \left(3 + \frac{1}{9}\right)e^{-3t} \end{aligned}$$

e.
$$\begin{aligned} a &= 1, \quad y_1 = 1, \quad p = \frac{1}{t}, \quad q = 1 \\ u &= \int_1^t \frac{1}{t}\,ds = \ln t, \quad v = \int_1^t e^{\ln s}\,1\,ds = \int_1^t s\,ds = \frac{s^2}{2}\Big|_1^t = \frac{t^2}{2} - \frac{1}{2} \\ y &= v(t)e^{-u(t)} + y_a e^{-u(t)} = \left(\frac{t^2}{2} - \frac{1}{2}\right)e^{-\ln t} + 1e^{-\ln t} \\ &= \left(\frac{t^2}{2} - \frac{1}{2}\right)\frac{1}{t} + \frac{1}{t} = \frac{t}{2} + \frac{1}{2t} \end{aligned}$$

$$\begin{aligned}
\text{g.}\quad a &= 0, \quad y_0 = 7, \quad p = 0.2, \quad q = e^{-0.2t} \\
u &= \int_0^t 0.2\, ds = 0.2t \quad v = \int_0^t e^{0.2s}\, e^{-0.2s}\, ds = \int_1^t 1\, ds = t \\
y &= v(t)e^{-u(t)} + y_a e^{-u(t)} \\
&= t\, e^{-t} + 7e^{-t}
\end{aligned}$$

$$\begin{aligned}
\text{i.}\quad a &= 0, \quad y_0 = 2, \quad p = \sin t, \quad q = \sin t \\
u &= \int_0^t \sin s\, ds = -\cos t + 1 \\
v &= \int_0^t e^{-\cos s + 1}\, \sin s\, ds = e\, e^{-\cos s}\Big|_0^t = e(e^{-\cos t} - e^{-1}) \\
y &= v(t)e^{-u(t)} + y_a e^{-u(t)} = e\left(e^{-\cos t} - e^{-1}\right)\, e^{-(-\cos t + 1)} + 2e^{-(-\cos t + 1)} \\
&= e\left(e^{-\cos t} - e^{-1}\right) e^{\cos t} e^{-1} + 2e^{\cos t} e^{-1} \\
&= 1 + 2e^{-1} e^{\cos t}
\end{aligned}$$

$$\begin{aligned}
\text{j.}\quad a &= 0, \quad y_0 = 5, \quad p(t) = 3, \quad q(t) = \sin 4t \\
u(t) &= \int_0^t p(s)\, ds = \int_0^t 3\, ds = 3t \\
v(t) &= \int_0^t e^{u(s)} q(s)\, ds = \int_0^t e^{3s} \sin 4s\, ds = \frac{3}{25} e^{3s} \sin 4s - \frac{4}{25} e^{3s} \cos 4s\Big|_0^t \\
&= \frac{3}{25} e^{3t} \sin 4t - \frac{4}{25} e^{3t} \cos 4t + \frac{4}{25} \\
y &= v(t)e^{-u(t)} + y_a e^{-u(t)} \\
&= \left(\frac{3}{25} e^{3t} \sin 4t - \frac{4}{25} e^{3t} \cos 4t + \frac{4}{25}\right) e^{-3t} + 5e^{-3t} \\
&= \frac{3}{25} \sin 4t - \frac{4}{25} \cos 4t + \frac{4}{25} e^{-3t} + 5e^{-3t}
\end{aligned}$$

Exercise 17.7.2.

$$\begin{aligned}
\text{a.}\quad a &= 0, \quad y_0 = 5, \quad u = \int_0^t 2\, ds = 2t, \quad v = 0 \\
y &= 5e^{-2t}
\end{aligned}$$

$$\begin{aligned}
\text{c.}\quad a &= 0, \quad y_0 = 4, \quad u = 3t, \\
&\qquad v = \int_0^t e^{3s} s\, ds = \frac{1}{3} s e^{3s} - \frac{1}{9} e^{3s}\Big|_0^t = \frac{1}{3} t e^{3t} - \frac{1}{9} e^{3t} + \frac{1}{9} \\
y &= \left(\frac{1}{3} t e^{3t} - \frac{1}{9} e^{3t} + \frac{1}{9}\right) e^{-3t} + 4e^{-3t} \\
y &= \frac{1}{3} t - \frac{1}{9} + \frac{37}{9} e^{-3t}
\end{aligned}$$

$$\text{e.}\quad a = 0, \quad y_0 = 0 \quad u = 0.2t, \quad v = \int_0^t e^{0.2s} e^{-0.2s}\, ds = t$$

$$y \quad = \quad te^{-02t} + 0$$

Exercise 17.7.4. The solution, $y(t)$, exists on all of $[a, b]$. $p(t)$ and $q(t)$ are defined and continuous on $[a, b]$ so that $u(t) = \int_a^t p(s)\, ds$ and $v(t) = \int_a^t e^{-u(s)}\ q(s)\, ds$ are also defined and continuous on $[a, b]$. Thus $y(t) = v(t)e^{-u(t)} + y_a e^{-u(t)}$ are defined and continuous on all of $[a, b]$.

Exercise 17.7.5.

a. The initial condition, $y(a) = y_a$:

$$u(a) = \int_a^a p(s)\, ds = 0, \quad v(a) = \int_a^a e^{u(s)} q(s)\, ds = 0,$$

$$y(a) = u(a)e^{-v(a)} + y_a e^{-v(a)} = 0 + y_a \cdot 1 = y_a$$

b. Show that $y(t)$ solves Equation 17.19. Note:

$$u(t) = \int_a^t p(s)\, ds, \quad u'(t) = p(t), \qquad v(t) = \int_a^t e^{u(s)} q(s)\, ds, \quad v'(t) = e^{u(t)} q(t)$$

$$\begin{aligned}
y(t) &= v(t)e^{-u(t)} + y_a e^{-u(t)} \\
y'(t) &= \left[v(t)e^{-u(t)} + y_a e^{-u(t)}\right]' \\
&= v(t)e^{-u(t)}(-u'(t)) + v'(t)e^{-u(t)} + y_a e^{-u(t)}(-u'(t)) \\
&= = -v(t) \cdot e^{-u(t)} \cdot p(t) + e^{u(t)} q(t) e^{-u(t)} - y_a e^{-u(t)} p(t) \\
&= -v(t)p(t)e^{-u(t)} + q(t) - y_a e^{-u(t)} p(t). \\
y'(t) + p(t)y(t) &= -v(t)p(t)e^{-u(t)} + q(t) - y_a e^{-u(t)} p(t) \\
&\quad +p(t)(v(t)e^{-u(t)} + y_a e^{-u(t)}) \\
&= -v(t)p(t)e^{-u(t)} + q(t) - y_a e^{-u(t)} p(t) \\
&\quad +p(t)v(t)e^{-u(t)} + p(t)y_a e^{-u(t)}) \\
&= q(t)
\end{aligned}$$

Exercise 17.7.8. Look at the profile for N_2. By taking the 30 meter portion first, an ascent to 10 meters is similar to a decompression stop, nitrogen is lost from the 10 minute half-life tissue during the

15 minutes at 10 meters. N_1 increases during the 15 minutes at 10 meters and then increases almost as much at 30 meters as it did in the N_2 profile.

Exercise 17.7.9. a. 2.08 b. 1.22

Exercise 17.7.10. See Figure 17.37.

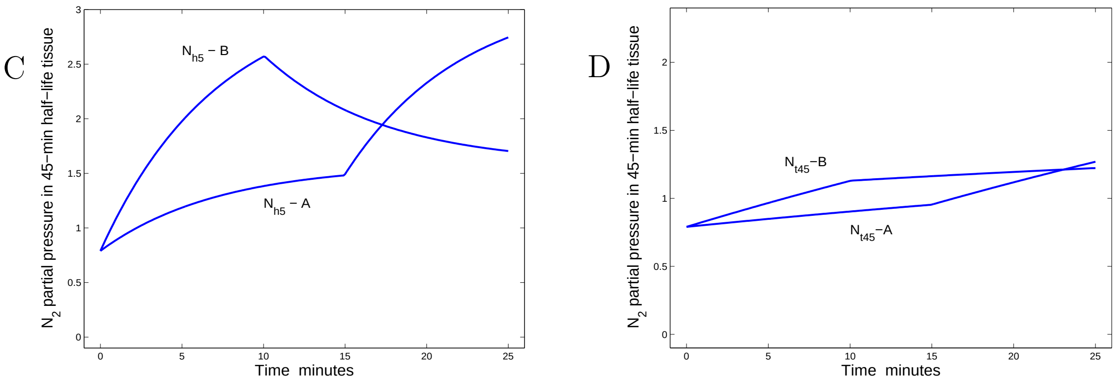


Figure 17.37: **Exercise 17.7.10**. Nitrogen content N_{h5} in 5-minute half-life tissue and N_{h45} in 40-minute half-life tissue for a diver A. who dives to 10 meters for 15 minutes and immediately descends to 30 meters for 10 mintes and for a diver B. who dives to 30 meters for 10 minutes and immediately ascends to 10 meters for 15 minutes.

Exercise 17.8.3. Suppose $\ln ty = g(t)h(y)$ for all t and all y. Then

$$g(1)h(1) = \ln 1 \cdot 1 = 0, \quad g(1)h(e) = \ln e = 1, \quad \text{and} \quad g(e)h(1) = \ln e = 1.$$

The first equation implies that one of $g(1)$ and $h(1)$ is 0. If $g(1) = 0$ then $g(1)h(e) = 0 \neq 1$, which contradicts the second equation. If $h(0) = 0$ then $g(e)h(1) = 0 \neq 1$, which contradicts the third equation.

Exercise 17.8.4.

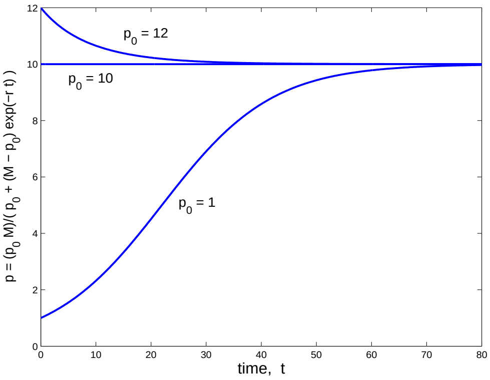


Figure 17.38: Graphs of $p(t) = p_0M/(p_0 + (M - p_0)e^{-rt})$ for $p_0 = 1$, $p_0 = 12$ and $p_0 = 10$.

a.
$$\begin{array}{rcll}
y' & = & \dfrac{t}{y} & y(0) = 2 \\
y\,y' & = & t & \\
\dfrac{y^2}{2} & = & \dfrac{t^2}{2} + C & \\
\dfrac{2^2}{2} & = & \dfrac{0^2}{2} + C & C = 2 \\
y & = & t^2 + 2 &
\end{array}$$

c.
$$\begin{array}{rcll}
y' & = & \dfrac{t-1}{y} & y(2) = 1 \\
y\,y' & = & t - 1 & \\
\dfrac{y^2}{2} & = & \dfrac{t^2}{2} - t + C & \\
\dfrac{1^2}{2} & = & \dfrac{2^2}{2} - 2 + C & C = 1/2 \\
\dfrac{y^2}{2} & = & \dfrac{t^2}{2} - t + 1/2 &
\end{array}$$

e.
$$\begin{array}{rcll}
y' & = & y - 1 & y(0) = 1.5 \\
\dfrac{y'}{y-1} & = & 1 & \\
\ln(y-1) & = & t + C & \\
\ln(1.5-1) & = & 0 + C & C = \ln 0.5 \\
\ln(y-1) & = & t + \ln 0.5 &
\end{array}$$

g.
$$\begin{array}{rcll}
y' & = & y^2 - 1 & y(0) = 2 \\
\dfrac{y'}{y^2-1} & = & 1 & \\
\dfrac{y'}{y+1} - \dfrac{y'}{y-1} & = & 2 & \\
\ln(y+1) - \ln(y-1) & = & 2t + C & \\
\ln(2+1) - \ln(2-1) & = & 2 \cdot 0 + C & C = \ln 3 \\
\ln(y+1) - \ln(y-1) & = & 2t + \ln 3 &
\end{array}$$

Exercise 17.8.8. See Figure 17.38.

Exercise 17.8.9.

$$\begin{aligned} P(t) &= p_0M/(p_0+(M-p_0)\,e^{-rt}) = p_0M(p_0+(M-p_0)\,e^{-rt})^{-1} \\ P'(t) &= rp_0M(M-p_0)(p_0+(M-p_0)\,e^{-rt})^{-2}\,e^{-rt} \\ P''(t) &= -r^2p_0M(M-p_0)e^{-rt}(p_0+(M-p_0)\,e^{-rt})^{-3} \\ &\times \Big(p_0+(M-p_0)e^{-rt}-2(M-p_0)e^{-rt}\Big) \\ P''(t) &= 0 \quad \text{when} \quad \Big(p_0+(M-p_0)e^{-rt}-2(M-p_0)e^{-rt}\Big)=0 \\ e^{-rt_{steep}} &= \frac{p_0}{M-p_0} \\ P(t_{steep}) &= p_0M/(p_0+(M-p_0)\,e^{-rt_{steep}}) \\ &= p_0M/(p_0+(M-p_0)\,\frac{p_0}{M-p_0}) \qquad = \qquad \frac{M}{2} \end{aligned}$$

$P''(t_{steep})$ is zero and $P'(t_{steep})$ is the maximum value of $P'(t)$, so $P(t)$ is increasing the fastest when $t = t_{steep}$.

Exercise 17.8.11.

$$\text{a.} \quad v' = ve^{-v}-\gamma_0 v \quad \frac{d\,v'}{d\,v} = -ve^{-v}+e^{-v}-\gamma_0 \quad \gamma_0 e^v = 1-v$$

For $\gamma_0 \leq 1$, there is a solution, $\overline{v}$ to $\gamma_0 e^v = 1-v$ and $0 < \overline{v} \leq 1$.

$$\frac{d^2\,v'}{d\,v^2} = ve^{-v}-2e^{-v} = (v-2)e^{-v} \qquad \left.\frac{d^2\,v'}{d\,v^2}\right|_{v=\overline{v}} = (\overline{v}-2)e^{-\overline{v}} < 0.$$

Therefore, v' is maximum when $v = \overline{v}$.

$$\begin{aligned} \text{c.} \qquad v' &= -v\ln(v) \quad \frac{d\,v'}{d\,v} = -1-\ln v \qquad = 0 \quad \text{when} \quad v = e^{-1} \\ \frac{d^2\,v'}{d\,v^2} &= -\frac{1}{v} \qquad < 0 \quad \text{for all} \quad v > 0. \end{aligned}$$

Therefore, v' is maximum when $v = e^{-1}$.

Exercise 17.8.12.

$$\begin{aligned} \text{a.} \qquad\qquad u' &= u \times \frac{u-\epsilon}{u+\epsilon} \times (1-u) \\ \frac{u+\epsilon}{u(u-\epsilon)(1-u)}u' &= 1 \\ \frac{u+\epsilon}{u(u-\epsilon)(1-u)}u' &= \frac{A}{u}+\frac{B}{u-\epsilon}+\frac{C}{1-u} \end{aligned}$$

$$
\begin{array}{llcll}
 & u+\epsilon & = & A(u-\epsilon)(1-u)+Bu(1-u)+Cu(u-\epsilon) & \\
u=0 & \epsilon & = & A(-\epsilon) & A=-1 \\
u=\epsilon & 2\epsilon & = & B\epsilon(1-\epsilon) & B=\frac{2}{1-\epsilon} \\
u=1 & 1+\epsilon & = & C(1)(1-\epsilon) & C=\frac{1+\epsilon}{1-\epsilon}
\end{array}
$$

b.
$$\left(\frac{-u'}{u}+\frac{2}{1-\epsilon}\frac{u'}{u-\epsilon}+\frac{1+\epsilon}{1-\epsilon}\frac{u'}{1-u}\right)=1$$

$$
\begin{array}{lrcl}
 & -\ln|u|+\frac{2}{1-\epsilon}\ln|u-\epsilon|-\frac{1+\epsilon}{1-\epsilon}\ln|1-u| & = & t+C \\
\epsilon=0.1 & -\ln|u|+\frac{2}{0.9}\ln|u-0.1|-\frac{1.1}{0.9}\ln|1-u| & = & t+C \\
u_0=0.12 & -\ln|0.12|+\frac{2}{0.9}\ln|0.12-0.1|-\frac{1.1}{0.9}\ln|1-0.12| & = & 0+C \\
 & C & = & -6.4169 \\
u=0.08 & -\ln|0.08|+\frac{2}{0.9}\ln|0.08-0.1|-\frac{1.1}{0.9}\ln|1-0.08| & = & 0+C \\
 & C & = & -6.0657
\end{array}
$$

c. $u_0=0.08$, $C=-6.0657$. Graphs are in Figure 17.39, curve A.

d. $u_0=0.12$, $C=-6.4169$. Graphs are in Figure 17.39, curve B.

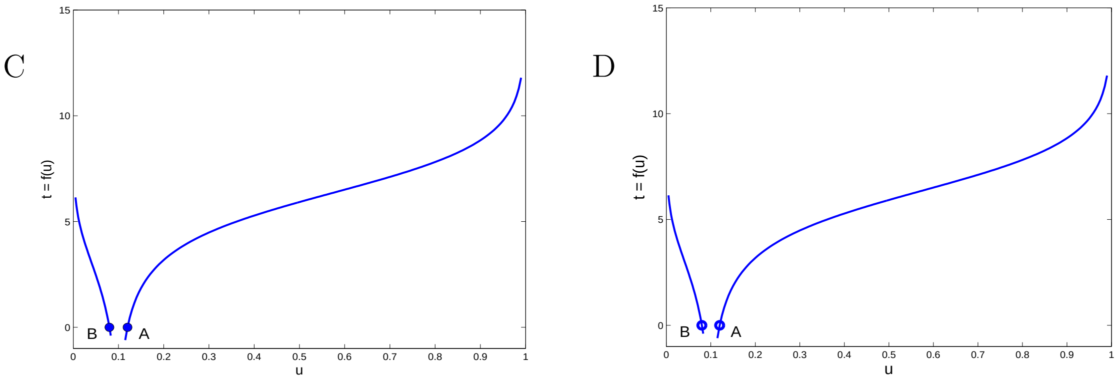


Figure 17.39: **Exercise 17.8.12**. A. The graphs of the inverse of the solution to $u' = u \times \frac{u-\epsilon}{u+\epsilon} \times (1-u)$ with $\epsilon = 0.1$ and $u_0 = 0.12$ (curve A) and $u_0 = 0.08$ (curve B). B. The inverses of the graphs in A, showing u as a function of t.

Exercise 17.9.1.

c. We would let $G(t) = 1.429(50t - 416)$. d. Solve

$$\Omega(0)=1, \quad \frac{d}{dt}\Omega(t)=K\times\left([O_2]-\frac{\Omega(t)}{V}RT\right)-1.429\times 1.28e^{0.45t}$$

on [0,10]. Evaluate $\Omega(10)=\Omega_{10}$. Then solve

$$\Omega(10)=\Omega_{10}, \quad \frac{d}{dt}\Omega(t)=K\times\left([O_2]-\frac{\Omega(t)}{V}RT\right)-1.429(50t-416)$$

on [10,20].

Exercise 17.9.3. $K = 1.82$.

Exercise 17.9.4. $n \times 5730$ years.

Exercise 17.9.5. It appears that 7098 years ago, δ as shown in Figure 17.18 is about 85. Then β^- emission at that time may be

$$\begin{aligned} 85 &= \left(\frac{E(-7098) - E(0)}{E(0)}\right) \times 1000 \\ E(-7098) &= E(0) \times 1.085 = 15.3 \times 1.085 = 16.60. \end{aligned}$$

Then the age $(-t_0)$ of the bone would be found by

$$\begin{aligned} 7 &= 16.6e^{-\frac{\ln 2}{5730}\times(-t_0)} \\ t_0 &= -7138 \end{aligned}$$

The change from -7098 to -7138 is much less than the change from -6464 to -7098 and -7138 is probably within experimental error. -7138 may be accepted as the age of the bone.

Exercise 17.9.6. By inspection of the sine graph in Figure 17.18 it appears that there is a peak at (-7000, 85) and a trough at (-1400,-15). One-half of the period is -1400 - (-7000) = 5600 and the period is 11,200. The amplitude is $(1/2)(85 - (-15)) = 50$. An inflection point of the graph will be at $0.5\{(-7000, 85) + (-1400, -15)\}$ = (-4200, 35) and the graph is decreasing there. We take

$$\delta(t) = 35 - 50\sin(\frac{2*\pi}{11,200}(t + 4200))$$

as an equation of the sine curve. Check:

$$\begin{aligned} \delta(-4200) &= 35 - 50\sin(\frac{2*\pi}{11,200}(-4200 + 4200)) = 35, \\ \delta(-7000) &= 35 - 50\sin(\frac{2*\pi}{11,200}(-7000 + 4200)) = 35 - 50\sin(\frac{-\pi}{2}) = 85, \\ \delta(-1400) &= 35 - 50\sin(\frac{2*\pi}{11,200}(-1400 + 4200)) = 35 - 50\sin(\frac{\pi}{2}) = -15 \end{aligned}$$

$$\delta(t) = \left(\frac{E(t) - E(0)}{E(0)}\right) \times 1000$$

$$\begin{aligned} E(t) &= E(0)(1 + \delta(t)/1000) \\ &= E(0)\left(1.035 - 0.050\sin\left(\frac{2*\pi}{11,200}(t + 4200)\right)\right) \end{aligned}$$

Exercise 17.9.7.

$$\begin{aligned}
E(t) &= 15.3\left(1.035 - 0.050\sin\left(\frac{2*\pi}{11200}(t+4200)\right)\right)\\
E'(t) &= -15.3\times 0.05\times\frac{2*\pi}{11200}\times\cos\left(\frac{2*\pi}{11200}(t+4200)\right)\\
&\geq -15.3\times 0.05\times\frac{2*\pi}{11200}\times 1 = -0.000429.\\
E_{t_0}(t) &= E(t_0)\, e^{-\frac{\ln 2}{5730}(t-t_0)}\\
E'_{t_0}(t) &= E(t_0)\left(-\frac{\ln 2}{5730}\right)\, e^{-\frac{\ln 2}{5730}(t-t_0)}\\
&\leq -15\times\left(\frac{\ln 2}{5730}\right)\, e^{-\frac{\ln 2}{5730}(0-(-10000))}\\
&\leq -0.0018\times 0.298 = -0.00054
\end{aligned}$$

Summary:

$$E'_{t_0}(t) \leq -0.00054 < -0.000429 \leq E'(t)$$

so the slopes can never be equal.

Exercise 17.9.8.

$$\begin{aligned}
\frac{dy(t)}{dt} &= -\hat{K}(y(t))^3\\
\frac{y'}{y^{-3}} &= -\hat{K}\\
\frac{1}{-2}y^{-2} &= -\hat{K}\,t + C\\
\left.\frac{-1}{2y^2}\right|_{t=0} = \frac{-1}{2y_0^2} &= C\\
\frac{-1}{2y^2} &= -\hat{K}\,t - \frac{1}{2y_0^2}\\
\frac{1}{y^2} &= 2\hat{K}\,t + \frac{1}{y_0^2}\\
y^2 &= \frac{1}{2\hat{K}\,t + 1/y_0^2} = \frac{y_0^2}{2y_0^2\hat{K}t+1}
\end{aligned}$$

$$y(t) \quad = \quad \frac{y_0}{\sqrt{2y_0^2\,\hat{K}\,t+1}}$$

Exercise 17.9.9. Reaction 2, Third order.

Exercise 17.9.10.

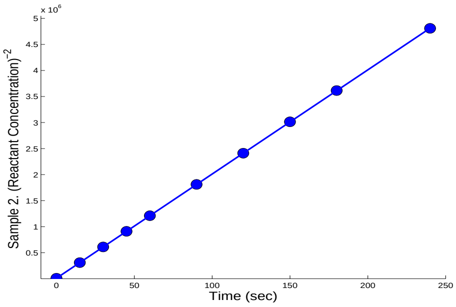


Figure 17.40: **Exercise 17.9.10.** Graph of the data for Reaction 2 with $[B] = 0.4$, plotted as for a third order reaction (m=3) in A, 1/Concentration(A) vs time. The line fitting the data has equation $y = 10000 + 20000t$, so $2\hat{K}_{[B]=0.2} = 5000$ (from above) and $2\hat{K}_{[B]=0.4} = 20000$. $\hat{K}_{[B]=0.4} = 10000 = 4 \times 2500 = 4 \times \hat{K}_{[B]=0.2}$ Thus $2^n = 4$, $n = 2$, the reaction is third order in B, and the reaction is $3A + 2B \longrightarrow A_3B_2$.

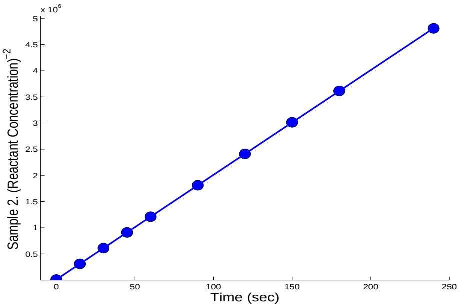


Figure 17.41: **Exercise 17.9.10.** Graph of the data for Reaction 3 with $[B] = 0.4$, plotted as for a first order reaction (m=1) in A, ln Concentration(A) vs time. The line fitting the data has equation $y = -4.6 - 0.02t$, so $\hat{K}_{[B]=0.1} = 0.01$ (from above) and $\hat{K}_{[B]=0.4} = 0.02 = 2 \times 0.01 = 2 \times \hat{K}_{[B]=0.2}$ Thus $2^n = 2$, $n = 1$, the reaction is first order in B, and the reaction is $A + B \longrightarrow AB$.

Chapter 18

Second order and systems of two first order differential equations.

Where are we going?

The goal of this chapter is a theorem that gives conditions under which equilibria of a system of two first order differential equations will be stable. Only stable equilibria are observed in nature. Typical first order systems are the competing species and the predator-prey equations:

$$\begin{array}{ccccccccccc}
 & & \text{Competing} & \text{Species} & & & & & \text{Predator} - \text{prey} & & \\
x'(t) & = & a\,x(t) & - & b\,x(t)y(t) & & x'(t) & = & a\,x(t) & - & b\,x(t)y(t) \\
y'(t) & = & c\,x(t) & - & d\,x(t)y(t) & & y'(t) & = & -c\,x(t) & + & d\,x(t)y(t)
\end{array}$$

In either system, equilibria with positive components signal that both species will persist in the environment. The theorem is similar to Theorem 16.3.1 that applies to systems of difference equations, and the stepping stones leading to the theorems are similar.

18.1 Constant coefficient linear second order differential equations.

A differential equation of the form

$$y(0) = y_0, \qquad y'(0) = y_0', \qquad y''(t) + py'(t) + qy(t) = f(t) \tag{18.1}$$

where p and q are constants and f is continuous is a second order linear constant coefficient differential equation. The equation

$$y(0) = y_0, \qquad y'(0) = y_0', \qquad y''(t) + py'(t) + qy(t) = 0 \tag{18.2}$$

is said to be homogeneous.

An easily understood physical problem leading to Equation 18.1 is that of a mass suspended from a spring whose motion would be described by the homogeneous Equation 18.2 if there is no external applied force and whose motion is described by Equation 18.1 if there is a force $f(t)$ applied to the mass. See Figure 18.1.

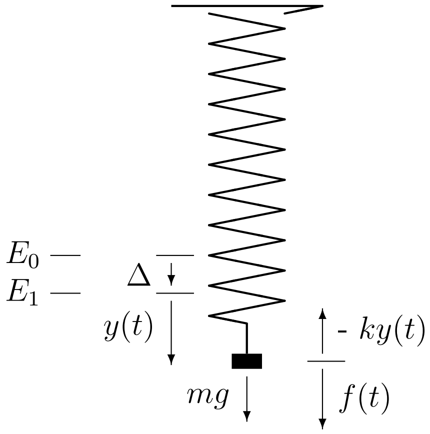


Figure 18.1: Diagram of a mass suspended from a spring and subject to an external force $f(t)$. E_0 is the equilibrium level of the spring with no weight attached. E_1 the equilibrium level of the spring with a weight of mass m attached.

The forces on the mass are the force mg due to gravity, the restoring force $-k(y(t)+\Delta)$ of the spring, and the external force $f(t)$.

The elongation of the spring due to the mass is Δ. At equilibrium E_1 the spring force is $-k\Delta$ and counters the gravitational force mg. There may be resistance in the system which is a force against the direction of motion and it is customary and convenient to assume that

$$\text{the force of resistance} \quad = \quad -r\,y'(t).$$

The forces acting on the mass are

$$mg - k(y(t)+\Delta) + f(t) - r\,y'(t) = -k\,y(t) + f(t) - r\,y'(t),$$

and by Newton's law of motion, $F = ma$,

$$my''(t) = -k\,y(t) + f(t) - ry'(t)$$

or

$$my''(t) + ry'(t) + k\,y(t) = f(t). \tag{18.3}$$

Division by m in Equation 18.3 produces the form of the nonhomogeneous Equation 18.1 with $p = r/m$ and $q = k/m$.

18.1.1 The homogeneous equation.

The homogeneous equation 18.2 is easily solved using methods similar to some used before. We 'guess' that there is a solution of the form $y = e^{rt}$. If so[1] then

$$y(t) = e^{rt}, \qquad y'(t) = r\,e^{rt} \qquad \text{and} \qquad y''(t) = r^2\,e^{rt}$$

Substitution of y, y' and y'' into Equation 18.2 yields

$$\begin{aligned} y''(t) + py'(t) + qy(t) &= 0 \\ r^2\,e^{rt} + p\,r\,e^{rt} + qe^{rt} &= 0 \\ \left(r^2 + p\,r + q\right) e^{rt} &= 0. \end{aligned}$$

Because $e^{rt} \neq 0$, if there is a solution of the form $y = e^{rt}$ to Equation 18.2 we should select r so that

$$r^2 + pr + q = 0. \tag{18.4}$$

Furthermore the previous steps may be reversed. If r is a root to $r^2 + pr + q = 0$, then $y = e^{rt}$ is a solution to $y''(t) + p\,y'(t) + qy(t) = 0$. Equation 18.4 is the *characteristic equation* and it roots are the *characteristic roots* of Equation 18.2.

The characteristic roots are either distinct and real, a repeated real root, or complex conjugate roots. The following theorem is helpful in building solutions to Equation 18.2.

Theorem 18.1.1 Superposition. Suppose $y_1(t)$ and $y_2(t)$ are two solutions to the homogeneous equation $y''(t) + py'(t) + qy(t) = 0$ and C_1 is a number and C_2 is a number. Then

$$y(t) = C_1 y_1(t) + C_2 y_2(t)$$

is a solution to $y''(t) + py'(t) + qy(t) = 0$

Proof. The proof of Theorem 18.1.1 is Exercise 18.1.1.

The solutions to the equations

$$y'(t) + p\,y'(t) + 2y(t) = 0, \quad \text{for} \quad p \quad = \quad 8, \quad 4, \quad \text{and} \quad 2. \tag{18.5}$$

with $y(0) = 1$, $y'(0) = 2$ in each are shown in Figure 18.2. They illustrate the transition from distinct real roots ($p = 8$) to a repeated real root ($p = 4$) to complex roots ($p = 2$) as the coefficient of $y'(t)$ (resistance to motion in Equation 18.3) decreases.

---

[1] You should suppose this to be a good guess or else we would not suggest it to you. It is motivated by the exponential solutions to the first order differential equations of Chapter 17 and hope (and the fact that it has been known to work for some 250 years).

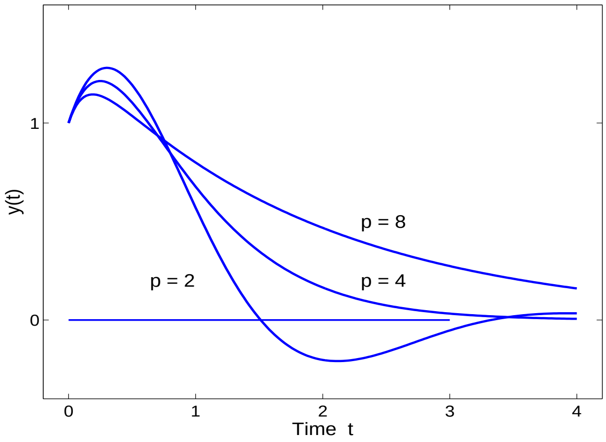


Figure 18.2: Graphs solutions to $y''+py'+4y=0$, $y(0)=1$, and $y'(0)=2$ for $p=8$ (distinct real negative roots), $p=4$ (a negative repeated root), and $p=2$ (complex roots).

By Theorem 18.1.1, if the characteristic roots of Equation 18.2 are distinct and real, r_1 and r_2, then for any number C_1 and number C_2

$$y(t)=C_1e^{r_1t}+C_2e^{r_2t} \tag{18.6}$$

is a solution to the homogeneous Equation 18.2. By using the initial conditions, $y(0)=y_0$ and $y'(0)=y_0'$, C_1 and C_2 can be evaluated. For

$$y(t)=C_1e^{r_1t}+C_2e^{r_2t}, \qquad y'(t)=C_1r_1e^{r_1t}+C_2r_2e^{r_2t}.$$

Then

$$\begin{array}{rclrcl} y(0) & = & C_1e^{r_1\times 0}+C_2e^{r_2\times 0} & C_1+C_2 & = & y_0 \\ y'(0) & = & C_1r_1e^{r_1\times 0}+C_2r_2e^{r_2\times 0} & C_1r_1+C_2r_2 & = & y_0' \end{array}$$

These equations may be solved for C_1 andC_2.

$$C_1=\frac{y_0'-r_2y_0}{r_1-r_2} \qquad C_2=\frac{r_1y_0-y_0'}{r_1-r_2} \tag{18.7}$$

A graph of the solution of Equation 18.5 with $p=8$, $r_1=-4+2\sqrt{3}$, $r_2=-4-2\sqrt{3}$,

$$y(t)=\frac{\sqrt{3}+1}{2}\exp((-4+2\sqrt{3})t)+\frac{\sqrt{3}-1}{2}\exp((-4-2\sqrt{3})t),$$

is shown in Figure 18.2.

If there is only one characteristic root r_1 of Equation 18.2 ($p^2-4q=0$) then both $y=e^{r_1t}$ and $y=t\,e^{r_1t}$ are solutions to Equation 18.2. (Exercise 18.1.2). Furthermore, for any numbers C_1 and C_2

$$y(t)=C_1e^{r_1t}+C_2te^{r_1t} \tag{18.8}$$

is a solution to the homogeneous Equation 18.2. As for distinct roots, C_1 and C_2 can be found using the initial conditions.

$$C_1=y_0 \qquad C_2=y_0'-r_1y_0 \tag{18.9}$$

A graph of $y(t) = e^{-2t} + 4te^{-2t}$, the solution of Equation 18.5 with $p = 4$, repeated root $r = -2$, is shown in Figure 18.2.

In the case of complex roots ($p^2 - 4q < 0$), r_1 and r_2 will be complex numbers $a + bi$ and $a - bi$ respectively, where $i = \sqrt{-1}$. The algebra and calculus of complex functions conform to the laws of exponents and derivative formulas of real valued functions, and the argument that Equation 18.6 solves the homogeneous equation has exactly the same steps in both real and complex arithmetic. We need to probe $e^{(a+bi)t}$ more deeply, however. It may stretch your credibility but we will define (actually Leonhard Euler found it in 1748 using Taylor's polynomials; see Exercise 9.4.6 of Volume I)

$$e^{(a+bi)t} \quad = \quad e^{at}\, e^{bt\, i} = e^{at}\cos bt + ie^{at}\sin bt.$$

What is really important to us is that each of the real and imaginary components of $e^{(a+bi)t}$ is a solution to the homogeneous equation. That is

$$y_1(t) = e^{at}\cos bt \quad \text{and} \quad y_2(t) = e^{at}\sin bt \quad \text{each solve}$$

$$y''(t) + py'(t) + qy(t) = 0.$$

We will show that $y = e^{at}\cos bt$ solves $y''(t) + py'(t) + qy(t) = 0$ when $a + bi$ is a root to $r^2 + pr + q = 0$.

$$\begin{aligned} y(t) &= e^{at}\cos bt \\ \\ y'(t) &= ae^{at}\cos bt - be^{at}\sin bt \\ \\ y''(t) &= a^2e^{at}\cos bt - 2abe^{at}\sin bt - b^2e^{at}\cos bt \end{aligned}$$

Substitute these functions into $y''(t) + py'(t) + qy(t) = 0$ and collect similar terms.

$$\begin{aligned} y''(t) + py'(t) + qy(t) &= a^2e^{at}\cos bt - 2abe^{at}\sin bt - b^2e^{at}\cos bt \\ &\qquad + p\left(ae^{at}\cos bt - be^{at}\sin bt\right) + qe^{at}\cos bt \\ \\ &= (a^2 - b^2 + pa + q)e^{at}\cos bt \\ &\qquad + (-2ab - pb)e^{at}\sin bt \end{aligned} \tag{18.10}$$

We need to show that the last expression is zero and the prospects are grim. Recall, however, that $a + bi$ is a root to $r^2 + pr + q = 0$. We use a little complex arithmetic.

$$\begin{aligned} r^2 + pr + q &= 0 \\ (a + bi)^2 + p(a + bi) + q &= 0 \\ a^2 + 2abi + b^2i^2 + pa + pbi + q &= 0 \qquad\qquad i^2 = -1 \\ (a^2 - b^2 + pa + q) + (2ab + pb)i &= 0 \end{aligned}$$

Now we got it! In order for a complex number to be zero, both the real and imaginary components must be zero. Therefore $a^2 - b^2 - pa + q = 0$ and $2ab + pb = 0$, expression 18.10 $= 0$, and $y(t) = e^{at}\cos bt$ is a solution to $y''(t) + py'(t) + qy(t) = 0$. The argument that $y(t) = e^{at}\sin bt$ is a solution to $y''(t) + py'(t) + qy(t) = 0$ is similar to the preceding steps and is omitted.

By Theorem 18.1.1, if $a + bi$ is a root to $r^2 + pr + q = 0$ and C_1 and C_2 are numbers then

$$y(t) = C_1 e^{at}\cos bt + C_2 e^{at}\sin bt \tag{18.11}$$

is a solution to $y''(t) - py'(t) + qy(t) = 0$. Using the initial conditions $y(0) = y_0$ and $y'(0) = y_0'$, C_1 and C_2 are determined.

$$C_1 = y_0 \qquad C_2 = \frac{y_0' - ay_0}{b} \tag{18.12}$$

A graph of $y(t) = e^{-t}\cos\sqrt{3}t + \sqrt{3}e^{-t}\sin\sqrt{3}t$, the solution of Equation 18.5 with $p = 2$, roots $= -1 \pm \sqrt{3}\,i$, is shown in Figure 18.2.

You can now solve every second order linear constant coefficient homogeneous differential equation with initial conditions. We have not proved that the solutions are unique, but they are unique – there are no other solutions. Please accept the uniqueness as a fact; its proof is beyond our scope. It is important that in each of the cases, distinct real roots, repeated real root, and complex roots $a \pm bi$, two solutions y_1 and y_2 of the homogeneous equation were found that are *linearly independent.*

Definition 18.1.1 Linear Independence of functions. Two functions y_1 and y_2 are linearly independent means that if C_1 is a number and C_2 is a number and

$$C_1 y_1(t) + C_2 y_2(t) \equiv 0$$

then $C_1 = C_2 = 0$. The assertion $C_1 y_1(t) + C_2 y_2(t) \equiv 0$ means that

$$\text{for all numbers } t, \qquad C_1 y_1(t) + C_2 y_2(t) = 0.$$

For example, in the case of distinct real roots, $y_1(t) = e^{r_1 t}$ and $y_2(t) = e^{r_2 t}$ are linearly independent. Suppose not, and that there are numbers C_1 and C_2 and

$$C_1 e^{r_1 t} + C_2 e^{r_2 t} \equiv 0$$

$$\text{Because} \quad C_1 e^{r_1 t} + C_2 e^{r_2 t} \equiv 0, \quad \left[r_1 C_1 e^{r_1 t} + r_2 C_2 e^{r_2 t}\right]' \equiv 0.$$

$$\text{Then} \quad C_1 e^{r_1 t} + C_2 e^{r_2 t} \equiv 0, \qquad \text{and} \quad r_1 C_1 e^{r_1 t} + r_2 C_2 e^{r_2 t} \equiv 0.$$

If we evaluate $C_1 e^{r_1 t} + C_2 e^{r_2 t} \equiv 0$ and $r_1 C_1 e^{r_1 t} + r_2 C_2 e^{r_2 t} \equiv 0$ at $t = 0$ we find that

$$\begin{aligned} C_1 + C_2 &= 0 \\ r_1 C_1 + r_2 C_2 &= 0 \end{aligned}$$

Because $r_1 \neq r_2$ the equations imply that $C_1 = C_2 = 0$.

You may choose to move directly to Section 18.2; that section and the material following it does not require the material in Subsection 18.1.2.

18.1.2 The nonhomogeneous equation.

There are three steps to solving the nonhomogeneous Equation 18.1

$$y(0) = y_0, \qquad y'(0) = y_0', \qquad y''(t) + py'(t) + qy(t) = f(t)$$

Step 1. Solve the homogeneous equation

$$y''(t) + py'(t) + qy(t) = 0$$

as in Subsection 18.1.1. That is, find $y_1(t)$ and $y_2(t)$ such that every solution to the homogeneous equation is of the form $C_1 y_1(t) + C_2 y_2(t)$.
Step 2. Find a *particular* solution $y_p(t)$ of the nonhomogeneous equation (ignore initial conditions for this step).
Step 3. $y_g(t) = y_p(t) + C_1 y_1(t) + C_2 y_2(t)$ is a *general* solution to the nonhomogeneous equation. Use the initial conditions $y(0) = y_0$ and $y'(0) = y_0'$ to evaluate C_1 and C_2. Specifically, solve

$$\begin{aligned} y(0) &= y_p(0) + C_1 y_1(0) + C_2 y_2(0) &= y_0 \\ y'(0) &= y_p'(0) + C_1 y_1'(0) + C_2 y_2'(0) &= y_0' \end{aligned}$$

for C_1 and C_2.

Step 2 is the new step. Step 1 is Subsection 18.1.1. That $y_g(t)$ in Step 3 solves the nonhomogeneous equation needs to be checked (Exercise 18.1.3). Important to Step 3 is to involve both $y_p(0)$ and $y_p'(0)$ in evaluating C_1 and C_2.

An approach to Step 2. Equation 18.1 is widely useful and a number of techniques to find a particular solution have been developed. We present only the most obvious which are based on the **assumption that $\mathbf{y_p(t)}$ should be similar to $\mathbf{f(t)}$**. If, for example, $f(t)$ is a polynomial, an exponential function or a sine function, then $y_p(t)$ should be, respectively, a polynomial, an exponential function or a sine function. This works well unless it happens that $f(t)$ is itself a solution to the homogeneous equation in which case a modification is required. Several examples will show the technique.

Problem. Find a solution $y_p(t)$ to

$$y''(t) - 3y'(t) + 2y(t) = t^2$$

Solution. Because $f(t) = t^2$ we guess that $y_p(t) = a_2t^2 + a_1t + a_0$. Even though $f(t)$ has only t^2, we need the t and constant terms in $y_p(t)$. Substitute:

$$\begin{aligned} \left[a_2t^2 + a_1t + a_0 \right]'' - 3\left[a_2t^2 + a_1t + a_0 \right]' + 2(\, a_2t^2 + a_1t + a_0 \,) &= t^2 \\ \\ 2a_2 - 3(2a_2t + a_1) + 2(a_2t^2 + a_1t + a_0) &= t^2 \\ \\ 2a_2t^2 + (-6a_2 + 2a_1)t + (2a_2 - 3a_1 + 2a_0) &= t^2 \end{aligned}$$

We now equate the polynomial coefficients and write

$$\begin{array}{rcrcrcl} 2a_2 & & & & & = & 1 \\ -6a_2 & + & 2a_1 & & & = & 0 \\ 2a_2 & - & 3a_1 & + & 2a_0 & = & 0 \end{array}$$

Then $a_2 = 1/2$, $a_1 = 3/2$ and $a_0 = 7/4$, and $y_p(t) = (1/2)t^2 + (3/2)t - 7/4$ is a particular solution to $y''(t) - 3y'(t) + 2y(t) = t^2$.

Problem. Find a solution $y_p(t)$ to

$$y''(t) - 3y'(t) + 2y(t) = e^{-t}$$

Solution. Try $y_p(t) = A\,e^{-t}$. Substitute

$$\begin{aligned} \left[A\,e^{-t}\right]'' - 3\left[A\,e^{-t}\right]' + A\,e^{-t} &= e^{-t} \\ A\,e^{-t} + 3A\,e^{-t} + 2A\,e^{-t} &= e^{-t} \\ 6A\,e^{-t} &= e^{-t} \end{aligned}$$

Then $A = 1/6$ and $y_p(t) = (1/6)e^{-t}$ is a particular solution to $y''(t) - 3y'(t) + 2y(t) = e^{-t}$.

Problem. Find a solution $y_p(t)$ to

$$y''(t) - 3\,y'(t) + 2\,y(t) = 5\,t^2 + 4\,e^{-t}$$

Solution. The particular solution should be the sum of a polynomial and an exponential function, of the form

$$y_p(t) = a_2\,t^2 + a_1\,t + a_0 + C\,e^{-t}.$$

Fortunately the solution is

$$y_p(t) = 5((1/2)\,t^2 + (3/2)\,t - 7/4) + 4\,((1/6)e^{-t}),$$

a predictable linear combination of the solutions to the previous two problems. See Exercise 18.1.4.

Theorem 18.1.2 If $y_{p,1}(t)$ solves $y''(t) - p\,y'(t) + q\,y(t) = f_1(t)$ and $y_{p,2}(t)$ solves $y''(t) - p\,y'(t) + q\,y(t) = f_2(t)$, then for any numbers A and B

$$A\,y_{p,1}(t) + B\,y_{p,2} \quad \text{solves} \quad y''(t) - p\,y'(t) + q\,y(t) = A\,f_1(t) + B\,f_2(t).$$

Proof. Exercise 18.1.4.

Problem. Find a solution $y_p(t)$ to

$$y''(t) - 3\,y'(t) + 2\,y(t) = e^t$$

Solution. Try $y_p(t) = Ae^t$. Substitute

$$\begin{aligned}\left[A\,e^t\right]'' - 3\left[A\,e^t\right]' + A\,e^t &= e^t \\ Ae^t - 3\,A\,e^t + 2\,A\,e^t &= e^t \\ 0 \times A\,e^t &= e^t\end{aligned}$$

Ugh! No such A. The problem is that $f(t) = e^t$ is a solution to the homogeneous equation $y''(t) - 3\,y'(t) + 2\,y(t) = 0$.

Try again. Try $y_p(t) = A \times t \times e^t$. Why? We would not suggest it if it will not work. Substitute:

$$\begin{aligned}\left[A \times t \times e^t\right]'' - 3\left[A \times t \times e^t\right]' + A \times t \times e^t &= e^t \\ A(te^t + 2e^t) - 3A(te^t + e^t) + 2Ate^t &= e^t \\ (A - 3A + 2A)te^t + (2A - 3A)e^t &= e^t\end{aligned}$$

Choose $A = -1$ and $y_p(t) = -t \times e^t$.

Problem. Find a solution $y_p(t)$ to

$$y''(t) - 3\,y'(t) + 2\,y(t) = \sin\omega t \tag{18.13}$$

Solution. Try $y_p(t) = A\cos\omega t + B\sin\omega t$. (Both the sine and cosine terms are needed.) Substitute:

$$\begin{aligned}[\,A\cos\omega t + B\sin\omega t\,]'' - 3\,[\,A\cos\omega t + B\sin\omega t\,]'& \\ +2(A\cos\omega t + B\sin\omega t) &= \sin\omega t \\ -A\omega^2\cos\omega t - B\omega^2\sin\omega t - 3(-A\omega\sin\omega t + B\omega\cos\omega t)& \\ +2(A\cos\omega t + B\sin\omega t) &= \sin\omega t \\ (-A\omega^2 - 3B\omega + 2A)\cos\omega t& \\ +(-B\omega^2 + 3A\omega + 2B)\sin\omega t &= \sin\omega t\end{aligned}$$

Match coefficients. Choose

$$-A\omega^2 - 3B\omega + 2A = 0 \qquad \text{and} \qquad -B\omega^2 + 3A\omega + 2B = 1$$

$$\begin{aligned}(2-\omega^2)A - 3\omega B &= 0 \\ 3\omega A + (2-\omega^2)B &= 1\end{aligned}$$

$$A = \frac{3\omega}{(2-\omega^2)^2 + 9\omega^2} \qquad \text{and} \qquad B = \frac{2-\omega^2}{(2-\omega^2)^2+9\omega^2}$$

$$y_p(t) = \frac{3\omega}{(2-\omega^2)^2 + 9\omega^2}\cos\omega t + \frac{2-\omega^2}{(2-\omega^2)^2+9\omega^2}\sin\omega t \tag{18.14}$$

Explore 18.1.1 Show that for the special case, $\omega = 1$, Equation 18.14 defines a solution to Equation 18.13 (with $\omega = 1$). ■

In the case that $\sin\omega t$ solves the homogeneous equation $y''(t) + p\,y'(t) + q\,y(t) = 0$, $y_p(t) = A\,\cos\omega t + B\,\sin\omega t$ will not solve the nonhomogeneous equation $y''(t) + py'(t) + qy(t) = \sin\omega t$, but $y_p(t) = t\,\times(A\,\cos\omega t + B\,\sin\omega t)$ will solve the nonhomogeneous equation. A summary of selections of $y_p(t)$ appears in Table 18.1.

Table 18.1: Selection of particular solutions $y_p(t)$ to the nonhomogeneous equation $y''(t) - py'(t) + qy(t) = f(t)$ for certain forms of $f(t)$.

$f(t)$	$y_p(t)$
$a_0 + a_1 t + \cdots a_n t^n$	$b_0 + b_1 t + \cdots b_n t^n$
e^{mt}	e^{mt} or te^{mt} or $t^2 e^{mt}$
$\cos\omega t$ or $\sin\omega t$	$A\cos\omega t + B\sin\omega t$ or $t\times(A\cos\omega t + B\sin\omega t)$
$e^{mt}\cos\omega t$ or $e^{mt}\sin\omega t$	$e^{mt}(A\cos\omega t + B\sin\omega t)$ or $t\times e^{mt}(A\cos\omega t + B\sin\omega t)$

Exercises for Section 18.1, Constant coefficient linear second order differential equations.

Exercise 18.1.1 Show by substitution that if $y_1(t)$ and $y_2(t)$ are solutions to $y''(t) + p\,y'(t) + q\,y(t) = 0$ and each of C_1 and C_2 is a number then

$$y(t) = C_1\,y_1(t) + C_2\,y_2(t)$$

is a solution to $y''(t) - p\,y'(t) + q\,y(t) = 0$.

Exercise 18.1.2 Show by substitution that if r_1 is the only root of $r^2 + p\,r + q = 0$ then $y = te^{r_1 t}$ is a solution to $y''(t) + p\,y'(t) + q\,y(t) = 0$. Note: If r_1 is the only root to $r^2 + p\,r + q = 0$ then $p^2 - 4q = 0$ and $r_1 = -p/2$.

Exercise 18.1.3 Show that if $y_h(t)$ solves $y''(t) + p\,y'(t) + q\,y(t) = 0$ and $y_p(t)$ solves $y''(t) + p\,y'(t) + q\,y(t) = f(t)$ then for any number C, $y_g(t) = y_p(t) + Cy_h(t)$ solves $y''(t) + p\,y'(t) + q\,y(t) = f(t)$.

Exercise 18.1.4 Show that if $y_{p,1}(t)$ solves $y''(t) + p\,y'(t) + q\,y(t) = f_1(t)$ and $y_{p,2}(t)$ solves $y''(t) + p\,y'(t) + q\,y(t) = f_2(t)$, then for any numbers A and B

$$A\,y_{p,1}(t) + B\,y_{p,2} \quad \text{solves} \quad y''(t) + p\,y'(t) + q\,y(t) = A\,f_1(t) + B\,f_2(t).$$

Exercise 18.1.5 Compute and graph the solutions to

$$y'' + p\,y' + y \;=\; 0 \qquad y(0) \;=\; 1 \qquad y'(0) \;=\; 0$$

for $p = 4$, $p = 2$, $p = 1$, $p = 0$, and $p = -2$.

Exercise 18.1.6 Compute and graph the solutions to

$$y'' + p\,y' + 9\,y \;=\; 0 \qquad y(0) \;=\; 0 \qquad y'(0) \;=\; 1$$

for $p = 10$, $p = 6$, $p = 4$, $p = 0$, and $p = -6$.

Exercise 18.1.7 Compute and graph the solution to

$$y'' + p\,y' + q\,y \;=\; 0 \qquad y(0) \;=\; 0 \qquad y'(0) \;=\; 0.$$

Exercise 18.1.8 Compute $y_h(t)$, the general solution to the homogeneous equation, and $y_p(t)$ a particular solution to the nonhomogeneous equation.

$$\begin{array}{llll} \text{a.} & y'' + 4\,y' + 3\,y = t & \text{b.} & y'' + 4\,y' + 3\,y = e^t \\[2ex] \text{c.} & y'' - 4\,y' + 3\,y = e^t & \text{d.} & y'' + 4\,y' + 3\,y = \cos t \end{array}$$

Exercise 18.1.9 Show that the for m, r, and k positive, the characteristic roots

$$r_1 = \frac{-r + \sqrt{r^2 - 4mk}}{2m} \qquad \text{and} \qquad r_2 = \frac{-r - \sqrt{r^2 - 4mk}}{2m}$$

for $my''(t) + r\,y'(t) + k\,y(t) = 0$ are both negative or have negative real parts.

Exercise 18.1.10 Consider a special case of a spring-mass system in which there is no resistance and with a a *harmonic forcing function*, $f(t) = \cos\omega t$. Thus examine

$$my''(t) + ky(t) = \cos\omega t \tag{18.15}$$

Let $\omega_0 = \sqrt{k/m}$.

It is routine to show that **if** $\omega \neq \omega_0$, the general solution to $my''(t) + ky(t) = \cos\omega t$ is

$$y(t) = \frac{1}{m(\omega_0^2 - \omega^2)}\cos\omega t + C_1\,\cos\omega_0 t + C_2\,\sin\omega_0 t.$$

a. Suppose that the mass is initially at rest so that $y(0) = 0$, and $y'(0) = 0$ and $\omega \neq \omega_0$. Show that the motion of the mass is approximated by

$$\begin{aligned} y(t) &= \frac{1}{m(\omega_0^2 - \omega^2)} \cos \omega t - \frac{1}{m(\omega_0^2 - \omega^2)} \cos \omega_0 t \\ &= \frac{1}{m(\omega_0^2 - \omega^2)} \left(\cos \omega t - \cos \omega_0 t \right). \end{aligned} \tag{18.16}$$

b. Sketch the graph of $y(t)$ in Equation 18.16 for the case $m = 1$, $\omega = 1$ and $k = 0.01$ (weak spring) ($\omega_0 = 0.1$). The impressed force $\cos t$ appears as the rapid oscillations, and the inherent system frequency appears as the overall gradual wave due to the term $\cos \omega_0 t = \cos 0.1t$

```
close all;clc;clear
t=[0:0.1:200];  m=1;  w=1; k=0.01; w0=sqrt{k/m};
fraction=1/(m*w0^2-w^2);
y=fraction*(cos(w0*t)-cos(w*t));
plot(t,y,'linewidth',2)
xlabel('Time','fontsize',16)
ylabel('Displacement','fontsize',16)
```

c. Sketch the graph of $y(t)$ in Equation 18.16 for the case $m = 1$, $\omega = 1$ and $k = 0.81$ (stiff spring) ($\omega_0 = 0.9$).

From the identity, $\cos x - \cos y = 2 \sin \frac{x+y}{2} \sin \frac{x-y}{2}$,

$$y(t) = \frac{1}{m(\omega_0^2 - \omega^2)} 2 \sin(\frac{\omega_0 + \omega}{2} t) \sin(\frac{\omega_0 - \omega}{2} t)$$

The amplitude of the rapid vibrations, $\sin(\frac{\omega_0+\omega}{2} t)$ is

$$\frac{1}{m(\omega_0^2 - \omega^2)} 2 \sin(\frac{\omega_0 - \omega}{2} t)$$

and results in a beat of frequency $4\pi/(\omega_0 - \omega)$(that may be heard in mechanical systems.

Exercise 18.1.11 *Resonance.* Consider the special case of Equation 18.15

$$m\, y''(t) + k\, y(t) = \cos \omega t, \quad \text{for which} \quad \omega = \sqrt{k/m} = \omega_0; \tag{18.17}$$

the frequency of the impressed force is equal to the inherent frequency of the spring-mass system. (The case $\omega = \omega_0$ of Exercise 18.1.10.)

a. Show that a particular solution to Equation 18.17 is

$$y_p(t) = \frac{1}{2m\omega_0} \times t \times \sin \omega_0 t.$$

Accept without proof that the general solution is

$$y(t) = C_1 \cos \omega_0 t + C_2 \sin \omega_0 t + \frac{1}{2m\omega_0} \times t \times \sin \omega_0 t.$$

b. Suppose that $y(0) = 0$ and $y'(0) = 0$. Show that

$$y(t) = \frac{1}{2m\omega_0} \times t \times \sin \omega_0 t$$

c. What happens to the mechanical system?

Exercise 18.1.12 Consider the equations,

$$y'' + 2\,y' + 2\,y = 5\,\cos t, \quad y(0) = 0, \quad y'(0) = 0.$$

a. Show that $y_h(t) = C_1\,e^{-t}\,\cos t + C_2\,e^{-t}\,\sin t$.

b. Show that $y_p(t) = \cos t + 2\sin t$.

c. Show that $y_g(t) = \cos t = 2\,\sin t + C_1\,e^{-t}\,\cos t + C_2\,e^{-t}\,\sin t$.

d. Evaluate C_1 and C_2.

Exercise 18.1.13 Show that

$$\text{Solution:}\quad y(t) = -\frac{50}{\sqrt{0.99}} e^{-0.1t} \sin \sqrt{0.99}t + 50\sin t \tag{18.18}$$

is the solution to

$$y(0) = 0, \quad y'(0) = 0, \quad y''(t) + 0.2y'(t) + y(t) = 10\cos t \tag{18.19}$$

Graph the solution. Note that the magnitude of the vibrations are five times those of the forcing function, $10\cos t$

```
close all;clc;clear
t=[0:0.1:40];
forcing=10*cos(t);
plot(t,forcing,'r','linewidth',2)
hold
y=-(50/sqrt(0.99))*exp(-0.1*t).*sin(sqrt(0.99)*t)+50*sin(t);
plot(t,y,'linewidth',2)
axis([0 40 -55 55])
```

Exercise 18.1.14 Consider now the case that there is resistance $r > 0$ in the spring-mass equation with a harmonic forcing function,

$$m\,y''(t) + r\,y'(t) + k\,y(t) = \cos \omega t. \tag{18.20}$$

The next few exercises examine the importance of y_p, the particular solution of Equation 18.20. We will need

$$\begin{aligned}
y_p &= A\cos 0.3\,t + B\sin 0.3\,t && \text{and one of}\\
y_h &= C_1\,e^{\mu_1\,t} + C_2\,e^{\mu_2\,t}, && \text{real roots,}\\
y_h &= C_1\,e^{\mu\,t} + C_2\,t\,e^{\mu\,t}, && \text{repeated root, and}\\
y_h &= e^{\mu\,t}\,(\,C_1\,\cos(\omega\,t) + C_2\,\sin(\omega\,t)) && \text{complex roots.}
\end{aligned}$$

a. Show that in the real root case, μ_1 and μ_2 are negative.

b. Show that in the repeated root case, the value of μ is negative.

c. Show that in the and complex roots case, the value of μ is negative.

d. Show that in all of the cases,

$$\lim_{t\to\infty} y_h(t) = 0$$

Exercise 18.1.15 Find the particular solution, $y_p = A\cos 0.3\,t + B\sin 0.3\,t$, and y_h, the solution to the homogeneous equation, for the systems below.

$$\text{a.}\qquad y'' + 2y' + 2y = \cos 0.3\,t$$

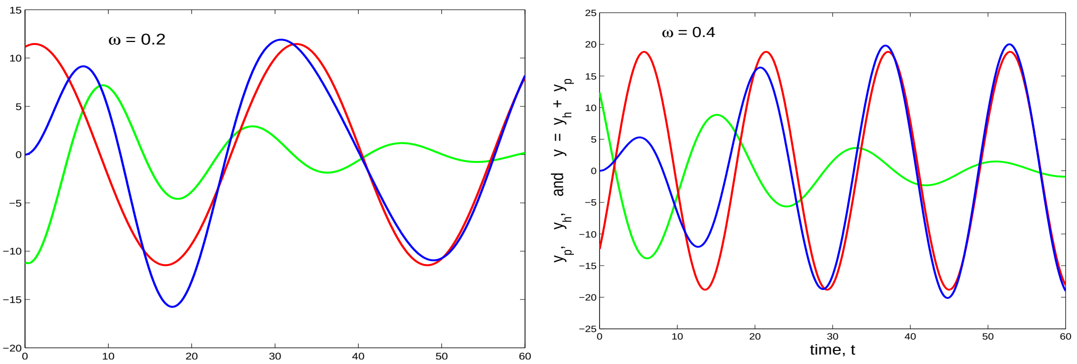
$$\text{b.}\qquad y'' + 0.1y' + 0.125y = \cos 0.3\,t$$

$$\text{c.}\qquad y'' + 0.05y' + 0.010625y = \cos 0.3\,t$$

$$\text{d.}\qquad y'' + 0.04y' + 0.0104y = \cos 0.3\,t$$

The algebra to solve these problems is extensive and you will find the MATLAB code below helpful at least to read and at best to run. It is written for the case of complex roots, and the system is initially at rest: $y(0) = 0$, $y'(0) = 0$. NOTE: $\omega \neq 0.3$ in the code.

```
close all;clear;clc
%% Solution to m y" + r y' + k y = cos(w t)
%%              y(0) = y'(0) = 0       Complex roots
m=1.0;r=0.1;k=0.125; w=0.4;
Z=k-m*w^2;  RW=r*w;  D= Z^2+RW^2;
A=Z/D
B=RW/D
t=[0:0.05:60];
Yp=A*cos(w*t)+B*sin(w*t);
mu = -r/(2*m)
w0= sqrt(4*k-r^2)/2
C1 = -A;  C2 = (mu*A - w*B)/w0;
Yh=exp(mu*t).*(C1*cos(w0*t) + C2*sin(w0*t));
Y=Yh+Yp;
plot(t,Yh,'g','linewidth',2)
hold
plot(t,Yp,'r','linewidth',2)
plot(t,Y,'linewidth',2)
text(8,22,'\omega = 0.4','fontsize',16)
```



Solutions to $y''+0.1y'+0.125y = \cos \omega\, t$, $y(0) = 0$, $y'(0) = 0$, for $\omega = 0.2$ and 0.4. The blue curves are the solution; the green curves are the solutions to the homogeneous equation, $y'' + 0.1y' + 0.125y = 0$, and the red curves are the particular solutions. The solution curves (blue) are asymptotic to the particular solutions (red).

Exercise 18.1.16 Find the particular solution, $y_p = A\cos 0.3\,t + B\sin 0.3\,t$, and y_h, the solution to the homogeneous equation, for the systems below.

a. $y'' + 3y' + 2y = \cos 0.3\,t$

b. $y'' + 0.1y' + 0.12y = \cos 0.3\,t$

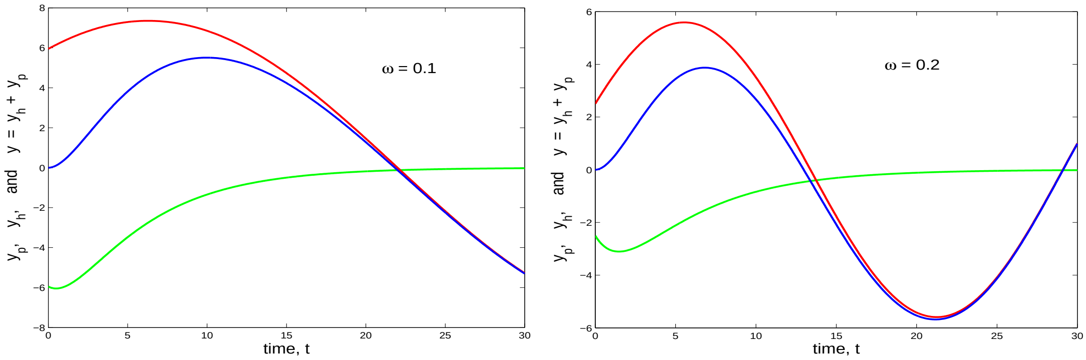
c. $y'' + 0.05y' + 0.0004y = \cos 0.3\,t$

d. $y'' + 0.13y' + 0.0036y = \cos 0.3\,t$

The algebra to solve these problems is extensive and you will find the MATLAB code below helpful at least to read and at best to run. It is written for the case of distinct real roots, and the system is initially at rest: $y(0) = 0$, $y'(0) = 0$. NOTE: $\omega \neq 0.3$ in the code.

```
close all;clear;clc
%% Solution to m y" + r y’ + k y = cos(w t)
%%             y(0) = y’(0) = 0      Distinct real roots.
m=1.0;r=0.1;k=0.12; w=0.2;
Z=k-m*w^2;  RW=r*w;  D= Z^2+RW^2;
A=Z/D
B=RW/D
t=[0:0.05:30];
Yp=A*cos(w*t)+B*sin(w*t);
mu1=(-r+sqrt(r^2-4*k))/2;
mu2=(-r-sqrt(r^2-4*k))/2;
C1 = (mu2*A-B*w)/(mu1-mu2);
C2 = (B*w-mu1*A)/(mu1-mu2)
Yh=C1*exp(mu1*t)+C2*exp(mu2*t);
Y=Yh+Yp;
plot(t,Yh,’g’,’linewidth’,2)
hold
plot(t,Yp,’r’,’linewidth’,2)
plot(t,Y,’linewidth’,2)
```

```
text(8,22,'\omega = 0.4','fontsize',16)
```



Solutions to $y''+0.1y'+0.12y = \cos\omega\, t$, $y(0) = 0$, $y'(0) = 0$, for $\omega = 0.1$ and 0.2. The blue curves are the solution; the green curves are the solutions to the homogeneous equation, $y'' + 0.1y' + 0.125y = 0$, and the red curves are the particular solutions. The solution curves (blue) are asymptotic to the particular solutions (red).

Exercise 18.1.17 The particular solution, y_p, of Equation 18.20 is of the form

$$y_p(t) = A\cos\omega t + B\sin\omega t. \tag{18.21}$$

Ugh! Your choice is to believe us or work it out yourself,

$$A = \frac{k - m\omega^2}{(k-m\omega^2)^2 + r^2\omega^2} \qquad \text{and} \qquad B = \frac{r\omega}{(k-m\omega^2)^2 + r^2\omega^2}.$$

a. Show that

$$A^2 + B^2 = \frac{1}{(k-m\omega^2)^2 + r^2\omega^2}.$$

b. Let ϕ be defined by

$$\phi = \arccos\,\frac{k - m\omega^2}{\sqrt{(k-m\omega^2)^2 + r^2\omega^2}}$$

Show that

$$\sin\phi = \frac{r\omega}{\sqrt{(k-m\omega^2)^2 + r^2\omega^2}}$$

c. Under what condition will $\phi = 0$?

d. Show that

$$\begin{aligned} y_p(t) &= \frac{1}{\sqrt{(k-m\omega^2)^2 + r^2\omega^2}}\,(\,\cos\phi\,\cos\omega t + \sin\phi\sin\omega t\,) \\ &= \frac{1}{\sqrt{(k-m\omega^2)^2 + r^2\omega^2}}\left(\cos(\,\omega(t - \frac{\phi}{\omega}\,)\right). \end{aligned} \tag{18.22}$$

e. **Conclusion.** Because $\lim_{h\to\infty} y_h(t) = 0$, $y(t)$ is asymptotic to $y_p(t)$. $y_p(t)$ is proportional to the forcing function, $\cos\omega t$ with a time lag of ϕ/ω. Because of resistance, the response of the system is asymptotic to a multiple of the forcing function with a time lag of ϕ/ω.

18.2 Stability and asymptotic stability of equilibria of pairs of autonomous differential equations.

Henceforth our interest will be pairs of differential equations of the form

$$\begin{array}{rclcrcl} x(0) & = & x_0 & \quad & x'(t) & = & f(x,y) \\ \\ y(0) & = & y_0 & \quad & y'(t) & = & g(x,y), \end{array} \tag{18.23}$$

We will assume that f and g are continuous and have continuous first and second partial derivatives with respect to both variables. This is sufficient to insure that for any initial values $x(0) = x_0$ and $y(0) = y_0$, Equations 18.23 have unique solutions $x(t)$ and $y(t)$ in a neighborhood of (x_0, y_0) and $t = 0$. The equations are said to be *autonomous* because $f(x,y)$ and $g(x,y)$ are independent of t. The homogeneous Equation 18.2, $y'' + py' + qy = 0$ is also autonomous where as the nonhomogeneous Equation 18.1, $y'' + py' + qy = f(t)$ is not autonomous.

An *equilibrium* point of Equations 18.23 is a number pair

$$(x_e, y_e) \qquad \text{such that} \qquad \begin{array}{rcl} f(x_e, y_e) & = & 0 \\ \\ g(x_e, y_e) & = & 0. \end{array} \tag{18.24}$$

If (x_e, y_e) is an equilibrium point of Equations 18.23 then the pair

$$x(t) \quad = \quad x_e \qquad y(t) \quad = \quad y_e \qquad \text{for all } t$$

has the property that

$$x'(t) \quad = \quad 0 \quad = \quad f(x_e, y_e) \qquad y'(t) \quad = \quad 0 \quad = \quad g(x_e, y_e).$$

Thus $x(t) \equiv x_e$ and $y(t) \equiv y_e$ is a solution to Equations 18.23.

Phase plane Direction Fields. Suppose (x_1, y_1) is a point in the x,y-plane and $x(t), y(t)$ is a pair of functions that solve Equations 18.23 and for some number t_1 $x(t_1) = x_1$ and $y(t_1) = y_1$. The solution pair $x(t), y(t)$ is said to pass through (x_1, y_1) at time t_1 (see Figure 18.3A). From Equations 18.23

$$x'(t_1) = f(x_1, y_1) \qquad \text{and} \qquad y'(t_1) = g(x_1, y_1).$$

The derivatives $x'(t_1)$ and $y'(t_1)$ are determined by (x_1, y_1) and do not depend on t_1. The slope of the tangent to the curve at $(x(t_1), y(t_1)) = y'(t_1)/x'(t_1)$ (see Figure 18.3B).

It is illuminating to draw small vectors *in the x-y plane* at some points x_1, y_1 that point in the direction determined by $x' = f(x_1, y_1)$ and $y' = g(x_1, y_1)$. The direction is an angle α where

$$\begin{array}{rcll} \alpha & = & \arctan \dfrac{y'}{x'} & \qquad \text{if } x' > 0 \\ \\ \alpha & = & \text{sign}(y') \times \dfrac{\pi}{2} & \qquad \text{if } x' = 0 \\ \\ \alpha & = & \pi + \arctan \dfrac{y'}{x'} & \qquad \text{if } x' < 0. \end{array}$$

(sign(z) = 1 if $z > 0$, = 0 if $z = 0$, and = -1 if $z < 0$.) See Figure 18.4A.

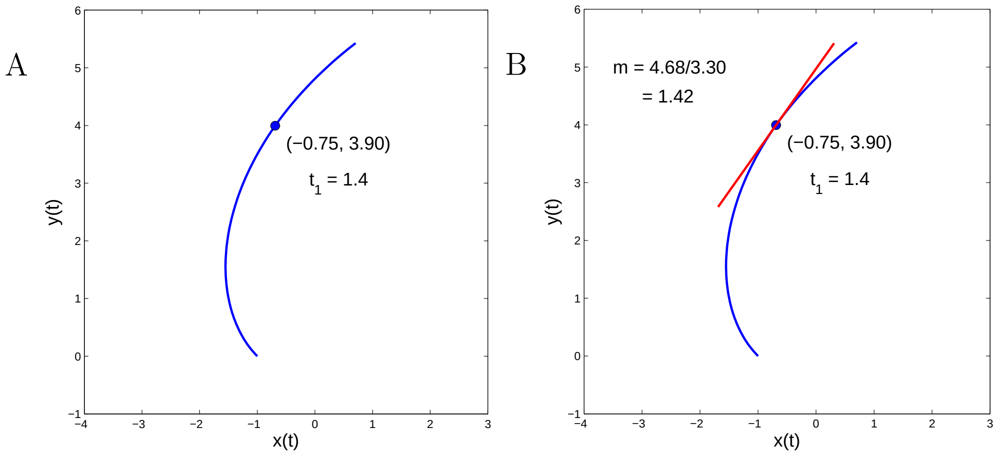


Figure 18.3: A. Solution for $x' = x + y$, $y' = y - x$, $x(0) = 1$, $y(0) = 0$ passes through the point $(x(t_1), y(t_1)) = (-0.75, 3.90)$ at the time $t_1 = 1.4$. B. The slope of the solution at time t_1 is $y'(1.4)/x'(1.4) = 4.68/3.30 = 1.42$.

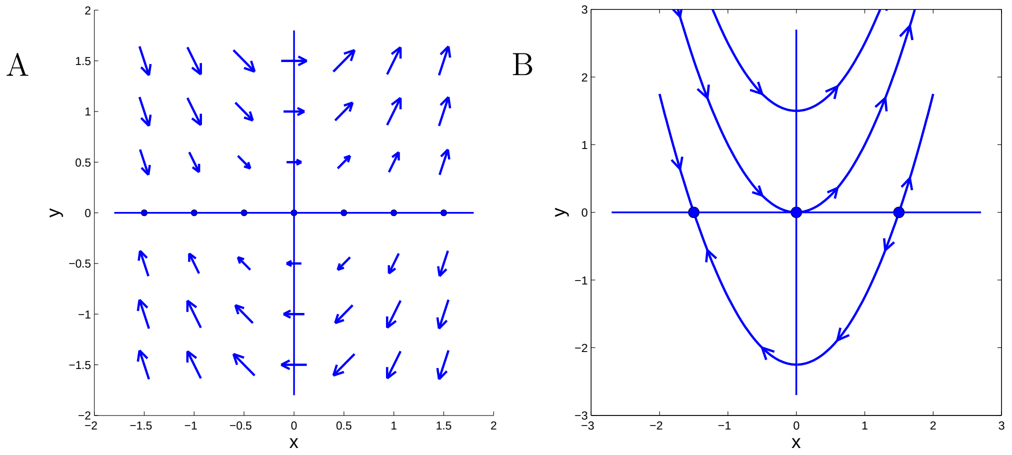


Figure 18.4: A. Direction field for $x' = y$, $y' = 2xy$. Each point $(x_e, 0)$ is an equilibrium point. B. Solution curves with directions of solutions marked.

Example 18.2.1 Consider the equations

$$\begin{array}{rclcrcl} x(0) & = & x_0 & \qquad & x'(t) & = & y \\ y(0) & = & y_0 & \qquad & y'(t) & = & 2xy \end{array} \tag{18.25}$$

First observe that every point with y-coordinate 0, $(x_e, 0)$, is an equilibrium point. Also observe the direction field in, Figure 18.4A; the direction of motion in the four quarters of the plane is determined by whether x' and y' are positive or negative. Now use Leibnitz notation; write

$$\frac{y'(t)}{x'(t)} = \frac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}} \quad \overset{!!}{=} \quad \frac{dy}{dx} = \frac{2xy}{y} = 2x \tag{18.26}$$

The equation, $dy/dx = 2x$ has an easy solution,

$$y = x^2 + C, \qquad y = x^2 + y_0 - x_0^2.$$

From this we conclude that every solution curve is part of a parabola, and we have drawn three such parabola's in Figure 18.4B.

The lower parabola in Figure 18.4B is $y = x^2 - 2.25$ and contains five nonintersecting solution curves to Equations 18.25 corresponding to

$$-\infty < x < -1.5, \qquad x = -1.5, \qquad -1.5 < x < 1.5,$$

$$x = 1.5, \qquad \text{and} \qquad 1.5 < x < \infty.$$

The parabola contains two equilibrium points, $(-1.5, 0)$ and $(1.5, 0)$.

Explore 18.2.1 Do This.

a. Show that for any positive number, a,

$$x(t) = a \tan at, \qquad y(t) = a^2 \sec^2 at, \qquad -\frac{\pi}{2} < t < \frac{\pi}{2}$$

solve Equations 18.25. Identify $(x(0), y(0))$ and characterize the solution curve.

b. Show that

$$x(t) = \frac{1}{1-t}, \qquad y(t) = \frac{1}{(1-t)^2}, \qquad -\infty < t < 1$$

solve Equations 18.25. Identify $(x(0), y(0))$ and characterize the solution curve.

c. Show that

$$x(t) = \frac{-1}{1+t}, \qquad y(t) = \frac{1}{(1+t)^2}, \qquad -1 < t < \infty$$

solve Equations 18.25. Identify $(x(0), y(0))$ and characterize the solution curve.

d. The previous two solutions lie on $y = x^2$. Find one more solution that lies on $y = x^2$.

e. If you find formulas for the five solution curves contained in $y = x^2 - 2.25$ you should at least get an A in this course. ■

The two equilibria $(-1.5, 0)$ and $(1.5, 0)$ in Figure 18.4B are quite different. Solution curves with points close to $(-1.5, 0)$ stay close to $(-1.5, 0)$ as time increases (see the arrows on the curves). There are solution curves with points close to $(1.5, 0)$, however, that move away from $(1.5, 0)$ as time increases. The equilibrium $(-1.5, 0)$ is *stable* (defined next) and the equilibrium $(1.5, 0)$ is not stable. ■

Definition 18.2.1 *Stable Equilibrium.* An equilibrium point, (x_e, y_e), of Equations 18.23 is a stable means that if ϵ is a positive number there is a positive number δ such that if $x(t), y(t)$ are the solutions to Equations 18.23 and for some t_1 $\sqrt{(x(t_1) - x_e)^2 + (y(t_1) - y_e)^2} < \delta$ then then for every number $t > t_1$, $\sqrt{(x(t) - x_e)^2 + (y(t) - y_e)^2} < \epsilon$.

Definition 18.2.2 *Asymptotically stable.* A stable equilibrium point, (x_e, y_e), of Equations 18.23 is asymptotically stable if there is a positive number δ such that if $x(t), y(t)$ are the solutions to Equations 18.23 and for some t_1 $\sqrt{(x(t_1) - x_e)^2 + (y(t_1) - y_e)^2} < \delta$ then

$$\lim_{t\to\infty} \sqrt{(x(t) - x_e)^2 + (y(t) - y_e)^2} \quad = \quad 0.$$

The equilibrium point $(-1.5, 0)$ of equations

$$x' = y, \qquad y' = 2xy$$

shown in Figure 18.4B is stable but not asymptotically stable. To show stability of $(-1.5, 0)$, suppose $\epsilon > 0$ and we may assume $\epsilon < 1$. Choose $\delta = 2\epsilon/3$. The case for $\epsilon = 0.9$ and $\delta = 0.6$ is shown in Figure 18.5A. The curves that enter the circle with radius δ stay within the circle of radius ϵ. It is necessary that δ be less than ϵ. There are curves that enter the circle with radius δ that do not stay in the circle of radius δ (but stay in the circle of radius ϵ), as shown in Figure 18.5B.

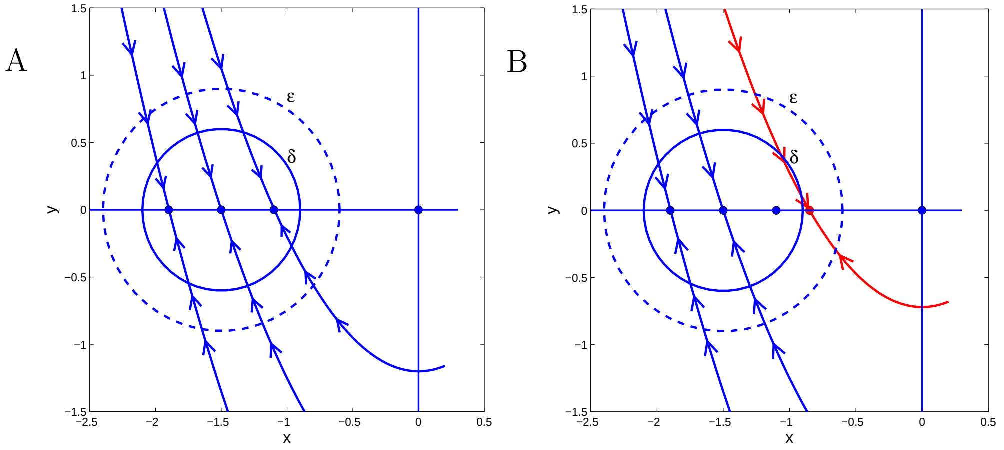


Figure 18.5: A. Solution curves of $x' = y$, $y' = 2xy$ stay close to the equilibrium point (-1.5,0). B. Curves (red) may enter the circle with radius δ but not stay within it. They do, however, stay within the circle of radius ϵ.

The equilibrium point (-1.5,0) of equations

$$x' = y, \qquad y' = 2xy$$

is not asymptotically stable, however. All of the equilibrium points $(x_e, 0)$ that lie within the circle of radius δ are graphs of solution curves that do not converge to (-1.5,0).

Example 18.2.2 It appears from the direction field in Figure 18.6, that the origin, (0,0), is an asymptotically stable equilibrium of

$$\begin{array}{rclcrcl} x(0) & = & x_0 & \qquad & x' & = & -x \\ y(0) & = & y_0 & & y' & = & -y. \end{array} \tag{18.27}$$

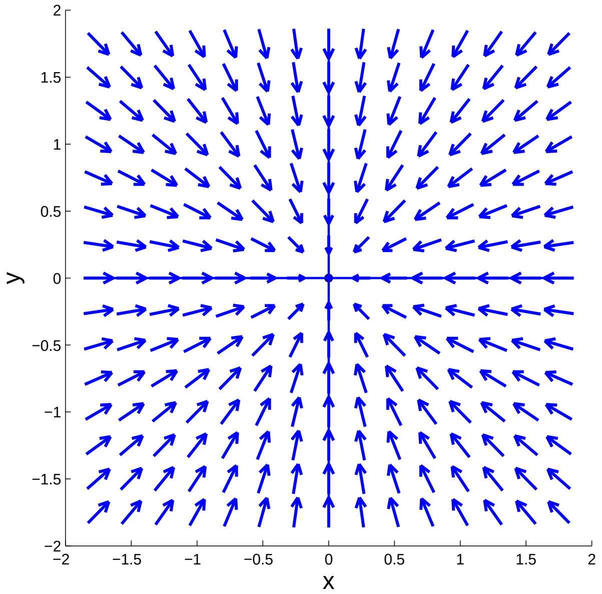


Figure 18.6: The direction field for $x' = -x$, $y' = -y$.

The solution to Equations 18.27 is

$$x(t) \quad = \quad x_0\, e^{-t}, \qquad y(t) \quad = \quad y_0\, e^{-t}$$

The origin is stable: Suppose $\epsilon > 0$, choose $\delta = \epsilon$. If for some t_1

$$\sqrt{(x(t_1) - 0)^2 + (y(t_1) - 0)^2} = \sqrt{x_1^2 + y_1^2}\, e^{-t_1} \quad < \quad \delta,$$

Then for all $t > t_1$,

$$\sqrt{(x(t) - 0)^2 + (y(t) - 0)^2} = \sqrt{x_1^2 + y_1^2}\, e^{-t} \quad < \quad \sqrt{x_1^2 + y_1^2}\, e^{-t_1} \quad < \delta.$$

Furthermore

$$\lim_{t \to \infty} \sqrt{(x(t) - 0)^2 + (y(t) - 0)^2} = \sqrt{x_1^2 + y_1^2}\, \lim_{t \to \infty} e^{-t} \quad = \quad 0,$$

so that (0,0) is asymptotically stable. ■

Exercises for Section 18.2, Stability and asymptotic stability of equilibria of pairs of autonomous differential equations.

Exercise 18.2.1 Do Explore 18.2.1.

Exercise 18.2.2 Is the origin a stable equilibrium of

$$x' = y, \qquad y' = 2xy\,? \qquad \text{Examine Figure 18.5.}$$

Exercise 18.2.3 a. Show that for any number, x_0, the solution to

$$\begin{aligned} x(0) &= x_0 & x' &= -y \\ y(0) &= 0 & y' &= 25x \end{aligned} \tag{18.28}$$

$$\text{is} \qquad x(t) = x_0 \cos 5t, \qquad y(t) = 5x_0 \sin 5t.$$

b. Draw the graph in the $x, y-$plane of the solutions for $x_0 = 1$ and $x_0 = 0.5$.

c. Show that the origin (0,0) is a stable equilibrium point of Equations 18.28.

d. Show that the origin (0,0) not an asymptotically stable equilibrium of Equations 18.28.

18.3 Two constant coefficient linear differential equations.

We examine the system

$$\begin{aligned} x(0) &= x_0 & x'(t) &= a_{1,1}x(t) + a_{1,2}y(t) \\ y(0) &= y_0 & y'(t) &= a_{2,1}x(t) + a_{2,2}y(t) \end{aligned} \tag{18.29}$$

where the coefficients, $a_{i,j}$, are constant. The origin is an equilibrium point. We will find conditions on the coefficients, $a_{i,j}$, that will insure that the origin is an asymptotically stable equilibrium.

We differentiate $x'(t) = a_{1,1}x(t) + a_{1,2}y(t)$ and write

$$\begin{aligned} x'(t) &= a_{1,1}x(t) + a_{1,2}y(t) \\ x''(t) &= a_{1,1}x'(t) + a_{1,2}y'(t) \\ y'(t) &= a_{2,1}x(t) + a_{2,2}y(t) \end{aligned} \tag{18.30}$$

Now eliminate $y(t)$ and $y'(t)$ between three Equations 18.30 and and find a second order equation that involves only $x(t)$.

$$x''(t) - (a_{1,1} + a_{2,2})x'(t) + (a_{1,1}a_{2,2} - a_{2,1}a_{1,2})x(t) = 0. \tag{18.31}$$

Equation 18.31 is a second order linear constant coefficient homogeneous differential equation of the form

$$x''(t) + px'(t) + qx(t) = 0.$$

discussed in Section 18.1.1. The characteristic equation

$$r^2 - (a_{1,1} + a_{2,2})r + (a_{1,1}a_{2,2} - a_{2,1}a_{1,2}) = 0 \qquad (18.32)$$

of Equation 18.31 and its characteristic roots are also defined to be the characteristic equation and roots of the system, Equation 18.29. The coefficient $a_{1,1} + a_{2,2}$ of Equation 18.32 is called the *trace* of the matrix

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2}, \end{bmatrix}$$

and the coefficient $a_{1,1}a_{2,2} - a_{2,1}a_{1,2}$ is the determinant of A. The characteristic equation 18.32 is sometimes written

$$r^2 - \text{trace}(A)r + \det(A) = 0 \qquad (18.33)$$

The same characteristic equation and roots are obtained if $x(t)$ and $x'(t)$ are eliminated from Equations 18.29 to obtain a second order differential equation in $y(t)$ (Exercise 18.3.1.)

For initial conditions

$$x(0) = x_0 \qquad y(0) = y_0$$

we can compute $x'(0)$

$$x'(0) = a_{1,1}x(0) + a_{1,2}y(0) = a_{1,1}x_0 + a_{1,2}y_0.$$

The characteristic equation, Equation 18.32, will have characteristic roots that are distinct and real r_1 and r_2, a repeated real, r, or complex, $a + bi$ and $a - bi$, and the solutions to the second order Equation 18.31 are, respectively,

$$\begin{array}{rrcl} \text{Distinct real roots,} & x(t) & = & C_1e^{r_1t} + C_2e^{r_2t}, \\ & C_1 & = & \dfrac{a_{1,1}x_0 + a_{1,2}y_0 - r_2x_0}{r_1 - r_2} \\ & C_2 & = & \dfrac{r_1x_0 - a_{1,1}x_0 - a_{1,2}y_0}{r_1 - r_2}, \end{array} \qquad (18.34)$$

$$\begin{array}{rrcl} \text{repeated real root,} & x(t) & = & C_1e^{rt} + C_2te^{rt}, \\ & C_1 & = & x_0 \\ & C_2 & = & a_{1,1}x_0 + a_{1,2}y_0 + r_1x_0, \end{array} \qquad (18.35)$$

$$\begin{array}{rrcl} \text{and complex roots.} & x(t) & = & C_1e^{at}\cos bt + C_2e^{at}\sin bt, \\ & C_1 & = & x_0 \\ & C_2 & = & \dfrac{a_{1,1}x_0 + a_{1,2}y_0 - ax_0}{b}. \end{array} \qquad (18.36)$$

Refer to Equations 18.6 - 18.9 and Equations 18.11 and 18.12.

From these solutions it is fairly easy to prove the following theorem.

Theorem 18.3.1 *Asymptotical Stability of a pair of constant coefficient homogeneous differential equations.* The origin (0,0) is an asymptotically stable equilibrium point of

$$\begin{aligned} x'(t) &= a_{1,1}x(t) + a_{1,2}y(t) \\ y'(t) &= a_{2,1}x(t) + a_{2,2}y(t) \end{aligned} \tag{18.37}$$

if the roots of the characteristic equation

$$r^2 - (a_{1,1} + a_{2,2})r + (a_{1,1}a_{2,2} - a_{1,2}a_{2,1}) = 0$$

satisfy one of the three conditions:

a. The roots are real and distinct and negative.

b. The root is a repeated root and is negative.

c. The roots are complex, $a + bi$ and $a - bi$ and a is negative.

Under conditions a. and b. the origin is called an asymptotically stable node and under condition c. the origin is called an asymptotically stable spiral point.

Proof. Suppose the roots are real and distinct and negative. We first show that (0,0) is a stable equilibrium. From Equations 18.34 we observe that

$$\begin{aligned} |x(t)| &= |C_1e^{r_1t} + C_2e^{r_2t}| \\ &\le |C_1| + |C_2| \\ &= \left|\frac{a_{1,1}x_0 + a_{1,2}y_0 - r_2x_0}{r_1 - r_2}\right| + \left|\frac{r_1x_0 - a_{1,1}x_0 - a_{1,2}y_0}{r_1 - r_2}\right| \\ &\le \left(\frac{|a_{1,1}| + |a_{1,2}| + |r_2|}{|r_1 - r_2|} + \frac{|r_1| - |a_{1,1}| + |a_{1,2}|}{|r_1 - r_2|}\right) \times max(|x_0|, |y_0|) \\ &\le K_x\sqrt{x_0^2 + y_0^2}. \end{aligned}$$

Similarly there is a constant K_y that depends only on the coefficients $a_{1,1}, \cdots, a_{2,2}$ such that $|y(t)| \le K_y\sqrt{x_0^2 + y_0^2}$. Therefore

$$\sqrt{(x(t))^2 + (y(t))^2} \le \sqrt{K_x^2 + K_y^2}\sqrt{x_0^2 + y_0^2} = K\sqrt{x_0^2 + y_0^2}$$

Suppose ϵ is to be a bound on $\sqrt{(x(t))^2 + (y(t))^2}$. Let $\delta = \epsilon/K$. Then if $\sqrt{x_0^2 + y_0^2} < \delta$,

$$\sqrt{(x(t))^2 + (y(t))^2} \le K\sqrt{x_0^2 + y_0^2} < K \times \epsilon/K = \epsilon$$

Therefore, (0,0) is stable.

From $x(t)C_1e^{r_1t} + C_2e^{r_2t}$ it is immediate that

$$\lim_{t\to\infty} x(t) = C_1 \lim_{t\to\infty} e^{r_1t} + C_2 \lim_{t\to\infty} e^{r_2t} = 0$$

because r_1 and r_2 are negative. Similarly $\lim_{t\to\infty} y(t) = 0$, and it follows that (0,0) is an asymptotically stable equilibrium of Equations 18.34.

The arguments for a repeated root and complex roots are similar and are omitted. *End of proof.*

Direction fields are shown in Figure 18.7 for three examples, all of which have real characteristic roots.:

A.	B.	C.
$\begin{array}{rcll} x' & = & -0.5x & \\ y' & = & & -0.3y \end{array}$	$\begin{array}{rcll} x' & = & -0.5x & \\ y' & = & & 0.3y \end{array}$	$\begin{array}{rcll} x' & = & 0.5x & \\ y' & = & & 0.3y \end{array}$
Two negative roots, -0.5 and -0.3	One negative, one positive, -0.5 and 0.3.	Two positive roots, 0.5 and 0.3.

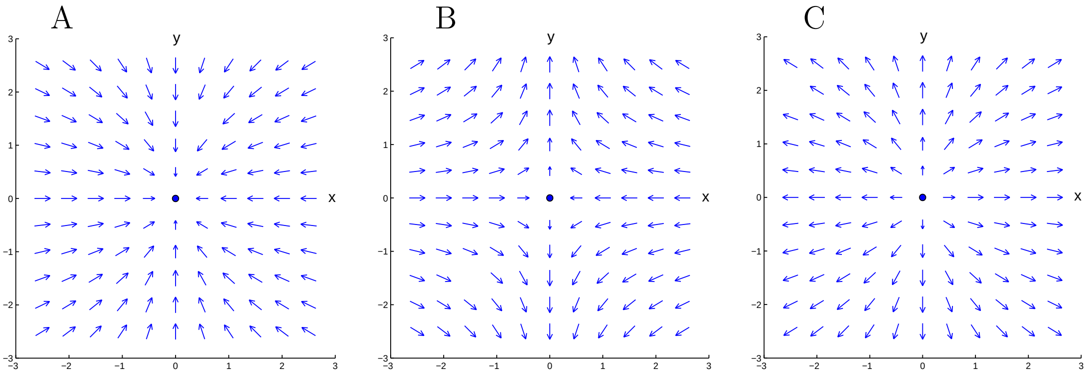


Figure 18.7: Direction fields drawn in the phase planes for Equation 18.29 with A. $A = [-0.5 \quad 0; \quad 0 \quad -0.3]$. B. $A = [-0.5 \quad 0; \quad 0 \quad 0.3]$. C. $A = [0.5 \quad 0; \quad 0 \quad 0.3]$.

It is obvious in Figure 18.7A that all paths lead to (0,0); (0,0) is a stable node of the equations A. There are two negative characteristic roots, -0.5 and -0.3, and the solution is $x(t) = x_0e^{-0.5t}$ and $y(t) = y_0e^{-0.3t}$. In Figure 18.7B only two paths lead to (0,0). Others paths start toward (0,0) but are diverted to either positive or negative y. The roots are -0.5 and 0.3 and the solution is $x(t) = x_0e^{-0.5t}$ and $y(t) = y_0e^{0.3t}$. The equations B are not stable, the origin is not a stable equilibrium point. In Figure 18.7C all paths lead away from (0,0). There are two positive roots, 0.5 and 0.3, and the equations C are not stable.

Solution curves for all of the systems D - E spiral around the origin which is typical of systems with complex roots.

In Figure 18.8D, because the real part of the root is negative the paths spiral inward toward (0,0) and the equations D are stable and (0,0) is a stable spiral point. In Figure 18.8E the real part of the root is zero; all paths are ellipses with center at (0,0). The paths do not converge to (0,0) and the system E is

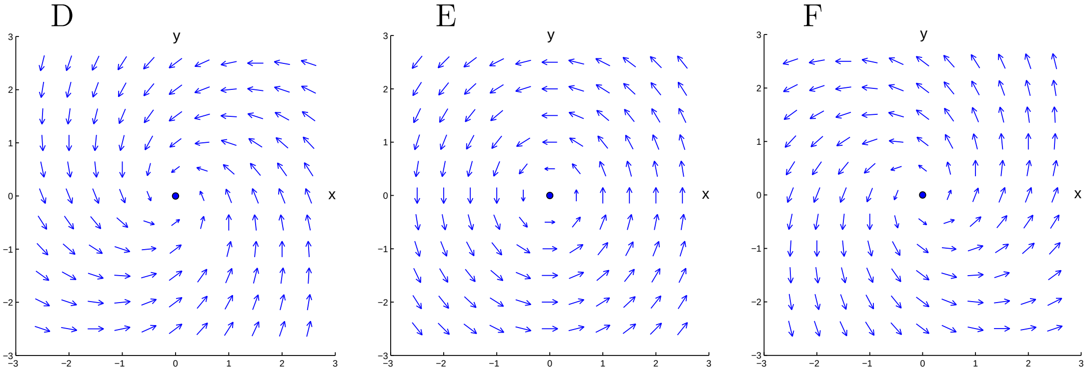


Figure 18.8: Direction fields drawn in the phase planes for Equation 18.29 with D. $A = [-0.2 \quad -0.4; \quad 0.5 \quad -0.3]$. E. $A = [0 \quad -0.4; \quad 0 \quad -0.3]$ F. $A = [0.2 \quad -0.4; \quad 0.5 \quad 0.3]$.

stable but not asymptotically stable. In Figure 18.8F the real part of the root is positive and the paths spiral outward. The system F is not stable.

Summary. The qualitative character of the direction fields and solutions to equations 18.29

$$\begin{array}{rclcrcl} x(0) & = & x_0 & \qquad & x'(t) & = & a_{1,1}x(t) + a_{1,2}y(t) \\ y(0) & = & y_0 & & y'(t) & = & a_{2,1}x(t) + a_{2,2}y(t) \end{array}$$

is determined by the roots to the characteristic equation

$$r^2 - (a_{1,1} + a_{2,2})r + a_{1,1}a_{2,2} - a2,1a_{1,2} = 0 \qquad \text{or} \qquad r^2 - \text{trace}(A)r + \det(A) = 0$$

The roots are

$$r_1 = \frac{\text{trace}(A) + \sqrt{(\text{trace}(A))^2 - 4\det(A)}}{2}$$

$$r_2 = \frac{\text{trace}(A) - \sqrt{(\text{trace}(A))^2 - 4\det(A)}}{2}$$

The roots are real if the discriminate $(\text{trace}(A))^2 - 4\det(A) \geq 0$ and complex otherwise. The conditions for stability of the equilibrium (0,0) are summarized:

1. $(\text{trace}(A))^2 - 4\det(A) \geq 0$. If $\text{trace}(A) < 0$ and $\det(A) > 0$ there are two negative real roots and (0,0) is a stable node. If $\det(A) \leq 0$ there is at least one positive or zero root and (0,0) is not a stable node.

2. $(\text{trace}(A))^2 - 4\det(A) = 0$. If $\text{trace}(A) < 0$ there is one negative real root and (0,0) is a stable node.

3. $(\text{trace}(A))^2 - 4\det(A) < 0$. If $\text{trace}(A) < 0$ there are two complex roots with negative real part and (0,0) is a stable spiral point.

18.3.1 Continuous model of penicillin clearance

In Chapter 1 you developed a discrete, *difference* equation, one-compartment model of the early stage of penicillin clearance after a bolus injection of penicillin into serum, and we developed a continuous, *differential* equation model in Chapter 5.

In Section 15.2 we developed a discrete, two-*difference*-equations, two-compartment model of the complete 6 hours of penicillin data. Here we find a continuous, two-*differential*-equations, two-compartment model of the complete penicillin data.

In Section 15.2 we developed a discrete, system of two difference equations, two-compartment model of the complete 6 hours of penicillin data. (See Equation 15.5),

$$\begin{aligned} A_{t+1} - A_t &= -0.23A_t + 0.1B_t \qquad \text{mg} \\ B_{t+1} - B_t &= 0.068A_t - 0.1B_t \qquad \text{mg}, \end{aligned}$$

where A_t and B_t are amounts of penicillin at integer values of t, $t = 0, 5, 10, 15, \cdots$. $A_t/1.0$ and $B_t/0.68$ are penicillin concentrations in serum and tissue, respectively.

In vector and matrix form it was written in Section 15.6.1

$$X_0 = \begin{bmatrix} 200 \\ 0 \end{bmatrix} \qquad X_{t+1} - X_t = MX_t \quad \text{with solution} \quad X_t = (I + M)^t X_0, \tag{18.38}$$

where t takes integer values, $t = 0, 5, 10, 15, \cdots$, and

$$X_t = \begin{bmatrix} A_t \\ B_t \end{bmatrix}, \quad M = \begin{bmatrix} -0.23 & 0.10 \\ 0.068 & -0.1 \end{bmatrix}, \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad I + M = \begin{bmatrix} 0.77 & 0.10 \\ 0.068 & 0.9 \end{bmatrix}.$$

We now let continuous functions $A(t)$ and $B(t)$ denote serum and tissue penicillin amounts for all values of $t \geq 0$, and let

$$V(t) = \begin{bmatrix} A(t) \\ B(t) \end{bmatrix}.$$

Assumption

The continuous analog of a one compartment discrete growth function,

$$P_{t+1} - P_t = K\,P_t, \quad \text{is} \quad P'(t) = \ln(1+K)\,P(t).$$

So we reason that the continuous analog of the two compartment discrete growth function,

$$X_{t+1} - X_t = MX_t, \quad \text{for discrete times} \quad t = 0, 5, 10, 15, \cdots \quad \text{is}$$

$$V'(t) = \ln(I+M)\;V(t) \tag{18.39}$$

where now $t \geq 0$, all positive values of time.

With this assumption, we will show that

$$V(t) = (I+M)^t\,V(0).$$

Whoa! M and I are matrices! What do you mean, $\ln(I+M)$? And $(I+M)^t$ for fractional values of t?

Warning: Agitation ahead. May cause insomnia, anxiety, and riots.

To see that there is a question, observe the outputs of the MATLAB programs:

```
close all;clc;clear
A=[0.77 0.1;0.068 0.9]
Z=sqrt(A)
X=Z*Z


OUTPUT
A = 0.7700  0.1000   Z = 0.8763   0.0548  X = 0.8525   0.5775
    0.0680  0.9000       0.2608   0.9487      0.4762   0.9825
```

We expect X = A in the OUTPUT because for a number, m, sqrt(m)*sqrt(m) = m, but $X \neq A$. However,

```
close all;clc;clear
A=[0.77 0.1;0.068 0.9]
Z=sqrtm(A)
```

```
X=Z*Z


OUTPUT
A = 0.7700    0.1000  Z = 0.8775    0.3162  X = 0.7700    0.1000
    0.0680    0.9000      0.0373    0.9476      0.0680    0.9000
```

works as expected.

For A a matrix, MATLAB'S sqrt(A) is the matrix of square roots of the entries in A; sqrtm(A) is more complex and very important.

Similarly, MATLAB has exp(A) and expm(A), log(A) and logm(A), sin(A) and sinm(A). The '___m' signals a crucial matrix operation. I will try to explain. (Note: For our use, think $A = I + M$.)

Characteristic Vectors. We first find a *non-zero* 2 by 1 vector, U for which

$$A\,U = r\,U, \qquad \text{where } r \text{ is a number and} \qquad U \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

If there is a solution, then we will see that r is a characteristic[2] root of A as found in Section 15.6.1 and previously used in Equation 15.16. The vector U is called a *characteristic vector* of the matrix A. To find r and U, we write

$$\begin{aligned} A\,U &= r\,U \\ A\,U - r\,U = A\,U - r\,I\,U &= (\,A - r\,I)\,U = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \\ \begin{bmatrix} a_{1,1} - r & a_{1,2} \\ a_{2,1} & a_{2,2} - r \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned} \tag{18.40}$$

If the determinant of $A - r\,I$ is not zero, the only solution to the last equation is $u_1 = u_2 = 0$. But we want a *non-zero* solution, U, so to find a non-zero solution, U, we must have $\det(A - rI\,) = 0$.

$$\det \begin{bmatrix} a_{1,1} - r & a_{1,2} \\ a_{2,1} & a_{2,2} - r \end{bmatrix} = r^2 - r(a_{1,1} + a_{2,2}) + a_{1,1}\,a_{2,2} - a_{2,1}\,a_{1,2} = 0$$

This is the familiar equation for the characteristic roots of a matrix. See Equation 15.16 in Section 15.4.

For concrete exposition we turn to our example

$$A = I + M \;= \begin{bmatrix} 0.77 & 0.10 \\ 0.068 & 0.90 \end{bmatrix}.$$

The characteristic equation of this matrix is

$$r^2 - (0.77 + 0.9)r + (0.77 \times 0.90 - 0.068 \times 0.10) = r^2 - 1.67r + 0.6862 = 0,$$

$$\text{with roots} \quad r_1 = 0.73, r_2 = 0.94,$$

---

[2] The words **eigenvalue** and **eigenvector** are also used. Eigen is a German word for "self."

which we also found in Section 15.4.

We consider the *two* eigenvector equations 18.40

$$\begin{array}{rcl} r_1 = 0.73 \quad \begin{bmatrix} 0.77 - 0.73 & 0.10 \\ 0.068 & 0.9 - 0.73 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} & = & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ & \text{and} & \\ r_2 = 0.94 \quad \begin{bmatrix} 0.77 - 0.94 & 0.10 \\ 0.068 & 0.9 - 0.94 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} & = & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{array}$$

which are

$$\begin{bmatrix} 0.04 & 0.10 \\ 0.068 & 0.17 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -0.17 & 0.10 \\ 0.068 & -0.04 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Each of these matrix equations has many solutions; we need one solution for each matrix equation, and in each matrix system we choose $u_1 = 1$ and solve for u_2 using the top equation (the bottom equations give the same answer).

$$\begin{array}{lrclrcl} \text{First system:} & 0.04 \cdot 1 + 0.10 \cdot u_2 & = & 0 & u_2 & = & -0.4 \\ \text{Second system:} & -0.17 \cdot 1 + 0.10 \cdot u_2 & = & 0 & u_2 & = & 1.7 \end{array}$$

so that two characteristic vectors are

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -0.4 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1.7 \end{bmatrix}$$

Explore 18.3.1 Show that the bottom equations of each system yield the same values of u_2.

$$0.068 \cdot 1 - 0.17 \cdot u_2 = 0 \quad \text{yields} \quad u_2 = -0.4 \quad \text{and}$$
$$0.068 \cdot 1 - 0.4 \cdot u_2 = 0 \quad \text{yields} \quad u_2 = 1.7 \qquad \blacksquare$$

Now we have

$$\begin{array}{rcl} \begin{bmatrix} 0.77 & 0.10 \\ 0.068 & 0.9 \end{bmatrix} \begin{bmatrix} 1 \\ -0.4 \end{bmatrix} & = & 0.73 \begin{bmatrix} 1 \\ -0.4 \end{bmatrix} \\ & \text{and} & \\ \begin{bmatrix} 0.77 & 0.10 \\ 0.068 & 0.9 \end{bmatrix} \begin{bmatrix} 1 \\ 1.7 \end{bmatrix} & = & 0.94 \begin{bmatrix} 1 \\ 1.7 \end{bmatrix}, \end{array}$$

which we can combine, being careful about the order of multiplication, into

$$\begin{bmatrix} 0.77 & 0.10 \\ 0.068 & 0.9 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -0.4 & 1.7 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -0.4 & 1.7 \end{bmatrix} \begin{bmatrix} 0.73 & 0 \\ 0 & 0.94 \end{bmatrix}. \tag{18.41}$$

$$\text{Let} \quad U = \begin{bmatrix} 1 & 1 \\ -0.4 & 1.7 \end{bmatrix} \quad \text{and} \quad R = \begin{bmatrix} 0.73 & 0 \\ 0 & 0.94 \end{bmatrix}.$$

Characteristic Value and Vector Decomposition. Equation 18.41 can be written

$$\begin{array}{c} AU = RU \quad \text{where} \quad R = \begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix}, \quad U = \left[\vec{U}_1 \; \vec{U}_2\right], \\ \text{and} \quad A\vec{U}_i = r_i \vec{U}_i, \quad \vec{U}_i \neq \vec{0}, \quad i = 1, 2, \quad \text{and} \quad r_1 \neq r_2. \end{array} \tag{18.42}$$

This is a general form for any matrix, A, with distinct characteristic values.

The inverse of U is a matrix, U^{-1}, such that $U\,U^{-1} = U^{-1}\,U = I$. Then

$$\begin{aligned} A\,U &= U\,R \\ (A\,U\,)\,U^{-1} &= (U\,R\,)\,U^{-1} \\ A\,(U\,U^{-1}\,) &= (U\,R\,)\,U^{-1} \\ A\,I &= (U\,R\,)\,U^{-1} \\ A &= U\,R\,U^{-1} \end{aligned}$$

For our special case,

$$A = I + M = \left[\begin{array}{cc} 0.77 & 0.10 \\ 0.068 & 0.9 \end{array}\right], \quad \text{and} \quad U = \left[\begin{array}{cc} 1 & 1 \\ -0.4 & 1.7 \end{array}\right].$$

$$\text{The inverse of} \quad \left[\begin{array}{cc} 1 & 1 \\ -0.4 & 1.7 \end{array}\right] \quad \text{is} \quad \left[\begin{array}{cc} 0.8095 & -0.4762 \\ 0.1905 & 0.4762 \end{array}\right].$$

$$\text{So} \quad I + M = \left[\begin{array}{cc} 1 & 1 \\ -0.4 & 1.7 \end{array}\right] \left[\begin{array}{cc} 0.73 & 0 \\ 0 & 0.94 \end{array}\right] \left[\begin{array}{cc} 0.8095 & -0.4762 \\ 0.1905 & 0.4762 \end{array}\right].$$

MATLAB CHECK.

```
close all;clc;clear                              OUTPUT

%%[V,R] = EIG(X) produces a diagonal    A =  0.7700   0.1000
%%matrix R of eigenvalues and a              0.0680   0.9000
%%matrix V whose columns are
%% corresonding eigenvectors            V = -0.9285  -0.5070
                                         0.3714  -0.8619
A=[0.77 0.1;0.068 0.9]
[V,R]=eig(A);                           R =  0.7300        0
                                              0   0.9400

%% Adjust the eigenvectors
U=V*[1/V(1,1) 0 ; 0 1/V(1,2)]           U =  1.0000   1.0000
                                        -0.4000   1.7000
Test = U*R*inv(U)
                                 Test =  0.7700   0.1000
                                         0.0680   0.9000
```

Now, for example, we define

$$\ln_{matrix}(I+M) = \left[\begin{array}{cc} 1 & 1 \\ -0.4 & 1.7 \end{array}\right] \left[\begin{array}{cc} \ln 0.73 & 0 \\ 0 & \ln 0.94 \end{array}\right] \left[\begin{array}{cc} 0.8095 & -0.4762 \\ 0.1905 & 0.4762 \end{array}\right]$$

$$= \left[\begin{array}{cc} -0.2666 & 0.1204 \\ 0.0819 & -0.1100 \end{array}\right]$$

Definition 18.3.1 *Functions of matrices.* Suppose A is a 2 by 2 matrix **with distinct eigenvalues**, r_1 and r_2, with corresponding eigenvectors U_1 and U_2. Let U be the matrix with columns U_1 and U_2,

$$U = [\,U_1 \quad U_2\,], \quad \text{and} \quad R = \left[\begin{array}{cc} r_1 & 0 \\ 0 & r_2 \end{array}\right]$$

Then $A = U\,R\,U^{-1}$. Define (note new notation expm)

$$\text{expm}\;(\,A) = U \left[\begin{array}{cc} e^{r_1} & 0 \\ 0 & e^{r_2} \end{array}\right] U^{-1}, \quad \text{and} \tag{18.43}$$

using familiar notation, we define for t a number

$$A^t = U \left[\begin{array}{cc} r_1^t & 0 \\ 0 & r_2^t \end{array}\right] U^{-1} \tag{18.44}$$

Suppose also that r_1 and r_2 are **positive real numbers**. Then

$$\text{lnm}\;(\,A\,) = U \left[\begin{array}{cc} \ln r_1 & 0 \\ 0 & \ln r_2 \end{array}\right] U^{-1} \tag{18.45}$$

Explore 18.3.2 If you are still alert and attentive, you have absorbed a lot of new (and important) information. You should think it over a bit.

a. Show that

$$\text{lnm (expm } W) = W.$$

b. Show that

$$\text{expm (}\,\text{lnm } W) = W$$

c. Show that

$$\text{expm (}\,t \times \text{lnm } W) = W^t$$

d. Note that positive real eigenvalues of W are important for lnm (W), but not so crucial for expm (W).

e. In MATLAB to get lnm (W) and expm (W) for a matrix W, use the MATLAB matrix functions, logm(W) and expm(W).

MATLAB will compute the 'scalar' $\ln W$ and $\exp W$, but returns

$$\ln \left[\begin{array}{cc} w_{1,1} & w_{1,2} \\ w_{2,1} & w_{2,2} \end{array}\right] = \left[\begin{array}{cc} \ln w_{1,1} & \ln w_{1,2} \\ \ln w_{2,1} & \ln w_{2,2} \end{array}\right]$$

and

$$\exp \left[\begin{array}{cc} w_{1,1} & w_{1,2} \\ w_{2,1} & w_{2,2} \end{array}\right] = \left[\begin{array}{cc} \exp w_{1,1} & \exp w_{1,2} \\ \exp w_{2,1} & \exp w_{2,2} \end{array}\right].$$

f. Run the following MATLAB code.

```
close all;clc;clear
W=[2 1;4 3];
A=exp(W)
B=expm(W)
C=log(W)
D=logm(W)
E=log(A)
F=logm(B)
G=exp(C)
H=expm(D)
```

g. Show that MATLAB's scalar functions $\ln W$ and $\exp W$ also yield

$$\ln(\exp W) = W$$

h. Now what are you going to use? $\ln W$ or lnm W? $\exp W$ or expm W? Run the following MATLAB code

```
A=[0.77 0.1;0.068 0.9]
[A exp(A) expm(A)]
[A log(A) logm(A)]

[exp(log(A)) expm(logm(A))]

[log(A*A*A) 3*log(A)]
[logm(A*A*A) 3*logm(A)]

[exp(2*A) exp(A)*exp(A)]
[expm(2*A) expm(A)*expm(A)]
```

i. Extend Definition 18.3.1 to define sinm A and sqrtm A.

j. Show that (sqrtm A) × (sqrtm A) = A. ∎

We now return to the discrete two compartment model, Equation 18.38, of penicillin clearance,

$$V_0 = \begin{bmatrix} 200 \\ 0 \end{bmatrix} \qquad V_{t+1} - V_t = MV_t \quad \text{with solution} \quad V_t = (I+M)^t \, V_0,$$

and our claim (assumption) that the continuous analog is Equation 18.39,

$$V(0) = \begin{bmatrix} 200 \\ 0 \end{bmatrix} \qquad V'(t) = \text{lnm}(I+M)\, V(t)$$

with solution

$$V(t) = \operatorname{expm}(\operatorname{lnm}(\,(I+M)\,t)\,)\,V(0) \quad = \quad (I+M)^t\,V(0)$$

Does

$$V(t) = (I+M)^t \qquad \text{solve} \qquad V'(t) = \operatorname{lnm}(I+M)\,V(t)\ ?$$

You probably hope so; you have waded through some difficult (but important) material.

Let $A = I + M$ and use previous notation, $A = U\,R\,U^{-1}$. Recall that for normal scalar variables,

$$\left[\,a^t\,\right]' = (\ln a)a^t.$$

We next show that for matrix A and scalar t,

$$\left[\,A^t\,\right]' = (\operatorname{lnm} A)A^t. \tag{18.46}$$

On first reading, accept Equation 18.46 as correct and skip the following proof. There are two mistakes in the proof, however, that you may wish to find

$$\begin{aligned}
\frac{d}{dt}\,A^t &= \lim_{h\to 0}\frac{A^{(t+h)} - A^t}{h} \\
&= \lim_{h\to 0}\frac{U\begin{bmatrix} r_1^{(t+h)} & 0 \\ 0 & r_2^{(t+h)}\end{bmatrix}U^{-1} - U\begin{bmatrix} r_1^t & 0 \\ 0 & r_2^t\end{bmatrix}U^{-1}}{h} \\
&= \lim_{h\to 0}\,U\,\frac{\begin{bmatrix} r_1^{(t+h)} & 0 \\ 0 & r_2^{(t+h)}\end{bmatrix} - \begin{bmatrix} r_1^t & 0 \\ 0 & r_2^t\end{bmatrix}}{h}\,U^{-1} \\
&= \lim_{h\to 0} U\,\frac{\begin{bmatrix} r_1^{(t+h)} - r_1^t & 0 \\ 0 & r_2^{(t+h)} - r_2^t\end{bmatrix}}{h}\,U^{-1} \\
&= U\left(\begin{bmatrix} \lim_{h\to 0}\frac{r_1^{(h)} - r_1}{h} & 0 \\ 0 & \lim_{h\to 0}\frac{r_2^{(t+h)} - r_2^t}{h}\end{bmatrix}\right)U^{-1} \\
&= U\begin{bmatrix} (\ln r_1)\,r_1^t & 0 \\ 0 & (\ln r_2)\,r_2^t\end{bmatrix}U^{-1} \\
&= U\begin{bmatrix} (\ln r_1) & 0 \\ 0 & (\ln r_2)\end{bmatrix}\begin{bmatrix} r_1^t & 0 \\ 0 & r_2^t\end{bmatrix}U^{-1} \\
&= U\begin{bmatrix} (\ln r_1) & 0 \\ 0 & (\ln r_2)\end{bmatrix}U\,U^{-1}\begin{bmatrix} r_1^t & 0 \\ 0 & r_2^t\end{bmatrix}U^{-1} \\
&= (\operatorname{lnm} A)\,A^t \qquad\qquad \textbf{Whew!, UGH!}
\end{aligned}$$

Now, with

$$V(t) = (I+M)^t\, V(0), \qquad \begin{aligned} V'(t) &= (\operatorname{lnm}(I+M))\,(I+M)^t\, V(0) \\ &= (\operatorname{lnm}(I+M))\, V(t), \end{aligned}$$

so $V(t) = (I+M)^t\, V(0)$ solves Equation 18.39 (which is $V'(t) = \operatorname{lnm}(I+M))\, V(t))$. Furthermore

$$V(0) = (I+M)^t\, V(0)\,|_{t=0} = (I+M)^0\, V(0) = I\, V(0) = V(0)$$

so the initial condition is satisfied.

It is interesting to compare M which quantifies movement per 5 minutes for the discrete system and lnm($I+M$) which quantifies instantaneous rate of movement for the continuous system.

$$M = \begin{bmatrix} -0.23 & 0.1 \\ 0.068 & -0.10 \end{bmatrix} \qquad \text{and} \qquad \operatorname{lnm}(I+M) = \begin{bmatrix} -0.2666 & 0.1204 \\ 0.0819 & -0.1100 \end{bmatrix}.$$

The movement between compartments are more intense in the continuous model than in the discrete model. A discrete one-compartment model with annual growth rate of 6% is matched by an instantaneous growth rate of $\ln(1.06) \doteq 5.83\%$, however; instantaneous growth rate in the one-compartment model is less than annual growth per year. The numbers in lnm $(I+M)$ are all *larger* in magnitude than their counterparts in M. Furthermore, the number -0.1100 is the fraction of B_t (penicillin in tissue) that leaves tissue and 0.1204 is the fraction of B_t that enters serum. Perhaps the basic model in which we assumed that serum volume is 1 liter and tissue volume is 0.68 liters (or, basically, that tissue volume is 0.68 times serum volume) is a fundamental error.

The continuous solution matches the observed data very well, however, with exactly the same values at the observed data times as those of the discrete solution, as shown in Figure 18.9.

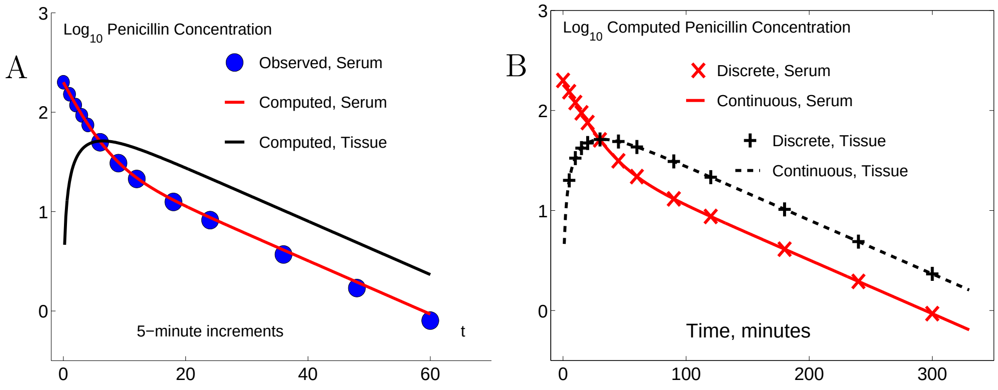


Figure 18.9: A. Graphs of the observed penicillin data from Figure 15.1 and $v_1(t)/1.0$ and $v_2(t)/0.68$, where v_1 and v_2 are the components of $V(t)$, and 1.0 and 0.68 adjust for compartment sizes. B. Comparison of the discrete and continuous approximations to serum and tissue penicillin concentrations. They agree at the discrete time values.

Exercises for Section 18.3, Two constant coefficient linear differential equations.

Exercise 18.3.1 Eliminate the functions $x(t)$ and $x'(t)$ between the two Equations 18.29 and find a second order equation that involves only $y(t)$. You should get

$$y''(t) - (a_{1,1} + a_{2,2})y'(t) + (a_{1,1}a_{2,2} - a_{2,1}a_{1,2})y(t) = 0. \tag{18.47}$$

The characteristic equation of this differential equation is also Equation 18.32.

Exercise 18.3.2 In each of the direction fields of Figure 18.7 there is an obvious omission. Compute the slopes and directions of each of the omitted vectors.

Exercise 18.3.3 In each of the direction fields of Figure 18.8 there is an obvious omission. Compute the slopes and directions of each of the omitted vectors.

Exercise 18.3.4 For each of the systems, determine whether the origin is stable, asymptotically stable, or unstable.

$$\text{a.} \quad \begin{array}{rcrcr} x' &=& 2x &-& 5y \\ y' &=& x &-& 2y \end{array} \qquad\qquad \text{b.} \quad \begin{array}{rcrcr} x' &=& 2x &-& 5y \\ y' &=& 2x &-& 4y \end{array}$$

$$\text{c.} \quad \begin{array}{rcrcr} x' &=& -6x &-& 2y \\ y' &=& 2x &-& 1y \end{array} \qquad\qquad \text{d.} \quad \begin{array}{rcrcr} x' &=& -9x &+& 4y \\ y' &=& -4x &-& 1y \end{array}$$

$$\text{e.} \quad \begin{array}{rcrcr} x' &=& 3x &-& 2y \\ y' &=& 2x &-& 1y \end{array} \qquad\qquad \text{f.} \quad \begin{array}{rcrcr} x' &=& & & y/2 \\ y' &=& -5x &-& 3y \end{array}$$

$$\text{g.} \quad \begin{array}{rcrcr} x' &=& -x &-& 5y \\ y' &=& 2x &-& 3y \end{array} \qquad\qquad \text{h.} \quad \begin{array}{rcrcr} x' &=& &-& 5y \\ y' &=& 2x &+& 2y \end{array}$$

$$\text{i.} \quad \begin{array}{rcrcr} x' &=& -3x &+& 1y \\ y' &=& 2x &-& 2y \end{array} \qquad\qquad \text{j.} \quad \begin{array}{rcrcr} x' &=& 3x &+& y \\ y' &=& 2x &+& 2y \end{array}$$

$$\text{k.} \quad \begin{array}{rcrcr} x' &=& x &-& 2y \\ y' &=& 2x &+& 1y \end{array} \qquad\qquad \text{l.} \quad \begin{array}{rcrcr} x' &=& 6x &+& 4y \\ y' &=& 2x &-& y \end{array}$$

Exercise 18.3.5 Do Explore 18.3.2.

Exercise 18.3.6 Interpret the output of the MATLAB program.

```
A=[0.77 0.1;0.068 0.9]


T1 = [exp(A) expm(A)]
T2 = [log(A) logm(A)]


T3 = [exp(log(A)) expm(logm(A))]


T4 = [log(A^2) 2*log(a)]
```

```
T5 = [logm(A^2) 2*logm(A^2)]

T6 = [exp(2*A) exp(A)*exp(A)]
T7 = [expm(2*A) expm(A)*expm(A)]
```

Output

```
T1 =  2.1598    1.1052    2.1674    0.2309
      1.0704    2.4596    0.1570    2.4676

T2 = -0.2614   -2.3026   -0.2666    0.1204
     -2.6882   -0.1054    0.0819   -0.1100

T3=   0.7700    0.1000    0.7700    0.1000
      0.0680    0.9000    0.0680    0.9000

T4 = -0.5113   -1.7898   -0.5227   -4.6052
     -2.1754   -0.2024   -5.3765   -0.2107

T5 = -0.5331    0.2408   -0.5331    0.2408
      0.1637   -0.2201    0.1637   -0.2201

T6 =  4.6646    1.2214    5.8475    5.1052
      1.1457    6.0496    4.9444    7.2326

T7 =  4.7341    1.0703    4.7341    1.0703
      0.7278    6.1254    0.7278    6.1254
```

Exercise 18.3.7 Equation 18.39 is

$$V'(t) = \ln(I+M)\, V(t) = A\, V(t) \quad \text{where} \quad A = \begin{bmatrix} 0.7700 & 0.1000 \\ 0.0680 & 0.9000 \end{bmatrix} \tag{18.48}$$

Assume

$$V(t) = \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix},$$

Then

$$\begin{array}{rclrcl} v_1(0) & = & 200 & v_1' & = & 0.7700v_1 + 0.1000v_2 \\ v_2(0) & = & 0 & v_2' & = & 0.0680v_1 + 0.9000v_2 \end{array}$$

a. Find a single second order equation in v_1'', v_1', and v_1 with initial conditions, $v_1(0)$ and $v_1'(0)$ whose solution is the same as $v_1(t)$ of the system, 18.48.

b. Solve for v_1 in your second order equation and compare your solution with the serum penicillin data.

18.4 Systems of two first order differential equations.

We return to two possibly nonlinear homogeneous differential equations

$$\begin{aligned} x'(t) &= f(x(t), y(t)) \\ y'(t) &= g(x(t), y(t)) \end{aligned} \qquad (18.49)$$

where f and g are continuous and have continuous first and second partial derivatives with respect to both variables.

A first global look at the solutions of Equations 18.49 is obtained from looking at the *null curves* defined by

$$f(x,y) = 0, \qquad \text{and} \qquad g(x,y) = 0.$$

Example 18.4.1 Consider the system

$$\begin{aligned} x'(t) &= (169 - x^2 - y^2)/10 \\ y'(t) &= 17 - x - y \end{aligned} \qquad (18.50)$$

The null curve $x' = 0$ is the circle $169 - x^2 - y^2 = 0$; the $y' = 0$ null curve is the line $17 - x - y = 0$. Both are shown in Figure 18.10. On the circle, $x' = 0$ and every direction field arrow is vertical. On the line, $y' = 0$ and every direction field arrow is horizontal. Inside the circle $169 - x^2 - y^2 = 0$, $x' > 0$ and direction arrows point to the right; outside the circle $x' < 0$ and direction arrows point to the left. Below the line $17 - x - y = 0$, $y' > 0$ and arrows point upward; above the line $y < 0$ and arrows point downward.

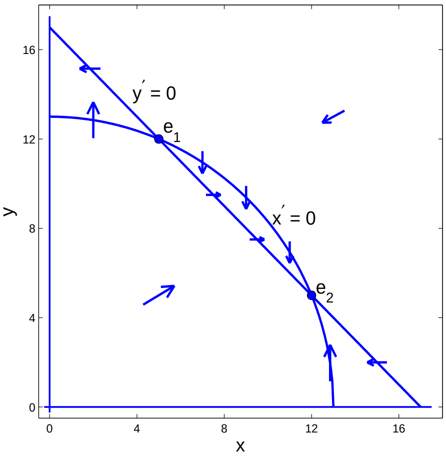


Figure 18.10: Null curves for $x' = (169 - x^2 - y^2)/10$, $y' = 17 - x - y$.

The two points, $e_1 = (5, 12)$ and $e_2 = (12, 5)$, where the null curves intersect are equilibrium points. Both $x' = 0$ and $y' = 0$ at the equilibrium points and both

$$\left\{ \begin{aligned} x_1(t) &\equiv 5 \\ y_1(t) &\equiv 12 \end{aligned} \right. \quad \text{for all } t, \qquad \text{and} \qquad \left\{ \begin{aligned} x_2(t) &\equiv 12 \\ y_2(t) &\equiv 5 \end{aligned} \right. \quad \text{for all } t,$$

are solutions to Equations 18.50.

Explore 18.4.1 Show that $x \equiv 5$, $y \equiv 12$ is a solution to

$$x' = (169 - x^2 - y^2)/10, \qquad y' = 17 - x - y.$$

■

Direction fields near $e_1 = (5, 12)$ and near $e_2 = (12, 5)$ for $x' = (169 - x^2 - y^2)/10$, $y' = 17 - x - y$ are shown in Figure 18.11A and B, respectively. They are quite different. Near $e_2 = (12, 5)$ the arrows point toward e_2; near $e_1 = (5, 12)$ some of the arrows do not point toward e_1. We will find that e_2 is *asymptotically stable*, and that e_1 does not meet the criterion that assures that it is asymptotically stable.

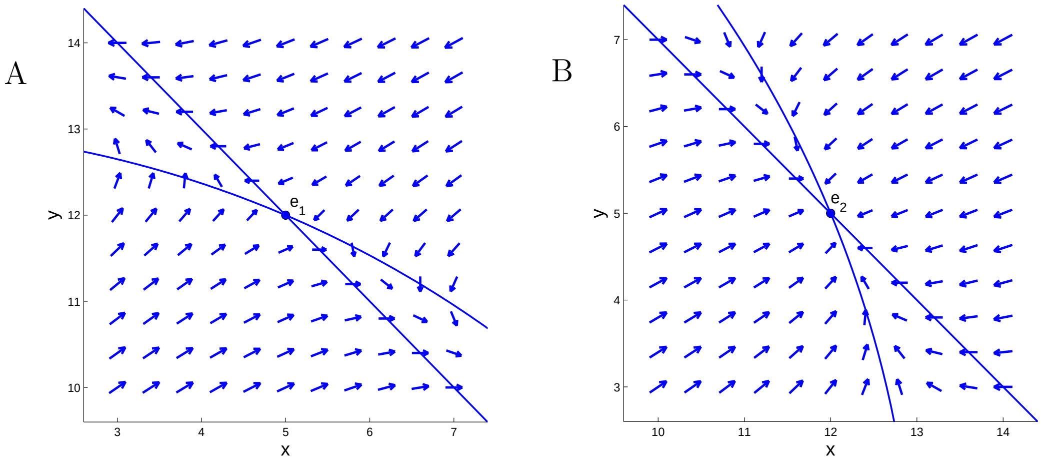


Figure 18.11: A. Direction field near the equilibrium point (5,12) of Equations 18.50. B. Direction field near the equilibrium point (12,5) of Equations 18.50.

■

Definition 18.4.1 *An equilibrium point,* (x_e, y_e) *of Equations 18.49* is a point of intersection of the null curves and satisfies

$$f(x_e, y_e) = 0 \qquad \text{and} \qquad g(x_e, y_e) = 0.$$

If (x_e, y_e) is an equilibrium point, $x(t) \equiv x_e$ and $y(t) \equiv y_e$ are solutions of Equations 18.49 and are called *equilibrium solutions.*

Definition 18.4.2 An equilibrium point (x_e, y_e) of Equations 18.49 is *asymptotically stable* means that

a. (It is *stable*:) If ϵ is a positive number there is a positive number δ such that if (x_0, y_0) is a point and the distance from (x_0, y_0) to (x_e, y_e) is less than δ then the solution $(x(t), y(t))$ to Equations 18.49 with initial conditions $x(0) = x_0$ and $y(0) = y_0$ has the property that for all t the distance from $(x(t), y(t))$ to (x_e, y_e) is less than ϵ.

b. (And:) There is a positive number δ_0 such that if (x_0, y_0) is a point and the distance from (x_0, y_0) to (x_e, y_e) is less than δ_0 then the solution $(x(t), y(t))$ to Equations 18.49 with initial conditions $x(0) = x_0$ and $y(0) = y_0$ has the property that

$$\lim_{t\to\infty} x(t) = x_e \qquad \text{and} \qquad \lim_{t\to\infty} y(t) = y_e$$

Definition 18.4.3 Suppose f and g are functions of two variables and (x_e, y_e) is in the domain of both f and g and suppose that f_1, f_2, g_1 and g_2 are all continuous on the interior of a circle with center at (x_e, y_e). The local linear approximation to the two homogeneous differential equations

$$\begin{aligned} x'(t) &= f(x(t), y(t)) \\ y'(t) &= g(x(t), y(t)) \end{aligned} \tag{18.51}$$

at an equilibrium point (x_e, y_e) is

$$\begin{aligned} x'(t) &= f_1(x_e, y_e) \times (x(t) - x_e) + f_2(x_e, y_e) \times (y(t) - y_e) \\ y'(t) &= g_1(x_e, y_e) \times (x(t) - x_e) + g_2(x_e, y_e) \times (y(t) - y_e) \end{aligned} \tag{18.52}$$

With $\xi(t) = x(t) - x_e$ and $\eta(t) = y(t) - y_e$ Equations 18.52 become

$$\begin{aligned} \xi'(t) &= f_1(x_e, y_e) \times \xi(t) + f_2(x_e, y_e) \times \eta(t) \\ \eta'(t) &= g_1(x_e, y_e) \times \xi(t) + g_2(x_e, y_e) \times \eta(t) \end{aligned} \tag{18.53}$$

The matrix

$$\begin{bmatrix} f_1(x,y) & f_2(x,y) \\ g_1(x,y) & g_2(x,y) \end{bmatrix} \tag{18.54}$$

is called the Jacobian matrix of the Differential Equations 18.51 at the point (x, y).

The theorem of this chapter is

Theorem 18.4.1 The differential equations 18.51 is asymptotically stable if the local linear approximation, Equations 18.53, is asymptotically stable.

This theorem together with Theorem 18.3.1 provide a direct computational means to determine whether an equilibrium point is asymptotically stable. Its proof is beyond our scope, but the theorem is easily understood and we use it. The one-dimensional version of this theorem is Theorem 18.4.1 for which we do gave a (fairly technical) proof. We will apply Theorem 18.4.1 to several systems in the next section.

Example 18.4.1 (Continued) Equations 18.50,

$$\begin{aligned} x' &= (169 - x^2 - y^2)/10 \\ y' &= 17 - x - y \end{aligned} \tag{18.55}$$

has two equilibrium points, $e_1 = (5,12)$ and $e_2 = (12,5)$. The Jacobian of Equations 18.55 is

$$\begin{bmatrix} f_1(x,y) & f_2(x,y) \\ g_1(x,y) & g_2(x,y) \end{bmatrix} \quad = \quad \begin{bmatrix} -x/5 & -y/5 \\ -1 & -1 \end{bmatrix}.$$

For $e_1 = (5,12)$ the Jacobian is

$$\begin{bmatrix} -x/5 & -y/5 \\ -1 & -1 \end{bmatrix}_{(x,y)=(5,12)} \quad = \quad \begin{bmatrix} -1 & -12/5 \\ -1 & -1 \end{bmatrix}.$$

The trace and determinant of the Jacobian are -2 and -7/5, respectively, and the characteristic roots of the local linear approximation to Equations 18.55 (the roots to $r^2 + 2r - 7/5 = 0$) are

$$\begin{aligned} r_1 &= \frac{-2 + \sqrt{4 + 49/25}}{2} \doteq 0.4413, \\ &\text{and} \\ r_2 &= \frac{-2 - \sqrt{4 + 49/25}}{2} \doteq -4.4413. \end{aligned}$$

Because one of the characteristic roots is positive we do not conclude that Equations 18.55 are asymptotically stable.

Explore 18.4.2 Show that the trace and determinant of the Jacobian of Equations 18.55 at $e_2 = (12,5)$ are respectively -17/5 and 7/5 and that both characteristic roots are negative. Thus conclude that Equations 18.55 are asymptotically stable at (12,5). ■

■

Exercises for Section 18.4 Systems of two first order differential equations.

Exercise 18.4.1 Find the local linear approximation to the system

$$\begin{aligned} x' &= x - x^2 - xy \\[2em] y' &= y - 0.5xy - 2y^2 \end{aligned}$$

a. At the equilibrium point (0,0).

b. At the equilibrium point (0,0.5).

c. At the equilibrium point (1,0).

d. At the equilibrium point (2/3,1/3).

For each of the local linear approximations, determine whether it is stable.

Exercise 18.4.2 For each of the systems, find all of the equilibrium points and determine whether the system is asymptotically stable at each equilibrium point.

$$\text{a.} \quad \begin{aligned} x' &= -xy \\ y' &= 1 - x - y \end{aligned} \qquad \text{b.} \quad \begin{aligned} x' &= x - 2xy \\ y' &= y + xy \end{aligned}$$

$$\text{c.} \quad \begin{aligned} x' &= 2 - x^2 - y^2 \\ y' &= 1 - xy \end{aligned} \qquad \text{d.} \quad \begin{aligned} x' &= 5 - x^2 - y^2 \\ y' &= 2 - xy \end{aligned}$$

Exercise 18.4.3 Show that (1,1), (2,2), and (3,3) are equilibrium points of

$$\begin{aligned} x' &= 1 - \frac{11}{6}x + \frac{10xy}{11 + xy} \\ y' &= 1 - \frac{11}{6}y + \frac{10xy}{11 + xy} \end{aligned}$$

Determine which of these equilibrium points are stable. This example is patterned after a model Anderson and May[3] present to suggest that antibody level may switch from a lower level stable *unexposed state* to an upper level stable *actively immune state.*

Exercise 18.4.4 For meditation. Is there a system, $x' = f(x, y)$, $y' = g(x, y)$ for which the origin, (0,0), is an equilibrium point and condition b. of Asymptotic Stability, Definition 18.4.2, is satisfied but condition a. is not satisfied? You might suppose that there is or condition a. would not be included in the definition, but we may just be insecure. It will be easier to look for a set of non-intersecting solution curves that fill the plane than to try to find expressions for $f(x, y)$ and $g(x, y)$.

---

[3] *ibid.*, p 36

18.5 Applications of Theorem 18.4.1 to biological systems.

Simple population models of competing species and predator-prey relations introduced by Alfred Lotka and Vito Volterra in the 1920's have been the basis of discussion of stability and local stability.

18.5.1 Competition.

Assume there are two species, x and y, with populations sizes $x(t)$ and $y(t)$ that compete for the same food. We write

$$\begin{aligned} x'(t) &= r_x \times x(t) \times (1 - ax(t) - by(t)) \\ y'(t) &= r_y \times y(t) \times (1 - cx(t) - dy(t)) \end{aligned} \tag{18.56}$$

If b and c are zero, there is no competition and $x(t)$ and $y(t)$ grow in a logistic manner with maximum population sizes $1/a$ and $1/d$, respectively. In some formulations a and d are zero so that without competition, $x(t)$ and $y(t)$ grow exponentially. The number b measures the effect of y on x and in a sense measures how much the maximum supportable x is reduced in the presence of y. Similarly, c measures the effect of x on y.

Depending on the parameter values, there may be either an equilibrium point (x_e, y_e) with both $x_e > 0$ and $y_e > 0$ so that both populations exist 'in harmony,' or one population may be so competitive that the other is driven to extinction.

The equilibrium points of Equations 18.56 are found by solving

$$\begin{array}{rclrclcl} x' &=& 0 \qquad & f(x,y) &=& r_x x(1 - ax - by) & = & 0 \\ y' &=& 0 \qquad & g(x,y) &=& r_y \times y \times (1 - cx - dy) & = & 0 \end{array}$$

There are four solutions if $ad - bc \neq 0$.

$$\begin{array}{rclrcl} x &=& 0, \qquad & y &=& 0 \\ x &=& 0, \qquad & y &=& 1/d \\ x &=& 1/a, \qquad & y &=& 0 \\ x &=& \dfrac{d-b}{ad-bc}, \qquad & y &=& \dfrac{a-c}{ad-bc}. \end{array}$$

The equilibrium $(0, 1/d)$ signals x is not present and y is at its maximum supportable level. Similarly $(1/a, 0)$ signals that x is at its maximum supportable population and y is not present.

The Jacobian matrix of

$$f(x,y) = r_x x - r_x a x^2 - r_x bxy \qquad \text{and} \qquad g(x,y) = r_y y - r_y cxy - r_y dy^2$$

is computed from the partial derivatives:

$$\begin{array}{rclrcl} f_x(x,y) &=& r_x - 2r_x ax - r_x by \qquad & f_y(x,y) &=& -r_x bx \\ g_x(x,y) &=& -r_y cy \qquad & g_y(x,y) &=& r_y - r_y cx - 2r_y dy \quad . \end{array}$$

The Jacobian matrices at the first three equilibrium points are

$$\begin{array}{ccc} (0,0) & (0,1/d) & (1/a,0) \\ \\ \left[\begin{array}{cc} r_x & 0 \\ 0 & r_y \end{array}\right] & \left[\begin{array}{cc} r_x(1-b/d) & 0 \\ -r_y(c/d) & -r_y \end{array}\right] & \left[\begin{array}{cc} -r_x & -r_x(b/d) \\ 0 & r_y(1-c/a) \end{array}\right] \end{array} .$$

For diagonal, lower triangular or upper triangular matrices, the diagonal numbers are the characteristic numbers for the matrix. In a 2 by 2 matrix the characteristic equation is

$$r^2 - (a_{1,1} + a_{2,2})\, r + a_{1,1}\, a_{2,2} = (r - a_{1,1})\,(r - a_{2,2}) = 0$$

$$\Longrightarrow \quad r = a_{1,1} \quad \text{or} \quad r = a_{2,2}.$$

The equilibrium (0,0) is nonstable. The trace of the Jacobian matrix is $r_x + r_y$ and is positive and if (x_0, y_0) is close to (0,0), (a small number of each of x and y is introduced into the environment), $(x(t), y(t)$ will move away from (0,0). (In fact, both $|\,x(t)\,|$ and $|\,y(t)\,|$ will initially increase.)

The equilibrium $(0, 1/d)$ is ambiguous. If $1 - b/d < 0$ then the trace of the Jacobian matrix, $r_x(1 - b/d) - r_y$ is negative and the determinant, $r_x(1 - b/d) \times (-r_y)$ is positive. In this case, $(0, 1/d)$ is a stable node. This is bad news for x. It means that if y is established (is at its maximum supportable population) and a few members of x immigrate, those members of x will not have enough offspring to become established; $x(t)$ will decrease to zero.

The condition $1 - b/d < 0$ implies that $b \times 1/d > 1$. Recall that b measures the competitive influence of y on x and $1/d$ is the maximum supportable y population. If the product of these two numbers is larger than 1, then y can exclude x from the region.

If $1 - b/d > 0$ the determinant of the Jacobian matrix is positive, both roots are real and at least one root is positive. In this case $(0, 1/d)$ is not stable and a few members of x entering the region will initially increase.

In a similar way, $(1/a, 0)$ may or may not be a stable equilibrium point of the system and introduction of a few members of y may not or may lead to initial growth.

In order to understand the equilibrium $(d - b)/(ad - bc), (a - c)/(ad - bc)$ we consider three examples.

Example 1 Two sheep.	Example 2 Pig and sheep.	Example 3 Two pigs.
$x' = 0.1x(1 - x - 0.5y)$	$x' = 0.1x(1 - x - 0.5y)$	$x' = 0.1x(1 - x - 1.4y)$
$y' = 0.2y(1 - 0.6x - y)$	$y' = 0.2y(1 - 1.4x - y)$	$y' = 0.2y(1 - 1.4x - y)$

The difference between these three examples is that x competes more severely with y in Example 2 (1.4) than in Example 1 (0.6), and both compete severely in Example 3.

In these examples, and generally, it is instructive to look at the *nullclines* of x' and y' shown in Figure 18.12. The nullclines are the curves along which $x' = 0$ or $y' = 0$.

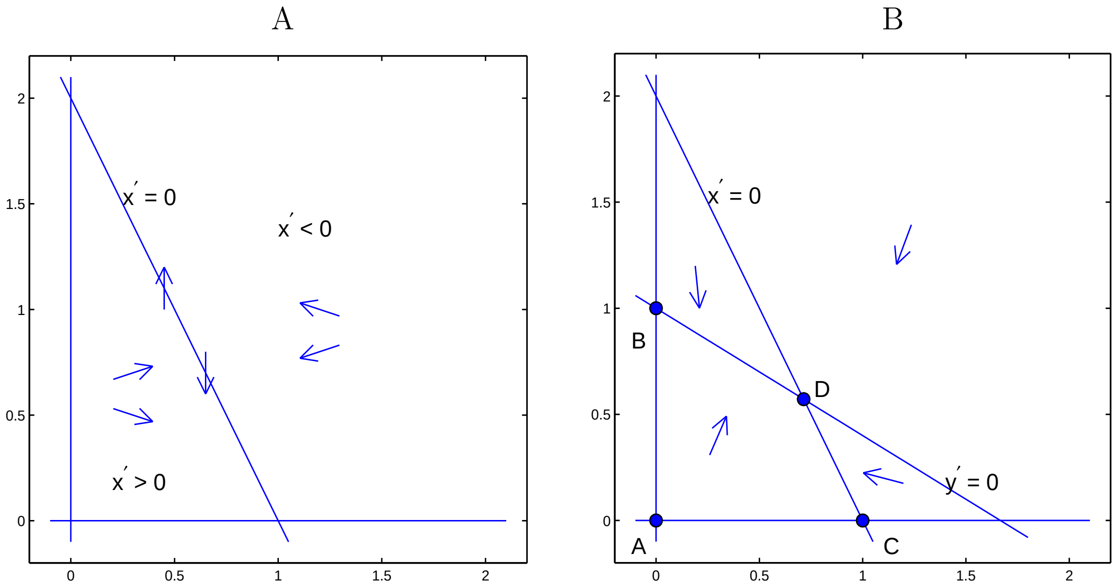


Figure 18.12: Nullcline of $x' = 0.1x(1-x-0.5y)$ in A, and of both $x' = 0.1x(1-x-0.5y)$ and $y' = 0.2y(1-0.6x-y)$ in B.

In Figure 18.12A is the single nullcline for $x' = 0.1x(1-x-0.5y) = 0$. The graph is the straight line with equation $1-x-0.5y = 0$ and is marked $x' = 0$ in Figure 18.12A. For a point above or to the right of that line, $1-x-0.5y$ is negative and x' is negative at that point. Consequently, direction field arrows point to the left above the line $1-x-0.5y = 0$. Whether they point to the left and up or to the left and down depends on y' at that location and one of each type is displayed. Similarly, direction field arrows below the line point to the right. Direction field arrows exactly on the line are vertical.

In Figure 18.12B, nullclines for both $x' = 0.1x(1-x-0.5y) = 0$ and $y' = 0.2x(1-0.6x-y) = 0$. They are the graphs of

$$\begin{array}{rcl} 1-x-0.5y & = & 0 \\ 1-0.6x-y & = & 0 \end{array} \qquad \text{which intersect at} \qquad (5/7, 4/7)$$

As in Figure 18.12A, direction field arrows above $1-x-0.5y = 0$ point to the left. If those arrows are also above $1-0.6x-y = 0$, then $y' < 0$ and the arrow point to the left and downward. Arrows below both lines point up and to the right. The four equilibrium points A, B, C, and D are (0,0), (0,1), (1,0) and (5/7,4/7).

A direction field for Example 1 appears in Figure 18.13A. It appears that the equilibrium points A, B, and C are nonstable and this follows from discussion above. It also appears that D is a stable node, and we show this next.

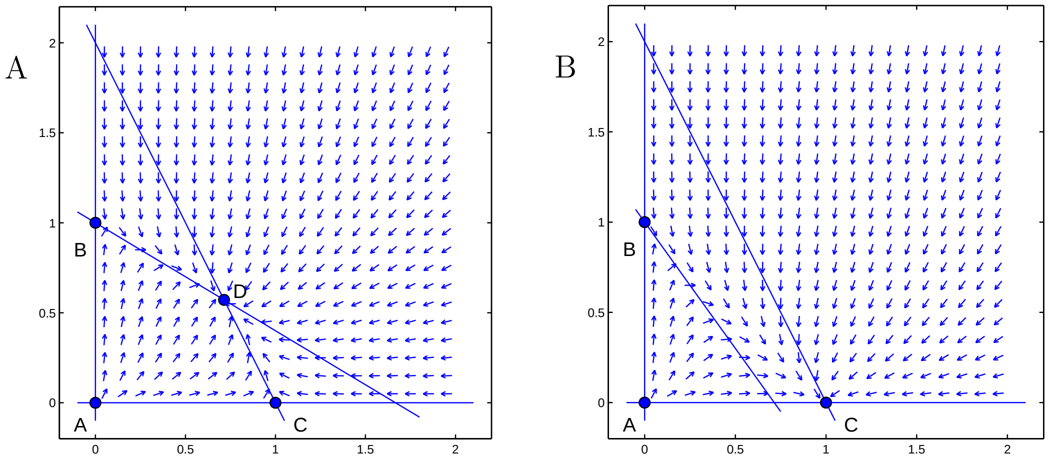


Figure 18.13: A. Direction field for Example 1. Direction field for Example 2.

The local linear approximation to Example 1 at $(5/7, 4/7)$ can be computed from:

$$\begin{array}{rcl@{\qquad\qquad}rcl} F(x,y) & = & 0.1x - 0.1x^2 - 0.05xy & G(x,y) & = & 0.2y - 0.12xy - 0.2y^2 \\ F_1(x,y) & = & 0.1 - 0.2x - 0.05y & F2(x,y) & = & -0.05x \\ G_1(x,y) & = & -0.12y & G_2(x,y) & = & 0.2 - 0.12x - 0.4y \end{array}$$

$$\begin{array}{rcl@{\qquad\qquad}rcl} F_1(5/7,4/7) & = & -0.5/7 & F_2(5/7,4/7) & = & -0.25/7 \\ G_1(5/7,4/7) & = & -0.48/7 & G_2(5/7,4/7) & = & -0.8/7 \end{array}$$

The Jacobian matrix is

$$M = \begin{bmatrix} -0.5/7 & -0.25/7 \\ -0.48/7 & -0.8/7 \end{bmatrix},$$

the trace is -0.5/7 - 0.8/7 = -1.3/7 is negative, and the determinant (-0.5/7)(-0.8/7 - (-0.48/7)(-0.25/7) = 0.04/7 is positive. According to the summary in the last section, M has two negative characteristic roots and $(5/7, 4/7)$ is a stable node of Example 1.

The direction field for Example 2 is shown in Figure 18.13B. The 'equilibrium' point D has moved out of the first quadrant, would be in the fourth quadrant with a unrealistic negative y coordinate. You can see from the direction field that all solutions lead to C. x competes severely with y and eventually only x remains in the region.

The equilibrium point D reappears in Example 3, as seen in Figure 18.14. But D is not stable. One solution actually leads to D from above the graph and another from below the graph but all other curves diverge either to B or C. In this Example only one of x and y has long term presence in the region, depending on the initial conditions x_0, y_0.

This last case may be the most realistic, for when two species enter head to head competition, often one of them will change its food source and may even migrate to another environment. Long term competition is rare. We are locked in a long term competition with The World, however, as hinted at in Figure 18.15.

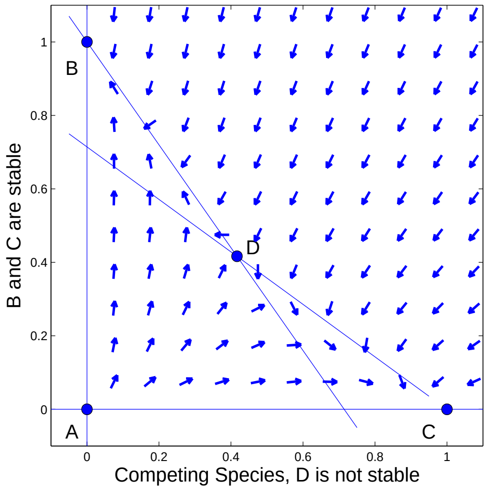


Figure 18.14: Direction field for Example 3. The equilibrium D is not stable; it is a saddle point.

18.5.2 Predator-prey

Volterra introduced the predator-prey model where x is the prey and y is the predator.

$$\begin{aligned} x'(t) &= ax - bxy \\ y'(t) &= -cy + dxy \end{aligned} \tag{18.57}$$

The model suggests that if the predator is not present $(y = 0)$ the prey will grow exponentially $x(t) = x_0 e^{at}$ and if the prey is not present $(x = 0)$ the predator will decay exponentially $y(t) = y_0 e^{-ct}$. The presence of y reduces the growth rate of x and the presence of x increases the growth rate of y.

There are two equilibrium points of Equations 18.57 found by solving

$$\begin{aligned} ax - bxy &= 0 \qquad & (x_1, y_1) &= (0, 0) \\ -cy + dxy &= 0 \qquad & (x_2, y_2) &= (c/d, a/b) \end{aligned}$$

The nullclines are the graphs of $x = c/d$ and $y = a/b$ and $x = 0$ and $y = 0$, and run vertically or horizontally through the equilibrium points. The Jacobian matrices are easily computed.

$$\begin{aligned} F(x,y) &= ax - bxy \qquad & G(x,y) &= -cy + dxy \\ F_1(x,y) &= a - by \qquad & F2(x,y) &= -bx \\ G_1(x,y) &= -cy \qquad & G_2(x,y) &= -c + dx \end{aligned}$$

At the equilibrium points the Jacobian matrices are

$$\text{At} \quad (0,0), \quad \begin{bmatrix} a & 0 \\ 0 & -c \end{bmatrix}. \qquad \text{At} \quad (c/d, a/b)), \quad \begin{bmatrix} 0 & -bc/d \\ -ca/b & 0 \end{bmatrix}.$$

Neither equilibrium point is stable. At (0,0) there is one positive root and one negative root. At $(c/d, a/b)$ the roots are complex with real part equal to 0; the solution curves of the local linear approximation are ellipses.

Figure 18.15: A. NASA photo of deforestation in Tierras Bajas project, Bolivia, from the International Space Station. http://earthobservatory.nasa.gov/Newsroom/NewImages/ images.php3?img_id=4842 B. Orang-utan (man of the forest) has been tranquilized to move it to a safe location after the forest he lived in had been destroyed to make way for a palm oil plantation. Photograph 18th November 2007 by Hardi Baktiantoro. The Centre for Orangutan Preservation was established in 2007 by Hardi Baktiantoro, specifically to tackle the cause of deforestation and the deaths of tens of thousands of orangutans. Their website is http://www.orangutanprotection.com/.

The direction field and equilibrium points of

$$\begin{aligned} x' &= 0.2x - 0.2xy \\ y' &= -0.1y + 0.08xy \end{aligned} \tag{18.58}$$

appear in Figure 18.17. The solution down the vertical axis to the origin shows the decline of predator without prey, the solution leading away from the origin along the horizontal axis shows the growth of

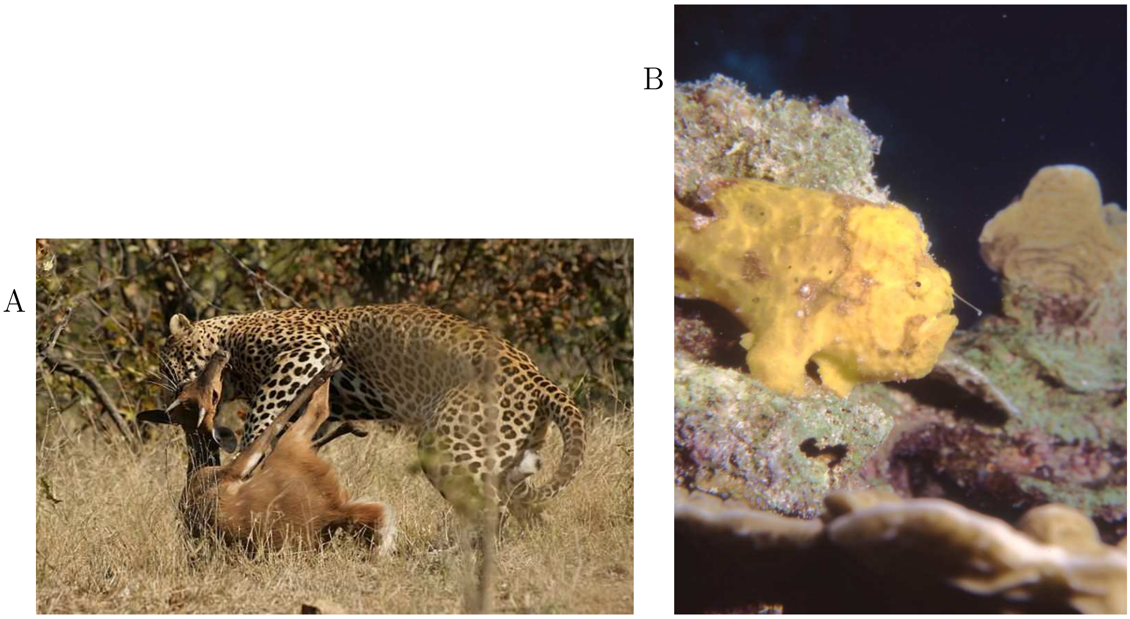


Figure 18.16: A. Leopard kill of a bush buck in Kruger National Park, bounded by South Africa, Zimbabwe and Mozambique. Photo by NJR ZA on http://en.wikipedia.org/wiki/Predation. Issued under CreativeCommons Attribution-ShareAlike 3.0 Unported (cc by-sa 3.0) license. B. Longlure Frogfish. The appendage emanating in front of the eyes attracts small fish which are rapidly engulfed by the frogfish. Photo by JLC in Bonaire.

prey without predators. The solutions near the equilibrium (1.25,1.0) appear to circle around (1.25,1.0). In fact, they do.

Danger: Outrageous manipulations ahead. Think of one of those curves as being described by a function s such that

$$y = s(x), \qquad \text{and write} \qquad y(t) = s(x(t)).$$

By the chain rule

$$y'(t) = s'(x(t))x'(t), \qquad \text{and} \qquad s'(x(t)) = \frac{y'(t)}{x'(t)}.$$

Now $s'(x)$ means the derivative of s with respect to x and would be written in Leibnitz notation as $\frac{ds}{dx}$, and because $y = s(x)$ it may also be written as $\frac{dy}{dx} = \frac{ds}{dx}$. We write

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{-0.1y + 0.08xy}{0.2x - 0.2xy}.$$

The variables are separable in this differential equation.

$$\begin{aligned} \frac{dy}{dx} &= \frac{-0.1y + 0.08xy}{0.2x - 0.2xy} \\ \left(\frac{0.2}{y} - 0.2\right)\frac{dy}{dx} &= \frac{-0.1}{x} + 0.08 \\ 0.2 \ln y - 0.2y &= -0.1 \ln x + 0.08x + C \end{aligned}$$

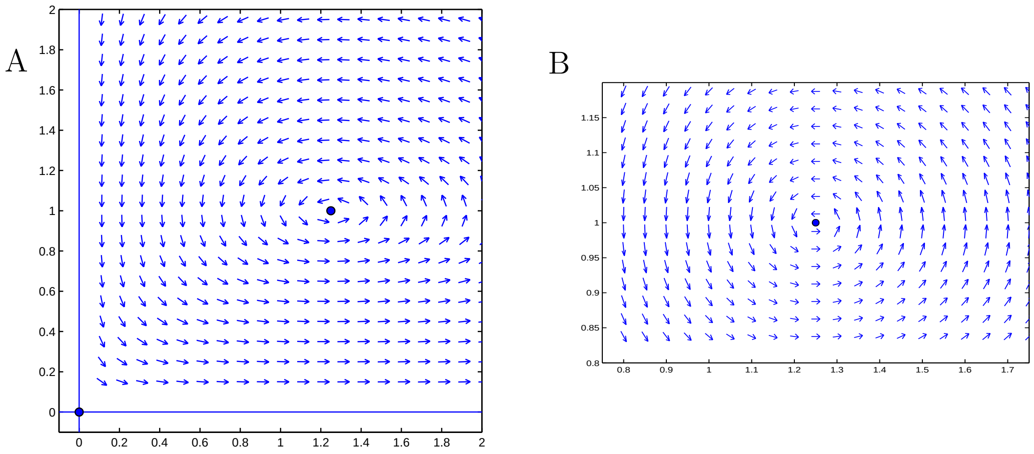


Figure 18.17: A. Direction field for Equations 18.58. B. Magnified view near the equilibrium $(1.25, 1)$.

$$0.2\ln y + 0.1\ln x - 0.2y = 0.08x \quad = \quad C \tag{18.59}$$

where C is a constant of integration.

Shown in Figure 18.18A is a graph in three dimensions of the function

$$F(x,y) = 0.2\ln y + 0.1\ln x - 0.2y - 0.08x, \qquad 1 \le x \le 1.4, \qquad 0.8 \le y \le 1.2 \tag{18.60}$$

The high point of F is $(1.25, 1.0, -0.2776)$, directly above the equilibrium point $(1.25, 1.0)$. A horizontal plane at $z = -0.279$ (corresponding to $C = -0.279$ in equation 18.59) intersects the graph of F in a closed curve in Figure 18.18; every such horizontal plane defines a closed solution curve to Equations 18.58 that surrounds the equilibrium point $(1.25, 1.0)$.

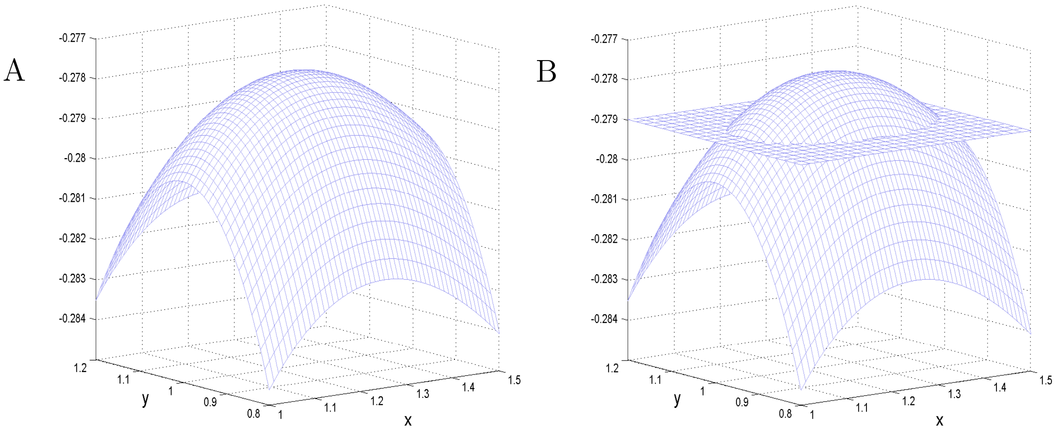


Figure 18.18: A. Graph of Equation 18.60, $F(x,y) = 0.2\ln y + 0.1\ln x - 0.2y - 0.08x$. B. Same graph with a horizontal plane, corresponding to $C = -0.29$ in Equation 18.59.

Explore 18.5.1 Show that the hight point of

$$F(x,y) = 0.2 \ln y + 0.1 \ln x - 0.2y - 0.08x, \qquad 1 \leq x \leq 1.4, \qquad 0.8 \leq y \leq 1.2$$

is $(1.25, 1.0, F(1.25, 1.0))$. Use Theorem 10.2.1 of Volume I. Begin by solving simultaneously,

$$\frac{\partial F(x,y)}{\partial x} = 0, \qquad \text{and} \qquad \frac{\partial F(x,y)}{\partial y} = 0$$

There is only one solution, (x_m, y_m). Show that

$$\left.\frac{\partial^2 F(x,y)}{\partial x^2}\right|_{(x_m,y_m)} < 0, \qquad \text{and} \qquad \left.\frac{\partial^2 F(x,y)}{\partial y^2}\right|_{(x_m,y_m)} < 0, \qquad \text{and}$$

$$\Delta = \left.\frac{\partial^2 F(x,y)}{\partial x^2}\right|_{(x_m,y_m)} \left.\frac{\partial^2 F(x,y)}{\partial y^2}\right|_{(x_m,y_m)} - \left(\left.\frac{\partial^2 F(x,y)}{\partial x \partial y}\right|_{(x_m,y_m)}\right)^2 > 0.$$

■

18.5.3 Symbiosis

Mutual benefit of individuals in different species is an important, widespread biological relationship referred to as symbiosis. Lichens are symbiotic organisms consisting of a fungus and an alga, and live in extremely diverse environments: arctic tundra, desert, rain forest branches and leaves. Cleaner fish may obtain most of their nutrition from removing dead skin and ectoparasites from large predatory fish. The goby fish sometimes lives together with a shrimp. The shrimp digs and cleans a burrow in the sand in which both the shrimp and the goby fish live. The shrimp is almost blind, leaving it vulnerable to predators when above ground. In case of danger the goby fish touches the shrimp with its tail to warn it. When that happens both the shrimp and goby fish quickly retreat into the burrow. (Examples taken from Wikipedia, http://en.wikipedia.org/wiki/Symbiosis).

Figure 18.19: A. Cleaner fish, Bluestreak cleaner wrass cleaning a Hawaiian White-spotted puffer. Posted by Mbz1 on http://en.wikipedia.org/wiki/Cleaner_fish. B. Lichen *Rhizocarpon geographicum* posted by fir0002 on http://en.wikipedia.org/wiki/Lichen.

A fascinating example of a goby *Gobidon histrio* mutualism with a coral *Acropora nasuta* is described in D. L. Dixon and M. E. Hay, Chorals chemically cue mutualistic fishes to remove competing seaweeds, *Science* **338**, 804 (2012). Close contact of the seaweed *Chlorodesmis fastigiata* transmits coral diseases and creates anoxic zones and enhance detrimental microbes on the coral. "Within minutes of seaweed contact $\cdots$ the coral releases an odor that recruits gobies to trim the seaweed and dramatically reduce coral damage that would otherwise occur." The gobies produce a toxic skin secretion which becomes more toxic after consuming the sea weed. A less technical account appears in Susan Milius, In the News, Story One, *Science News*, December 1, 2012, 5-6.

The analysis of symbiosis is included in Exercises 18.5.5 - 18.5.7

18.5.4 Susceptible, Infected, Recovered.

Again, an elementary model of infectious disease can form the basis of discussions. Assume there is a virus that is spread by contact from infected individuals to susceptible individuals and that infected individuals recover and are immune to the virus. Let $s(t)$, $i(t)$ and $r(t)$ denote the numbers of susceptible, infected, and recovered individuals, respectively. Assume that the rate at which infection spreads is proportional to the number of susceptible and to the number of infected. We write the equations:

$$\begin{aligned} s'(t) &= -\beta \times s(t) \times i(t) \\ i'(t) &= \beta \times s(t) \times i(t) - \gamma i(t) \\ r'(t) &= \gamma i(t) \end{aligned} \tag{18.61}$$

The parameter β is crucial. It reflects the probability of transmission of the disease and depends on the frequency of contacts between people and the ease of transmission of the disease.

Although three functions are written, the first two equations are independent of $r(t)$ and can be analyzed alone. The equilibrium points are $(0, 0)$ and $(s_0, 0)$. The interesting equilibrium is $(s_0, 0)$, and at $(s_0, 0)$ the Jacobian matrix is

$$M = \begin{bmatrix} 0 & -\beta s_0 \\ 0 & \beta s_0 - \gamma \end{bmatrix} \tag{18.62}$$

The roots of M are real and are zero and negative if $\beta\, s_0 < \gamma$ and zero and positive if $\beta\, s_0 > \gamma$.

$\beta S_0 = \gamma$ marks an important threshold. If $\beta s_0 < \gamma$, then $(s_0, 0)$ is a stable equilibrium. If a small number of infected individuals enter the populations with $\beta s_0 < \gamma$, the disease does not spread. If $\beta s_0 > \gamma$, then $(s_0, 0)$ is an unstable equilibrium. If a small number of infected individuals enter the populations with $\beta s_0 > \gamma$, the disease may become an epidemic.

Exercises for Section 18.5, Applications of Theorem 18.4.1 to biological systems.

Exercise 18.5.1 Draw the nullclines and some direction arrows and analyze the equilibria of the following competition models.

$$\begin{array}{rrcl} \text{a.} & x'(t) & = & 0.2 \times x(t) \times (1 - 0.2x(t) - 0.4y(t)) \\ & y'(t) & = & 0.1 \times y(t) \times (1 - 0.4x(t) - 0.5y(t)) \end{array}$$

$$\begin{array}{rrcl} \text{b.} & x'(t) & = & 0.2 \times x(t) \times (1 - 0.2x(t) - 0.8y(t)) \\ & y'(t) & = & 0.1 \times y(t) \times (1 - 0.4x(t) - 0.5y(t)) \end{array}$$

$$\begin{array}{rrcl} \text{c.} & x'(t) & = & 0.2 \times x(t) \times (1 - 0.6x(t) - 0.4y(t)) \\ & y'(t) & = & 0.1 \times y(t) \times (1 - 0.4x(t) - 0.5y(t)) \end{array}$$

$$\begin{array}{rrcl} \text{d.} & x'(t) & = & 0.2 \times x(t) \times (1 - 0.4x(t) - 0.4y(t)) \\ & y'(t) & = & 0.1 \times y(t) \times (1 - 0.3x(t) - 0.5y(t)) \end{array}$$

Exercise 18.5.2 Show that a stable equilibrium exists for the competition Equations 18.56,

$$\begin{aligned} x'(t) &= r_x \times x(t) \times (1 - ax(t) - by(t)) \\ y'(t) &= r_y \times y(t) \times (1 - cx(t) - dy(t)), \end{aligned}$$

if $a > c$ and $d > b$. (Suggestion: Draw the nullclines and candidate direction arrows in each of the four regions bounded by the nullclines.)

Exercise 18.5.3 Is the equilibrium point, $(1.25, 1.0)$, of predator-prey equations 18.58,

$$\begin{aligned} x' &= 0.2x - 0.2xy \\ y' &= -0.1y + 0.08xy \end{aligned} \qquad \text{stable?}$$

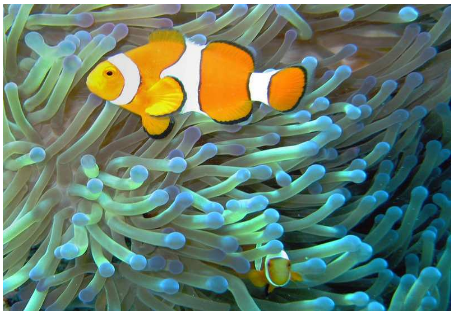
Is that equilibrium point asymptotically stable? As usual, you should explain your answers.

Exercise 18.5.4 The predator-prey equations assume that with no predator, the prey grows exponentially. Alternatively one might assume that with no predator, the prey grow according to a logistic (Verhultz) model. Write the predator-prey equations so that without predators the prey grows according to a logistic model. Find conditions for there to be an equilibrium for which both predator and prey exist, and determine the character of that equilibrium.

Exercise 18.5.5 Consider two interacting populations, x and y, that are mutually symbiotic: the presence of x enhances the growth of y and the presence of y enhances the growth of x. A dynamic relation between x and y may take the form

$$\begin{aligned} x'(t) &= r_x \times x(t) \times (1 - ax(t) + by(t)) \\ y'(t) &= r_y \times y(t) \times (1 + cx(t) - dy(t)) \end{aligned} \qquad (18.63)$$

Describe the roles of the parameters a, b, c, and d in Equations 18.63.



In a symbiotic mutualism, the clownfish feeds on small invertebrates that otherwise have potential to harm the sea anemone, and the fecal matter from the clownfish provides nutrients to the sea anemone. The clownfish is additionally protected from predators by the anemone's stinging cells, to which the clownfish is immune. -Wikipedia.

Exercise 18.5.6

Draw the nullclines and some direction arrows and analyze the equilibria of the following symbiosis models.

$$\text{a.}\quad \begin{aligned} x'(t) &= 0.2 \times x(t) \times (1 - 0.5x(t) + 0.4y(t)) \\ y'(t) &= 0.1 \times y(t) \times (1 + 0.4x(t) - 0.5y(t)) \end{aligned}$$

$$\text{b.}\quad \begin{aligned} x'(t) &= 0.2 \times x(t) \times (1 - 0.8x(t) + 0.4y(t)) \\ y'(t) &= 0.1 \times y(t) \times (1 + 0.4x(t) - 0.5y(t)) \end{aligned}$$

$$\text{c.}\quad \begin{aligned} x'(t) &= 0.2 \times x(t) \times (1 - 0.5x(t) + 0.4y(t)) \\ y'(t) &= 0.1 \times y(t) \times (1 + 0.4x(t) - 0.2y(t)) \end{aligned}$$

$$\text{d.}\quad \begin{aligned} x'(t) &= 0.2 \times x(t) \times (1 - 5x(t) + 10y(t)) \\ y'(t) &= 0.1 \times y(t) \times (1 + 2x(t) - 5y(t)) \end{aligned}$$

$$\text{e.}\quad \begin{aligned} x'(t) &= 0.2 \times x(t) \times (1 - 1.1x(t) + y(t)) \\ y'(t) &= 0.1 \times y(t) \times (1 + x(t) - y(t)) \end{aligned}$$

$$\text{f.}\quad \begin{aligned} x'(t) &= 0.2 \times x(t) \times (1 - 0.9x(t) + y(t)) \\ y'(t) &= 0.1 \times y(t) \times (1 + x(t) - y(t)) \end{aligned}$$

Exercise 18.5.7 Symbiotic relationships are common and persist for long periods. It is curious that there are no known or very few symbiotic relationships between mammals. There are, however, many symbiotic relationships between mammals and other organisms, *Escherichia coli*, for example.

Analysis of the equations for symbiosis, Equations 18.63:

$$\begin{aligned} x'(t) &= r_x \times x(t) \times (1 - ax(t) + by(t)) \\ y'(t) &= r_y \times y(t) \times (1 + cx(t) - dy(t)) \end{aligned}$$

a. Show that the equilibrium point without zeros is

$$x_1 = \frac{b+d}{ad-bc}, \qquad y_1 = \frac{a+c}{ad-bc} \qquad \text{if} \quad ad - bc \neq 0.$$

b. x_1 and y_1 are positive only if $ad - bc > 0$. This is a surprise to us. Set $b = c = d = 1$ and examine the equilibrium point for $a > 1$ and $a < 1$.

c. Assume $ad - bc > 0$ so that x_1 and y_1 are positive. Either work it out (no!) or accept our analysis that the Jacobian at (x_1, y_1) is

$$J = \begin{bmatrix} -a\,\dfrac{b+d}{ad-bc} & b\,\dfrac{b+d}{ad-bc} \\ \\ c\,\dfrac{a+c}{ad-bc} & -d\,\dfrac{a+c}{ad-bc} \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} -a\,(b+d) & b\,(b+d) \\ \\ c\,(a+c) & -d(a+c) \end{bmatrix}$$

Argue that if $ad - bc > 0$, (x_1, y_1) is an asymptotically stable equilibrium.

d. With persistence you might show that the characteristic roots are not complex. It requires showing that the discrimiant

$$\begin{aligned} \left(\,a(b+d) + d(a+c)\,\right)^2 - 4\left(\,a(b+d)d(a+c) - c(a+c)d(b+d)\,\right) \;&=\; \\ \\ \left(\,a(b+d) - d(a+c)\,\right)^2 + 4c(a+c)d(b+d) \;&\geq\; 0. \end{aligned}$$

Exercise 18.5.8 Anderson and May[4] give the following model of immune effector cells (helper and cytotoxic T-cells), E, that limit viral population, V, growth in a human body.

$$\begin{aligned} dE/dt &= \Lambda - \mu E + \epsilon V E \\ \\ dV/dt &= rV - \sigma V E \end{aligned} \tag{18.64}$$

a. Λ is intrinsic production rate of effector cells from bone marrow. Give similar meaning to each of the other four terms on the RHS of Equations 18.64.

b. Find the equilibrium effector cell population, $\hat{E}$, in the absence of virus.

c. Suppose an inoculum V_0 of virus is introduced into the body with $E = \hat{E}$. Find conditions on r, σ, and $\hat{E}$ in order that the viral population will increase.

d. The Jacobian matrix at any (E, V) is

$$J(E,V) = \begin{bmatrix} -\mu + \epsilon V & \epsilon E \\ -\sigma V & r - \sigma V \end{bmatrix}.$$

Show that

$$J(\hat{E}, 0) = \begin{bmatrix} -\mu & \sigma\dfrac{\Lambda}{\mu} \\ 0 & r - \sigma\dfrac{\Lambda}{\mu} \end{bmatrix}.$$

e. The characteristic values of the upper diagonal matrix $J(\hat{E}, 0)$ are the diagonal entries, $-\mu$ and $r - \sigma\frac{\Lambda}{\mu}$. What happens to a small introduction of virus into a healthy individual if both characteristic values are negative?

f. In order that the viral population to expand it is necessary that $r - \sigma\frac{\Lambda}{\mu} > 0$. What is the role of r in the model?

---

[4] R. M. Anderson and R. M. May, *Infectious Diseases of Humans*, Oxford University Press, Oxford, 1991, p33

g. If that condition is met and the viral population increases, show that there will be an equilibrium state,

$$E^* \quad = \quad r/\sigma, \qquad V^* \quad = \quad \frac{\mu r - \Lambda\sigma}{\epsilon r}.$$

It is clear that in order for V^* to be positive, we must have (again)

$$\mu r - \Lambda\sigma > 0 \qquad \text{so that} \qquad r > \frac{\Lambda\sigma}{\mu}.$$

h. Anderson and May report this system to be asymptotically stable, at (E^*, V^*), but only weakly so, meaning that it is subject to wide oscillations. Show that

$$J(E^*, V^*) = \begin{bmatrix} -\frac{\Lambda\sigma}{r} & \epsilon\frac{r}{\sigma} \\ -\sigma\frac{\mu r - \Lambda\sigma}{\epsilon r} & 0 \end{bmatrix}.$$

i. The characteristic equation of a 2×2 matrix M is is $s^2 - trace(M)\, s + det(M) = 0$. Show that the characteristic equation of $J(E^*, V^*)$ is

$$s^2 + \frac{\Lambda\sigma}{r}\, s + \mu r - \Lambda\sigma = 0. \tag{18.65}$$

j. The roots of the characteristic equation 18.65 are

$$\frac{-\frac{\Lambda\sigma}{r} \pm \sqrt{\left(\frac{\Lambda\sigma}{r}\right)^2 - 4(\mu r - \Lambda\sigma)}}{2}.$$

Argue that the real part is negative so that the system is stable. Note: Two cases:

$$\left(\frac{\Lambda\sigma}{r}\right)^2 - 4(\mu r - \Lambda\sigma) > 0 \quad \text{and} \quad \left(\frac{\Lambda\sigma}{r}\right)^2 - 4(\mu r - \Lambda\sigma) < 0$$

k. Use $\Lambda = 1$, $\mu = 0.5$, $\epsilon = 0.02$, $r = 0.25$, and $\sigma = 0.01$ and compute E^*, V^*, and the stability at $(E^*,\ V^*)$.

l. Let $E_0 = 2$, $V_0 = 1$, $\Lambda = 1$, $\mu = 0.5$, $\epsilon = 0.02$, $r = 0.25$, and $\sigma = 0.01$. Approximate the solutions to Equations 18.64 using the trapezoid rule. Observe that the rise in viral load precedes the increase in effector cells.

A MATLAB program to approximate the solution to Equations 18.64 using the trapezoid method:

```
close all;clc;clear
lam=1;mu=0.5;eps=0.02;rrr=0.25;sig=0.01;
e0=2;  v0=1;  E(1)=e0;  V(1)=v0;
for i = 1:60
    ME=lam - mu*E(i)+eps*V(i)*E(i); MV= rrr*V(i) - sig*V(i)*E(i);
    EP=E(i)+ME;    VP=V(i)+MV;
    MEP=lam - mu*EP+eps*VP*EP;    MVP=rrr*VP - sig*VP*EP;
```

```
%%      MEP=ME; MVP=MV;  %%Use this to revert to Euler's method
        E(i+1)=E(i)+(ME+MEP)/2;    V(i+1)=V(i) + (MV+MVP)/2;
    end
    plot(E,'+','linewidth',2)
    hold
    plot(V,'o','linewidth',2)
    xlabel('Time','fontsize',16)
    ylabel('Effector cells +, Virus o','fontsize',16)
```

Anderson and May argue that if

$$\left(\frac{\Lambda\sigma}{r}\right)^2 - 4(\mu r - \Lambda\sigma) \ll 0$$

then the solutions are periodic with period approximately $2\pi/(\mu r - \Lambda\sigma)$. They then discuss that $1/\mu$ is the life expectancy of the effector cells, and if life expectancy is in the order of years, continued immunity to the virus will follow.

Exercise 18.5.9 The local linear approximation to SIR Equations 18.61 at the equilibrium $(s_0, 0)$ uses the Jacobian matrix 18.62 and may be written

$$\begin{array}{rclcrcl} x'(t) & = & -\beta s_0 y & \qquad & x_0 & = & s_0 \\ y'(t) & = & (\beta s_0 - \gamma)\, y & \qquad & y_0 & = & \epsilon \end{array}$$

Think of ϵ as a small number of infected introduced to the population.

a. Show that

$$\begin{aligned} x(t) &= s_0 + \frac{\beta s_0 \epsilon}{\beta s_0 - \gamma}(1 - e^{(\beta s_0 - \gamma)t}) \\ y(t) &= \epsilon\, e^{(\beta s_0 - \gamma)t} \end{aligned}$$

solve the equations.

b. In the case that $\beta s_0 - \gamma < 0$ find $\lim_{t\to\infty} x(t)$ and $\lim_{t\to\infty} y(t)$, and discuss their meaning.

Exercise 18.5.10 It may be that recovered individuals do not have life time immunity; they become susceptible after a period p,and one may write

$$\begin{aligned} s'(t) &= \alpha r(t-p) - \beta \times s(t) \times i(t) \\ i'(t) &= \beta \times s(t) \times i(t) - \gamma i(t) \\ r'(t) &= \gamma i(t) - \alpha r(t-p) \end{aligned}$$

This system is considerably more complex than Equations 18.61, and is simplified by letting $p = 0$.

$$\begin{aligned} s'(t) &= \alpha r(t) - \beta \times s(t) \times i(t) \\ i'(t) &= \beta \times s(t) \times i(t) - \gamma i(t) \\ r'(t) &= \gamma i(t) - \alpha r(t) \end{aligned}$$

This system involves three functions and three equations and is beyond our exposition. However, you may be able to analyze it.

a. Show that $(s_0, 0, 0)$ is an equilibrium point, for any s_0.

b. Guess or compute the Jacobian matrix at $(s_0, 0, 0)$.

c. The characteristic values of the Jacobian matrix at $(s_0, 0, 0)$ are $\beta s_0 - \gamma$ and $-\alpha$. What is the criterion for an epidemic (the number of infected will increase when a small number, ϵ, of infected individuals enter a population of s_0 susceptible).

d. Solve the equations

$$\begin{aligned} x'(t) &= -\beta s_0\, y(t) + \alpha z(t) & \qquad x(0) &= s_0 \\ y'(t) &= (\beta s_0 - \gamma)\, y(t) & y(0) &= \epsilon \\ z'(t) &= \gamma y(t) - \alpha z(t) & z(0) &= 0 \end{aligned}$$

Solve first for $y(t)$, then for $z(t)$ and then for $x(t)$.

Exercise 18.5.11 Suppose in the SIR Equations 15.4 that $\beta s_0 > \gamma$ so that $(s_0, 0)$ is a non-stable equilibrium and $i_0 = \epsilon$ where ϵ is "small" and positive. Then the infection will spread. This problem shows that there will be a positive lower bound, b, on the susceptible, $s(t) \geq b$ for all t – not everyone gets sick.

We begin with the first two of Equations 15.4

$$\begin{aligned} s'(t) &= -\beta \times s(t) \times i(t) \\ i'(t) &= \beta \times s(t) \times i(t) - \gamma i(t). \end{aligned} \tag{18.66}$$

Shown in Figure 18.5.11 is a direction field of $(s(t), i(t))$ with $\beta = 0.01$ and $\gamma = 0.012$, and a solution to Equations 18.66 with $(s_0, i_0) = (1, 0.001)$ marked in red. The midpoint of each interval of the direction field is (s_k, i_k) and the slope of the interval is the ratio

$$\frac{i'}{s'} = \frac{\beta \times s_k \times i_k - \gamma i_k}{-\beta \times s_k \times i_k}.$$

a. The arrows on the solution in Figure 18.5.11 point towards increasing time. Describe the numbers of susceptible and infective as time progresses.

b. The point marked E on the graph is (0.685,0). What are the dynamics of the infection at E?

c. Compute the ratio

$$\frac{\frac{di}{dt}}{\frac{ds}{dt}} = \frac{\beta \times s(t) \times i(t) - \gamma i(t)}{-\beta \times s(t) \times i(t)}$$

and argue that

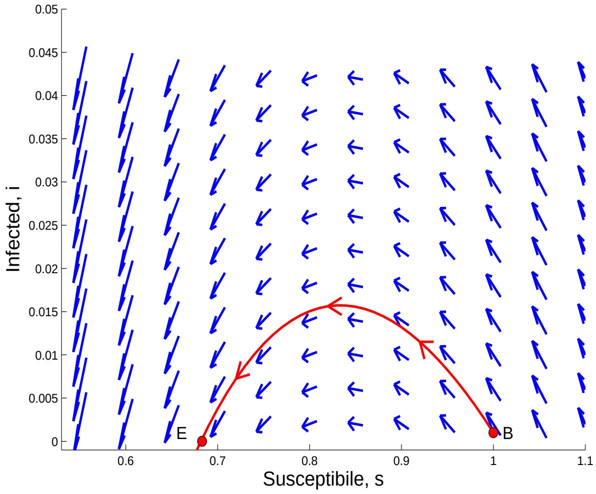
$$\frac{di}{ds} = -1 + \frac{\gamma}{\beta}\frac{1}{s}$$

d. Show that the solution to the previous equation is

$$i = i_0 + s_0 - s + \frac{\gamma}{\beta} \ln \frac{s}{s_0}.$$

e. Argue that there is a positive value of s for which $i = 0$.

Figure for Exercise 18.5.11 Direction field for Equations 18.66 with $\beta = 0.01$ and $\gamma = 0.012$, and a solution to Equations 18.66 with $(s_0, i_0) = (1, 0.001)$.



18.6 Chapter 18 Suggested solutions.

The answers that follow border on being complete solutions. Use only those parts that you need, or, better, use this to check your solution.

Explore 18.1.1. With $\omega = 1$,

$$y_p(t) = \frac{3\omega}{(2-\omega^2)^2 + 9\omega^2} \cos \qquad \omega t + \frac{2-\omega^2}{(2-\omega^2)^2 + 9\omega^2} \sin \omega t = 0.3 \cos t + 0.1 \sin t$$

and

$$y''(t) - 3y'(t) + 2y(t) = \sin \omega t \qquad = \sin t.$$

$$\begin{aligned} y_p(t) &= 0.3 \cos t + 0.1 \sin t \\ y_p''(t) &= -0.3 \sin t + 0.1 \cos t \\ y_p''(t) &= -0.3 \cos t - 0.1 \sin t \end{aligned}$$

$$\begin{aligned} & y_p''(t) - 3y_p'(t) + 2y_p(t) \\ = \; & (-0.3\cos t - 0.1\sin t) - 3(-0.3\sin t + 0.1\cos t) + 2(0.3\cos t + 0.1\sin t) \\ = \; & \sin t \end{aligned}$$

Exercise 18.1.2. Suppose r_1 is the only root of $r^2 + pr + q = 0$ and $y = te^{r_1 t}$. Then

$$\begin{aligned} y &= te^{r_1 t} \\ y' &= t\,r_1\,e^{r_1 t} + e^{r_1 t} \\ y'' &= t\,r_1^2\,e^{r_1 t} + 2r_1 e^{r_1 t}, \quad \text{and} \end{aligned}$$

$$\begin{aligned} y''(t) + py'(t) + qy(t) &= \\ (t\,r_1^2\,e^{r_1 t} + 2r_1 e^{r_1 t}) + p(t\,r_1\,e^{r_1 t} + e^{r_1 t}) + qte^{r_1 t} &= \\ (r_1^2 + pr_1 + q)\,t\,e^{r_1 t} + (2r_1 + p)e^{r_1 t} &= 0. \end{aligned}$$

Why is the last expression 0?

Exercise 18.1.4. Suppose

$$\begin{aligned} y_{p,1}''(t) + py_{p,1}'(t) + qy_{p,1}(t) &= f_1(t), \quad \text{and} \\ y_{p,2}''(t) + py_{p,2}'(t) + qy_{p,2}(t) &= f_2(t), \end{aligned}$$

and A and B are numbers. Then

$$\begin{aligned} A(y_{p,1}''(t) + py_{p,1}'(t) + qy(t)) &= A\,f_1(t), \quad \text{and} \\ B\left(y_{p,2}''(t) + py_{p,2}'(t) + qy_{p,2}(t)\right) &= B\,f_(t) \end{aligned}$$

$$\begin{aligned} Ay_{p,1}''(t) + By_{p,2}''(t) + Apy_{p,1}'(t) + Bpy_{p,2}'(t) + & \\ Aqy_{p,1}(t) + Bqy_{p,2}(t) &= A\,f_1(t) + B\,f_2(t) \end{aligned}$$

$$\begin{aligned} (Ay_{p,1} + By_{p,2})'' + p(Ay_{p,1} + By_{p,2})' + & \\ q(Ay_{p,1} + By_{p,2}) &= A\,f_1(t) + B\,f_2(t) \end{aligned}$$

Exercise 18.1.6. $p = 10$,

$$y'' + py' + y = 0 \qquad y(0) = 1 \qquad y'(0) = 0$$

$$\begin{array}{rclrcl}
p & = & 10, & y & = & \frac{12+5\sqrt{6}}{24}e^{(-5+2\sqrt{6})\,t}+\frac{12-5\sqrt{6}}{24}e^{(-5-2\sqrt{6}\,)t} \\
p & = & 6, & y & = & \frac{4+3\sqrt{2}}{8}e^{(-3+2\sqrt{2})\,t}+\frac{4-3\sqrt{2}}{8}e^{(-3-2\sqrt{2}\,)t} \\
p & = & 1, & y & = & e^{-t/2}\left(\frac{\sqrt{3}}{3}\sin((\sqrt{3}/2)\,t)+\cos((\sqrt{3}/2)\,t)\right) \\
p & = & 0, & y & = & \cos t \\
p & = & -6, & y & = & \frac{4-3\sqrt{2}}{8}e^{(3+2\sqrt{2})\,t}+\frac{4+3\sqrt{2}}{8}e^{(3-2\sqrt{2}\,)t}
\end{array}$$

Exercise 18.1.8.

$$\begin{array}{lrcllrcl}
\text{a.} & y_p(t) & = & 4/9+t/3 & & y_h(t) & = & C_1e^{-t}+C_2e^{-3t} \\
\text{c.} & y_p(t) & = & -\frac{1}{2}\,t\,e^{t} & & y_h(t) & = & C_1e^{t}+C_2e^{3t}
\end{array}$$

Exercise 18.1.10. a. Suppose $y(0)=0$, and $y'(0)=0$.

$$\begin{aligned}
y &= \frac{1}{m(\omega_0^2-\omega^2)}\cos\omega t+C_1\,\cos(\omega_0\,t)+C_2\,\sin(\omega_0\,t) \\
y' &= \frac{-\omega}{m(\omega_0^2-\omega^2)}\sin\omega t-C_1\,\omega_0\sin(\omega\,t)+C_2\,\omega_0\cos(\omega\,t)
\end{aligned}$$

$$\begin{aligned}
y(0)=0 &= \frac{1}{m(\omega_0^2-\omega^2)}\cos\omega 0+C_1\,\cos(\omega_0\,0)+C_2\,\sin(\omega_0\,) \\
0 &= \frac{1}{m(\omega_0^2-\omega^2)}+C_1=0, \\
C_1 &= \frac{-1}{m(\omega_0^2-\omega^2)}
\end{aligned}$$

$$\begin{aligned}
y'(0)=0 &= \frac{-\omega}{m(\omega_0^2-\omega^2)}\sin\omega 0-C_1\,\omega_0\sin(\omega_0\,0)+C_2\,\omega_0\cos(\omega_0\,0) \\
0 &= C_2
\end{aligned}$$

$$\begin{aligned}
y &= \frac{1}{m(\omega_0^2-\omega^2)}\cos\omega t+\frac{-1}{m(\omega_0^2-\omega^2)}\cos(\omega_0\,t) \\
&= \frac{1}{m(\omega_0^2-\omega^2)}\left(\cos(\omega t)-\cos(\omega_0 t)\right)
\end{aligned}$$

Exercise 18.1.11.

$$y_p(t) = \frac{1}{2m\omega_0}\times t\times\sin\omega_0 t$$

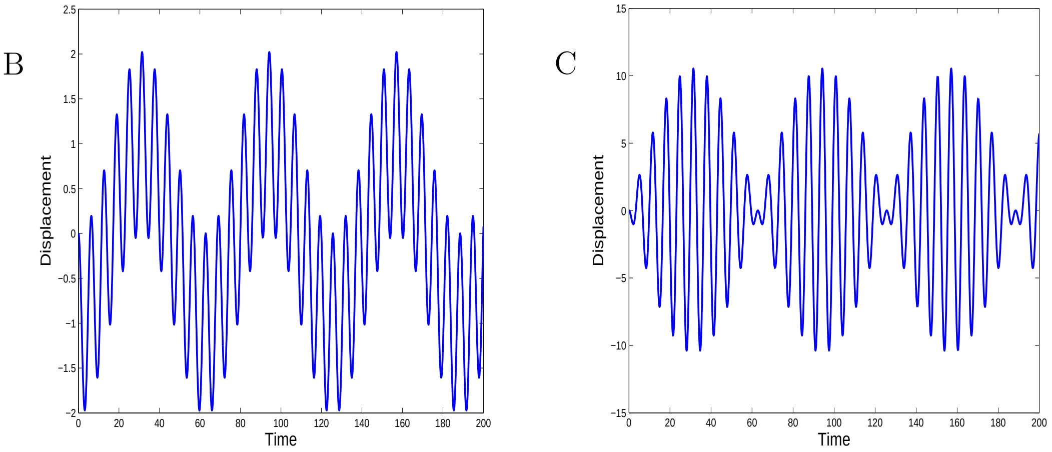


Figure 18.20: **Exercise 18.1.10.**(b. and c.) Graphs of $y = (1/(m\omega_0^2 - \omega^2))\,(\cos(\omega t) - \cos(\omega_0 t))$. B. Harmonic forcing with a weak spring, $m = 1$, $\omega = 1$, $\omega_0 = \sqrt{k/m} = 0.1$. C. Harmonic forcing with a stiff spring, $m = 1$, $\omega = 1$, $\omega_0 = \sqrt{k/m} = 0.9$.

$$
\begin{aligned}
y_p'(t) &= \frac{1}{2m\omega_0} t\omega_0 \cos\omega_0 t + \frac{1}{2m\omega_0}\sin\omega_0 t \\
y_p''(t) &= -\frac{1}{2m\omega_0} t\omega_0^2 \sin\omega_0 t + 2\frac{1}{2m\omega_0}\omega_0\cos\omega_0 t \\
y_p''(t) + \omega_0^2 y_p(t) &= -\frac{1}{2m\omega_0} t\omega_0^2 \sin\omega_0 t + 2\frac{1}{2m\omega_0}\omega_0\cos\omega_0 t + \\
&\qquad\qquad \omega_0^2\frac{1}{2m\omega_0} \times t \times \sin\omega_0 t \\
&= \frac{1}{m}\cos\omega_0 t
\end{aligned}
$$

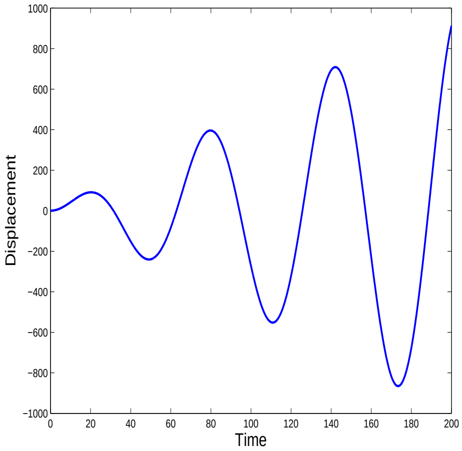


Figure 18.21: **Exercise 18.1.11.** Graph of $y = (1/(2m\omega_0))\sin(\omega_0 t)$. Resonant harmonic forcing in which the system flies apart.

Exercise 18.1.12.

$$my''(t) + ry'(t) + ky(t) = \cos \omega t.$$

For the homogeneous equation,

$$my''(t) + ry'(t) + ky(t) = 0, \qquad y = e^{st}, \qquad ms^2 + rs + k = 0.$$

The roots are

$$s_1 = \frac{-r + \sqrt{r^2 - 4mk}}{2m}, \qquad s_2 = \frac{-r - \sqrt{r^2 - 4mk}}{2m}$$

If $r^2 - 4mk > 0$, $\sqrt{r^2 - 4mk} < r$ and s_1 and s_2 are real and distinct and negative and

$$y_h(t) = C_1 e^{s_1 t} + C_2 e^{s_2 t} \qquad \to 0 \quad \text{as} \quad t \to \infty.$$

If $r^2 - 4mk = 0$, the roots are both $s_1 = s_2 = -r/(2m) < 0$ and

$$y_h(t) = C_1 e^{s_1 t} + C_2 t e^{s_1 t} \qquad \to 0 \quad \text{as} \quad t \to \infty.$$

If $r^2 - 4mk < 0$, s_1 and s_2 are both complex with real part $-r/(2m)$ and

$$\begin{aligned} y_h(t) &= e^{(-r/2m)t}\left(C_1 \cos(\frac{\sqrt{4mk - r^2}}{2m}t) + C_2 \cos(\frac{\sqrt{4mk - r^2}}{2m}t)\right) \\ &\to 0 \quad \text{as} \quad t \to \infty \end{aligned}$$

From the discussion,

$$y(t) = y_h(t) + A\cos\omega t + B \sin \omega t,$$

and from the previous steps, $y_h(t) \to 0$ as $t \to \infty$, so that $y(t)$ is asymptotic to $\frac{1}{r\omega}\cos\omega(t - 4/(\pi\omega))$.

Exercise 18.1.13.

$$\begin{aligned} y(t) &= -\frac{50}{\sqrt{0.99}} e^{-0.1t} \sin \sqrt{0.99}t + 50 \sin t \\ y' &= -50 e^{-0.1t} \cos \sqrt{0.99}t + \frac{5}{\sqrt{0.99}} e^{-0.1t} \sin \sqrt{0.99}t + 50 \cos t \\ y'' &= -50\sqrt{0.99} e^{-0.1t}(-\sin\sqrt{0.99}t) - 50(-0.1)e^{-0.1t}\cos\sqrt{0.99}t \\ &\quad + \frac{5}{\sqrt{0.99}} e^{-0.1t}\sqrt{0.99}\cos\sqrt{0.99}t \\ &\quad + \frac{5}{\sqrt{0.99}}(-0.1)e^{-0.1t}\sin\sqrt{0.99}t - 50 \sin t \\ &= \left(50\sqrt{0.99} - \frac{0.5}{\sqrt{0.99}}\right) e^{-0.1t}(-\sin\sqrt{0.99}t) \\ &\quad + (5+5)\, e^{-0.1t}\cos\sqrt{0.99}t - 50 \sin t \end{aligned}$$

$$\begin{aligned}
y'' + 0.2\,y' + y &= \left(50\sqrt{0.99} - \frac{0.5}{\sqrt{0.99}} + \frac{1}{\sqrt{0.99}} - \frac{50}{\sqrt{0.99}}\right) e^{-0.1t}(-\sin\sqrt{0.99}t) \\
&= (10 - 10 + 0)\, e^{-0.1t} \cos\sqrt{0.99}t \\
&\qquad + (-50 + 50)\sin t + 0.2 \times 50\cos t \\
&= 10\cos t
\end{aligned}$$

$$\begin{aligned}
y(0) &= -\frac{50}{\sqrt{0.99}} e^{-0.1\cdot 0}\sin\sqrt{0.99}\cdot 0 + 50\sin\cdot 0 = 0 \\
y'(0) &= -50e^{-0.1\cdot 0}\cos(\sqrt{0.99}\cdot 0) + \frac{5}{\sqrt{0.99}} e^{-0.1\cdot 0}\sin(\sqrt{0.99}\cdot 0) + 50\cos 0 = 0
\end{aligned}$$

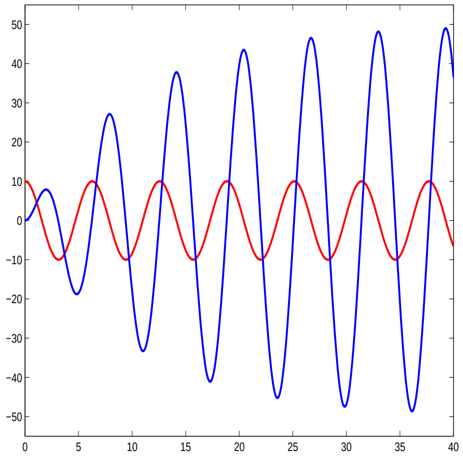


Figure 18.22: **Exercise 18.1.13.** Graph of $y = 10\cos t$ (red) and $y = -(50/\sqrt{0.99})e^{-0.1t}\sin\sqrt{0.99}t + 50\sin t$.

Explore 18.2.1.

$$\begin{aligned}
x(0) &= x_0 & \qquad x'(t) &= y(t) \\
y(0) &= y_0 & \qquad y'(t) &= 2x(t)y(t)
\end{aligned}$$

$$\begin{aligned}
\text{a.}\quad x(t) &= a\tan at & \qquad x'(t) &= a^2\sec^2 at \\
y(t) &= a^2\sec^2 at & \qquad y'(t) &= a^2(2\sec at)\frac{d}{dt}\sec at \\
& & &= 2a^3\sec at\sec at\tan at \\
& & x'(t) &= y(t) \\
& & y'(t) &= 2x(t)y(t) \\
x(0) &= 0 & & \\
y(0) &= a^2 & &
\end{aligned}$$

The graph of $(x(t), y(t))$ with $a = 1.2$ is the solid blue parabola in Figure 18.23.

Exercise 18.2.1. See the solution to Explore 18.2.1.

Exercise 18.2.2. The origin is not a stable equilibrium point of $x'(t) = y(t)$, $y'(t) = 2x(t)y(t)$. For any positive number δ, consider $a =$ the minimum of δ and 1. The solution $x(t) = a\tan at$, $y(t) = a^2\sec^2 at$

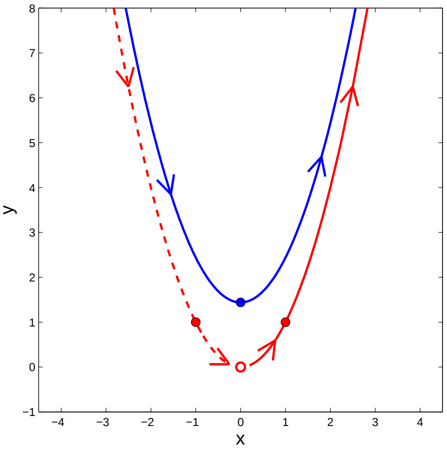


Figure 18.23: **Explore 18.2.1.** Graphs of solutions to $x'(t) = y(t)$, $y'(t) = 2x(t)y(t)$.

passes through the point $(0, a^2)$ at $t = 0$ and is within δ of the origin. But as $t \to \pi/2$, $(x(t), y(t)) \to (\infty, \infty)$.

Exercise 18.2.3. The origin is a stable equilibrium point of $x'(t) = -y(t)$, $y'(t) = 25x(t)$. The equations $x(t) = a\cos 5t$, $y(t) = 5a\sin 5t$ for $a \geq 0$ all solve the system and their graphs are ellipses that fill the x, y-plane. (To find these solutions, note that $x''(t) = -y'(t) = -25x(5)$, $x''(t) + 25x(t) = 0$, so that $x(t) = A\cos 5t + B\sin 5t$. We have a choice and choose $A = a$ and $b = 0$.) See Figure 18.24. There are no other solutions. Suppose $\epsilon > 0$ and choose $\delta = \epsilon/5$. If(x_0, y_0) is an initial point within δ of (0,0), it lies on an ellipse that lies entirely within a circle of radius 5δ with center at the origin. Therefore the solution to $x'(t) = -y(t)$, $y'(t) = 25x(t)$ with initial point (x_0, y_0) stays within ϵ of (0,0). It can be seen that (0,0) is not an asymptotically stable point; none of the ellipses get closer to the origin than the length of their minor axis.

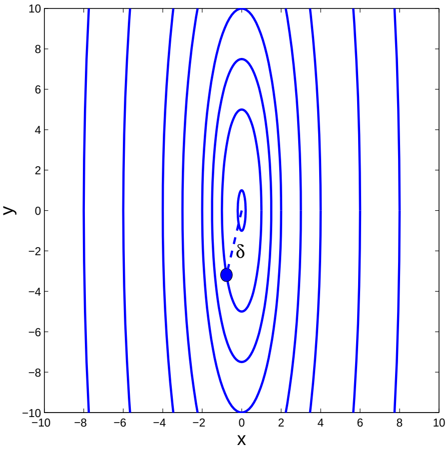


Figure 18.24: **Explore 18.2.1.** Graphs of solutions to $x'(t) = -y(t)$, $y'(t) = 25x(t)$.

Exercise 18.3.2. a. At (1,2), $x' = -1/2$, $y' = -0.6$. Draw a vector from (1,2) toward (1,2)+(-0.5,-0.6)

= (0.5,1.4).

Exercise 18.3.3. a. At (0.5,-1), $x' = 0.3$, $y' = 0.55$. Draw a vector from (0.5,1) toward (0.5,1)+(0.3,0.55) = (0.8,1.55).

Exercise 18.3.4. $r^2 - (a_{1,1} + a_{2,2})r + (a_{1,1}a_{2,2} - a_{1,2}a_{2,1}) = 0$.

$$\begin{array}{llrclrrl}
\text{a.} & \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} & r^2 + 0r + 1 & = & 0, & i & -i & \begin{array}{l}\text{Stable}\\ \text{not A-stable.}\end{array} \\
\text{c.} & \begin{bmatrix} -6 & -2 \\ 2 & -1 \end{bmatrix} & r^2 + 7r + 10 & = & 0, & -5 & -2 & \text{A-stable.} \\
\text{e.} & \begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix} & r^2 - 2r + 1 & = & 0, & 1 & 1 & \text{Unstable.} \\
\text{g.} & \begin{bmatrix} -1 & -5 \\ 2 & -3 \end{bmatrix} & r^2 + 4r + 13 & = & 0, & -2+3i & -2-3i & \text{A-stable.} \\
\text{i.} & \begin{bmatrix} -3 & 1 \\ 2 & -2 \end{bmatrix} & r^2 + 5r + 4 & = & 0, & -1 & -4 & \text{A-stable.} \\
\text{k.} & \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} & r^2 - 2r + 5 & = & 0, & 1+2i & 1-2i & \begin{array}{l}\text{Stable}\\ \text{not A-stable.}\end{array}
\end{array}$$

Exercise 18.4.1.

$$F(x,y) = x - x^2 - xy, \quad F_1(x,y) = 1 - 2x - y, \quad F_2(x,y) = -x$$

$$G(x,y) = y - 0.5xy - 2y^2, \quad G_1(x,y) = -0.5y \quad G_2(x,y) = 1 - 0.5x - 4y$$

$$J = \begin{bmatrix} 1 - 2x - y & -x \\ -0.5y & 1 - 0.5x - 4y \end{bmatrix}$$

$$J(0,0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{The LLA at (0,0) is} \qquad \begin{array}{rcl} x'(t) & = & x(t) \\ y'(t) & = & y(t) \end{array}$$

The roots are $r_1 = 1$ and $r_2 = 1$ and the LLA is not stable.

$$J(1,0) = \begin{bmatrix} 0.5 & 0 \\ -0.25 & -1 \end{bmatrix} \qquad \text{The LLA at (0,0) is} \qquad \begin{array}{rcl} x'(t) & = & 0.5x(t) \\ y'(t) & = & -0.25x(t) - y(t) \end{array}$$

The roots are $r_1 = 0.5$ and $r_2 = -1$ and the LLA is not stable.

Exercise 18.4.2.

$$\text{a.} \quad \begin{array}{rcrcl} F(x,y) & = & -xy & = & 0 \\ G(x,y) & = & 1 - x - y & = & 0 \end{array} \qquad (x,y) = (0,1) \quad \text{or} \quad (x,y) = (1,0)$$

$$\begin{array}{rclrcl} F_1(x,y) & = & -y & F_2(x,y) & = & -x \\ G_1(x,y) & = & -1 & G_2(x,y) & = & -1 \end{array} \qquad J = \left[\begin{array}{cc} -y & -x \\ -1 & -1 \end{array}\right]$$

$$J(0,1) = \left[\begin{array}{cc} -1 & 0 \\ -1 & -1 \end{array}\right], \qquad r_1 = -1, \quad r_2 = -1$$

The system is asymptotically stable at (0,1).

$$J(1,0) = \left[\begin{array}{cc} 0 & -1 \\ -1 & -1 \end{array}\right], \qquad r_1 = \frac{-1+\sqrt{5}}{2} > 0, \quad r_2 = \frac{-1-\sqrt{5}}{2} < 0.$$

The system is not asymptotically stable at (1,0).

$$\text{c.} \quad \begin{array}{rclcl} F(x,y) & = & 2-x^2-y^2 & = & 0 \\ G(x,y) & = & 1-xy & = & 0 \end{array} \qquad (x,y) = (1,1) \quad \text{or} \quad (x,y) = (-1,-1)$$

$$\begin{array}{rclrcl} F_1(x,y) & = & -2x & F_2(x,y) & = & -2y \\ G_1(x,y) & = & -y & G_2(x,y) & = & -x \end{array} \qquad J = \left[\begin{array}{cc} -2x & -2y \\ -y & -x \end{array}\right]$$

$$J(1,1) = \left[\begin{array}{cc} -2 & -2 \\ -1 & -1 \end{array}\right], \qquad r_1 = 0, \quad r_2 = -3$$

The system is not asymptotically stable at (1,1).

$$J(-1,-1) = \left[\begin{array}{cc} 2 & 2 \\ 1 & 1 \end{array}\right], \qquad r_1 = 0, \quad r_2 = 3$$

The system is not asymptotically stable at (1,1).

$$\text{d.} \quad \begin{array}{rclcl} F(x,y) & = & 5-x^2-y^2 & = & 0 \\ G(x,y) & = & 2-xy & = & 0 \end{array} \qquad \begin{array}{rcl} (x,y) & = & (2,1), \quad (-2,-1), \\ & & (1,2), \quad \text{or} \quad (-1,-2) \end{array}$$

Exercise 18.4.3.

$$\begin{array}{rcl} F(x,y) & = & 1 - \frac{11}{6}x + \dfrac{10xy}{11+xy} \\[2ex] G(x,y) & = & 1 - \frac{11}{6}y + \dfrac{10xy}{11+xy} \\[2ex] F(1,1) & = & 1 - \frac{11}{6}\cdot 1 + \dfrac{10\cdot 1\cdot 1}{11+1\cdot 1} \quad = \quad 1 - \frac{11}{6} + \frac{5}{6} \quad = \quad 0 \\[2ex] G(1,1) & = & 1 - \frac{11}{6}\cdot 1 + \dfrac{10\cdot 1\cdot 1}{11+1\cdot 1} \quad = \quad 1 - \frac{11}{6} + \frac{5}{6} \quad = \quad 0 \end{array}$$

$$\begin{array}{rclrcl} F_1(x,y) & = & -\frac{11}{6} + \dfrac{110y}{(11+xy)^2} & F_2(x,y) & = & \dfrac{110x}{(11+xy)^2} \\ G_1(x,y) & = & \dfrac{110y}{(11+xy)^2} & G_2(x,y) & = & -\frac{11}{6} + \dfrac{110x}{(11+xy)^2} \end{array}$$

$$J(1,1) = \left[\begin{array}{cc} -\frac{11}{6} + \frac{110}{144} & \frac{110}{144} \\[2ex] \frac{110}{144} & -\frac{11}{6} + \frac{110}{144} \end{array}\right] \doteq \left[\begin{array}{cc} -1.0694 & 0.7639 \\[2ex] 0.7639 & -1.0694 \end{array}\right]$$

$r_1 \doteq -0.3056$, $r_2 \doteq -1.8333$ and the system **is** stable at (1,1).

Exercise 18.5.1.

$$\begin{array}{lrcccl} \textbf{a}. & x'(t) & = & F(x,y) & = & 0.2 \times x(t) \times (1 - 0.2x(t) - 0.4y(t)) \\ & y'(t) & = & G(x,y) & = & 0.1 \times y(t) \times (1 - 0.4x(t) - 0.5y(t)) \end{array}$$

$$x' = 0 \quad \text{and} \quad y' = 0 \quad \Longrightarrow \quad (x,y) = \begin{array}{ll} (0,0), & (0,2), \\ (5,0) & \text{or} \quad (15/11, 10/11). \end{array}$$

The nullclines and direction field arrows are shown in Figure 18.25A.

$$\begin{array}{rclcrcl} F_1(x,y) & = & 0.2 - 0.08x - 0.04y & & F_2(x,y) & = & -0.04x \\ G_1(x,y) & = & -0.04y & & G_2(x,y) & = & 0.1 - 0.04x - 0.01y \end{array}$$

$$J(0,0) = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.1 \end{bmatrix}$$

The roots are $r_1 = 0.2$ and $r_2 = 0.1$ and the system **is not** stable at (0,0).

$$J(0,2) = \begin{bmatrix} 0.12 & 0 \\ -0.08 & -0.1 \end{bmatrix}$$

The roots are $r_1 = 0.12$ and $r_2 = -0.1$ and the system **is not** stable at (0,2).

$$J(5,0) = \begin{bmatrix} -0.2 & -0.4 \\ 0 & -0.1 \end{bmatrix}$$

The roots are $r_1 = -0.2$ and $r_2 = -0.1$ and the system **is** stable at (5,0).

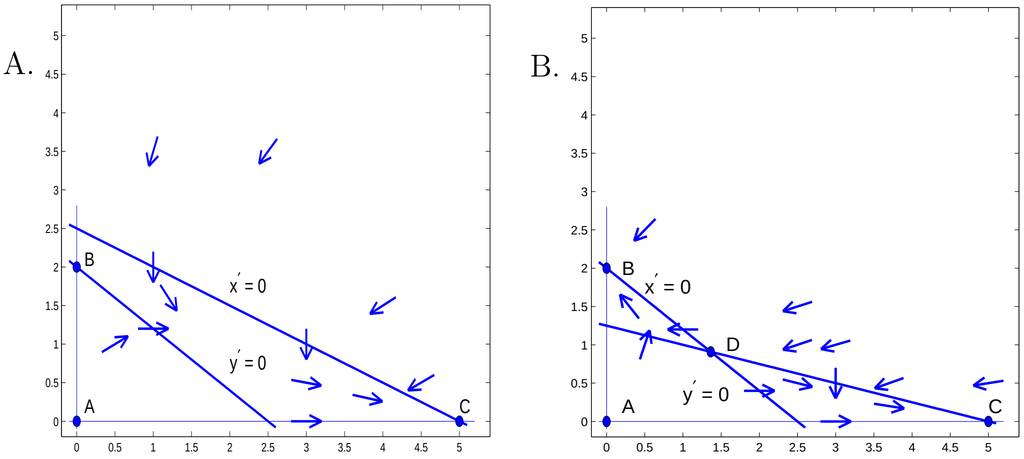


Figure 18.25: **Exercise 18.5.1**a and b. Nullclines and some direction arrows for A

$$\begin{array}{lrcccl} \textbf{c}. & x'(t) & = & F(x,y) & = & 0.2 \times x(t) \times (1 - 0.6x(t) - 0.4y(t)) \\ & y'(t) & = & G(x,y) & = & 0.1 \times y(t) \times (1 - 0.4x(t) - 0.5y(t)) \end{array}$$

$$x' = 0 \quad \text{and} \quad y' = 0 \quad \Longrightarrow \quad (x,y) = \begin{array}{ccc} (0,0), & (0,2), & \\ (5/3,0) & \text{or} & (5/7,10/7). \end{array}$$

A MATLAB program to solve this problem follows this solution. The nullclines and direction field arrows are shown in Figure 18.26C.

$$\begin{array}{rclcrcl} F_1(x,y) & = & 0.2 - 0.24x - 0.08y & \qquad & F_2(x,y) & = & -0.08x \\ \\ G_1(x,y) & = & -0.04y & & G_2(x,y) & = & 0.1 - 0.04x - 0.1y \end{array}$$

$$J(0,0) = \left[\begin{array}{cc} 0.2 & 0 \\ \\ 0 & 0.1 \end{array} \right]$$

The roots are $r_1 = 0.2$ and $r_2 = 0.1$ and the system **is not** stable at (0,0).

$$J(0,2) = \left[\begin{array}{cc} 0.04 & 0 \\ \\ -0.08 & -0.1 \end{array} \right]$$

The roots are $r_1 = -0.1$ and $r_2 = 0.04$ and the system **is not** stable at (0,2).

$$J(5/3,0) \doteq \left[\begin{array}{cc} -0.2 & -0.1333 \\ \\ 0 & 0.0333 \end{array} \right]$$

The roots are $r_1 = -0.2$ and $r_2 = 0.0333$ and the system **is not** stable at (5/3,0).

$$J(5/7,10/7) \doteq \left[\begin{array}{cc} -0.0857 & -0.0571 \\ \\ -0.0571 & -0.0714 \end{array} \right]$$

The roots are $r_1 = -0.1362$ and $r_2 = -0.0210$ and the system **is** stable at (5/7,10/7).

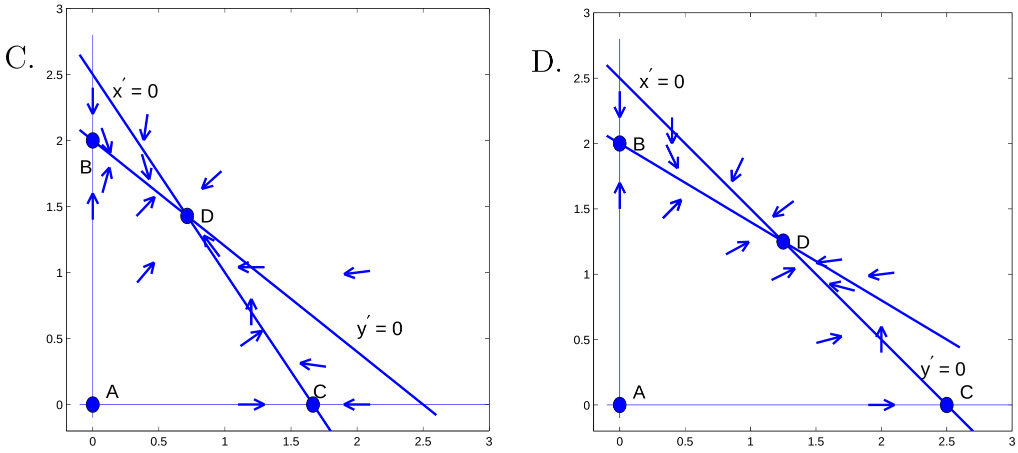


Figure 18.26: **Exercise 18.5.1** c and d. Nullclines and some direction arrows for A

MATLAB program to solve Exercise 18.5.1c

```
close all;clc;clear
equilib=[0 0;0 2;5/3 0; 5/7 10/7];
for k = 1:4
    xxx=equilib(k,1);    yyy=equilib(k,2);
    xp  = 0.2*xxx*(1-0.6*xxx-0.4*yyy);
    yp  = 0.1*yyy*(1-0.4*xxx-0.5*yyy);
    xp1 = 0.2 - 0.24*xxx-0.08*yyy;   xp2 = -0.08*xxx;
    yp1 = -0.04*yyy;                 yp2 = 0.1-0.04*xxx-0.1*yyy;
    J=    [xp1 xp2;yp1 yp2];
    E=eig(J);  %%Computes the roots of J.
    Val=[xxx xp xp1 xp2 E(1); yyy  yp yp1 yp2 E(2)]
end
x1=[-0.1:0.1:2.0];   y1=2.5 - 1.5*x1;  %%Nullcline
x2=[-0.1:0.1:2.6];   y2 = 2 - 0.8*x2;  %%Nullcline
plot(x1,y1,'linewidth',2); hold; plot(x2,y2,'linewidth',2)
axis([-0.2 3 -0.2 3]);   axis('square')
plot([-0.1 5.2],[0 0]);  plot([0 0],[-0.1 2.8])
th = 2*pi*[0:0.05:1]; r = 0.05; xx=r*cos(th); yy=1.2*r*sin(th);
fill(xx,yy,'b');
fill(xx,2+yy,'b');
fill(5/3+xx,yy,'b');
fill(5/7+xx,10/7+yy,'b')  %%Equilibrium points
text(0.1,0.1,'A','fontsize',16)
text(0.1,2,'B','fontsize',16)
text(5/3,0.1,'C','fontsize',16)
text(5/7+0.1,10/7,'D','fontsize',16)
text(0.15,2.4,'x^{\prime} = 0','fontsize',16)
text(2.0,0.6,'y^{\prime} = 0','fontsize',16)

d = 0.1;
xv=[-d d;0 0]; av=[0.2*d d;0.4*d 0]; bv=[0.2*d d;-0.4*d 0];
arrow=[0.0 1.5; 0.0 2.3; 0.4 1.0; 0.4 1.5;0.4 1.8; ...
   0.4 2.1;0.9 1.2; 0.9 1.7;1.2 0; 1.2 0.5; 1.2 0.7; ...
   1.2 1.04; 2.0 1.0; 2.0 0.0; 5/3 0.3;];
for k = 1:length(arrow)
    xxx=arrow(k,1);    yyy=arrow(k,2);
    xp= 0.2*xxx*(1-0.6*xxx-0.4*yyy);
    yp= 0.1*yyy*(1-0.4*xxx-0.5*yyy);
```

```
    alph = atan2(yp,xp);  %%MATLAB, Strange, but works

    rot=[cos(alph) -sin(alph);sin(alph) cos(alph)];
    vp = rot*xv; va = rot*av;  vb = rot*bv;
    plot(xxx+vp(1,:),yyy+vp(2,:),'linewidth',2)
    plot(xxx+va(1,:),yyy+va(2,:),'linewidth',2)
    plot(xxx+vb(1,:),yyy+vb(2,:),'linewidth',2)
end
```

Exercise 18.5.2. See Figure 18.27. All of the direction arrows in each region point toward the equilibrium point, e. e is a stable node.

A more tedious analysis shows that $e = ((d-b)/\Delta, (a-c)/\Delta)$ where $\Delta = ad - bc$. Both Δ and the coordinates of e are positive. The Jacobian is

$$J = \begin{bmatrix} \dfrac{-r_x a(d-b)}{\Delta} & \dfrac{-r_x b(d-b)}{\Delta} \\ \\ \dfrac{-r_y c(a-c)}{\Delta} & \dfrac{-r_y d(a-c)}{\Delta} \end{bmatrix}$$

Let $T = \text{Trace}(J)$ and $D = \text{Determinant}(J)$. The characteristic equation and roots (written in terms of s rather than the usual r) are

$$s^2 - T\,s + D = 0 \qquad s_1 = \frac{T + \sqrt{T^2 - 4D}}{2} \qquad s_2 = \frac{-T - \sqrt{T^2 - 4D}}{2}.$$

We will show that

$$T < 0, \qquad T^2 - 4D > 0, \qquad \text{and} \qquad D > 0.$$

It will follow than that both s_1 and s_2 are negative real roots and that e is a stable node. Remember that $a > c$ and $d > b$ and $\Delta > 0$.

$$T = \frac{-r_x a(d-b)}{\Delta} + \frac{-r_y d(a-c)}{\Delta} < 0.$$

$$\begin{aligned} T &= \frac{-r_x a(d-b)}{\Delta} + \frac{-r_y d(a-c)}{\Delta} < 0 \\ \\ T^2 - 4D &= \left(\frac{-r_x a(d-b)}{\Delta} \quad + \quad \frac{-r_y d(a-c)}{\Delta}\right)^2 \\ &\quad -4\left(\frac{-r_x a(d-b)}{\Delta}\frac{-r_y d(a-c)}{\Delta} - \frac{-r_y d(a-c)}{\Delta}\frac{-r_x b(d-b)}{\Delta}\right) \\ \\ &= \left(\frac{-r_x a(d-b)}{\Delta} - \frac{-r_y d(a-c)}{\Delta}\right)^2 + 4\frac{r_y d(a-c)}{\Delta}\frac{r_x b(d-b)}{\Delta} > 0 \end{aligned}$$

$$\begin{aligned} D &= \frac{-r_x a(d-b)}{\Delta}\,\frac{-r_y d(a-c)}{\Delta} - \frac{-r_y d(a-c)}{\Delta}\,\frac{-r_x b(d-b)}{\Delta} \\ &= \frac{r_x\, r_y\, (a-c)\,(d-b)\,(ad-bc)}{\Delta^2} > 0 \end{aligned}$$

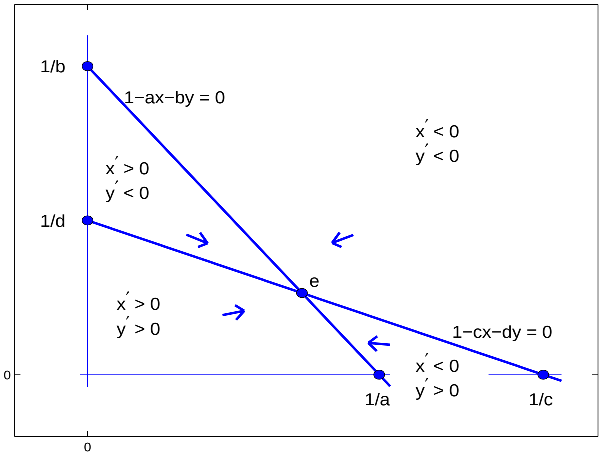


Figure 18.27: **Exercise 18.5.2**. Nullclines and some direction arrows for the competition model, $x'(t) = r_x \times x(t) \times (1 - ax(t) - by(t))$ and $y'(t) = r_y \times y(t) \times (1 - cx(t) - dy(t))$, where $1/c > 1/a$ and $1/b > 1/d$.

Exercise 18.5.4. The prey is x and the predator is y and logistic growth of the prey without predator together with predator effect is represented by

$$\begin{aligned} x'(t) &= ax\left(1 - \frac{x}{M}\right) - bxy \\ y'(t) &= -cy + dxy, \end{aligned} \qquad (18.67)$$

where M is the maximum prey the environment will support. An equilibrium point of the system and Jacobian at that point are

$$\left(\frac{c}{d}, \frac{a}{b}(1 - \frac{c}{Md})\right) \qquad \text{and} \qquad \begin{bmatrix} \dfrac{-a\,c}{M\,d} & \dfrac{-bc}{d} \\ \dfrac{ad}{b} - \dfrac{a\,c}{M\,b} & 0 \end{bmatrix}$$

For the equilibrium point to have positive coordinates, we must have $Md > c$.

The characteristic equation is

$$r^2 - \frac{-a\,c}{M\,d}r + \left(\frac{a\,d}{b} - \frac{a\,c}{M\,b}\right) \times \frac{bc}{d} = 0$$

$$r^2 + \frac{a\,c}{M\,d}r + ac\left(1 - \frac{c}{Md}\right) = 0$$

Both coefficients are positive (assuming that $Md > c$) and it follows that either both roots are negative, there is only one root and it is negative, or the roots are complex with negative real part. Thus the equilibrium point

$$\left(\frac{c}{d}, \frac{a}{b}(1 - \frac{c}{Md})\right) \quad \text{is asymptotically stable} \quad if Md > c.$$

We chose parameters $a = 0.2$; $M = 10$; $b = 0.2$; $c = 0.1$; and $d = 0.08$; $md = 0.8 > 0.1 = c$; $x_e = c/d = 1.25$; and $ye = a/b - (a * c)/(b * d * M) = 0.875$.

A direction field is shown in Figure 18.28, and it appears that the equilibrium is a stable spiral. Indeed, the roots for this case are $-0.0125 + 0.1317i$ and $-0.0125 - 0.1317i$ which confirm a stable spiral.

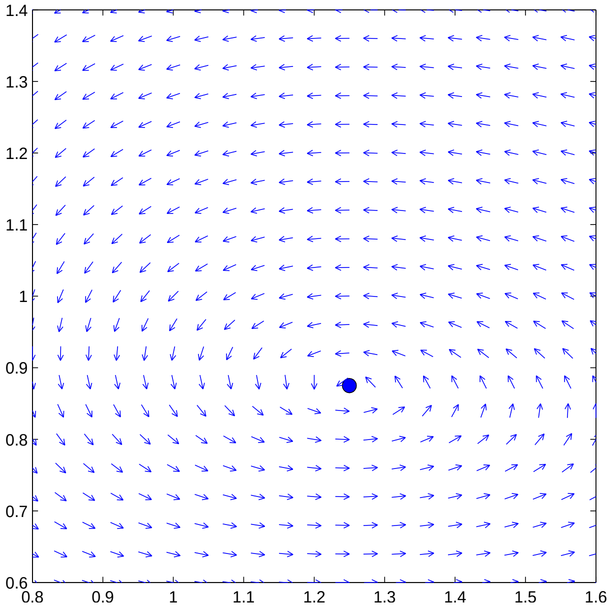


Figure 18.28: **Exercise 18.5.4**. A direction field for Equations 18.67 with parameters $a = 0.2$; $M = 10$; $b = 0.2$; $c = 0.1$; and $d = 0.08$; $x_e = c/d$; and $ye = a/b - (a * c)/(b * d * M)$.

Exercise 18.5.8.

$$\begin{aligned} dE/dt &= \Lambda - \mu E + \epsilon V E \\ dV/dt &= rV - \sigma V E \end{aligned} \tag{18.68}$$

a. $-\mu E$ is the exponential decrease of effector cells.
$V\dot{E}$ is the stimulation of production of effector cells due to the presence of virus.
rV is the exponential growth of virus.
$-\sigma\, V\, E$ is the destruction of virus by effector cells.

b. With $V = 0$ and $dE/dt = 0$, $\Lambda - \mu E + \epsilon V\, E = \Lambda - \mu\hat{E} = 0$, $\hat{E} = \Lambda/\mu$.

c. In order that the viral population increase when $E = \hat{E}$, dV/dt must be positive. Then

$$rV - \sigma V E = V(r - \sigma\hat{E}) > 0, \quad \Longrightarrow r - \sigma\hat{E} > 0.$$

d. $J(\hat{E}, 0)$ is $J(\Lambda/\mu, 0)$.

e. If both roots of $J(\hat{E}, 0)$ are negative, the system 18.68 is locally asymptotically stable at $(\hat{E}, 0)$.

f.

g. Solve

$$\begin{aligned} 0 &= \Lambda - \mu E^* + \epsilon V^* E^* \\ 0 &= rV^* - \sigma V^* E^* \end{aligned}$$

for E^* and V^*.

h. Evaluate

$$J(E,V) = \begin{bmatrix} -\mu + \epsilon V & \epsilon E \\ -\sigma V & r - \sigma V \end{bmatrix} \text{ for}$$

$$(E,V) = (E^*, V^*) = (r/\sigma, (\mu r - \Lambda\sigma)/\epsilon r).$$

i. $trace(M) = m_{1,1} + m_{2,2}, \qquad det(M) = m_{1,1}m_{2,2} - m_{2,1}m_{2,1}.$

j.

$$\text{Case}: \quad \left(\frac{\Lambda\sigma}{r}\right)^2 - 4(\mu r - \Lambda\sigma) > 0, \quad \sqrt{\left(\frac{\Lambda\sigma}{r}\right)^2 - 4(\mu r - \Lambda\sigma)} < \frac{\Lambda\sigma}{r}.$$

k. The Jacobian at (e^*, V^*) is

$$\begin{bmatrix} -0.04 & 0.5 \\ -0.2300 & 0 \end{bmatrix},$$

the roots of which are $-0.0200 \pm 0.3381i$. The system is stable and oscillates.

l. See Figure 18.29.

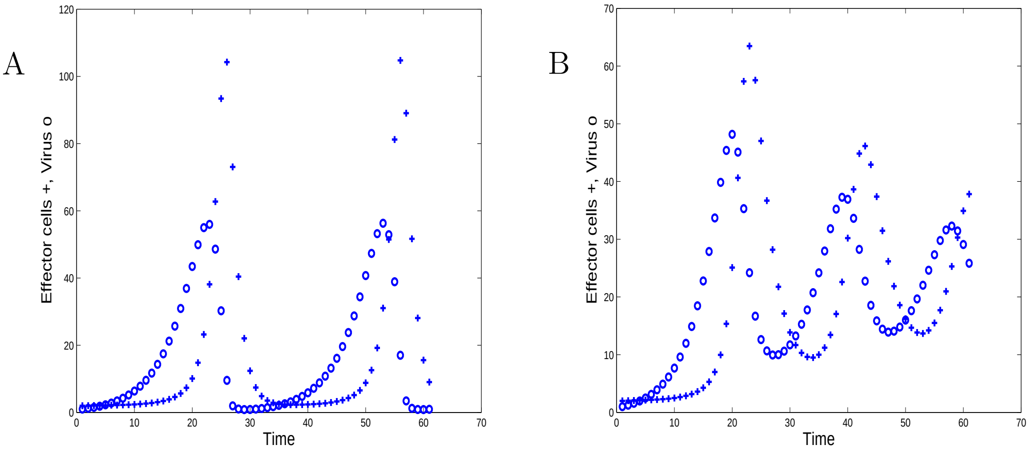


Figure 18.29: **Exercise 18.5.8.** Solutions to Equations 18.64 using $E_0 = 2$, $V_0 = 1$, $\Lambda = 1$, $\mu = 0.5$, $\epsilon = 0.02$, $r = 0.25$, and $\sigma = 0.01$. A. Euler's method. B. Trapezoid method.

Exercise 18.5.9.

$$\begin{aligned} \text{a.} \quad x(t) &= s_0 + \frac{\beta S_0 \epsilon}{\beta s_0 - \gamma}(1 - e^{(\beta s_0 - \gamma)t}) \\ y(t) &= \epsilon e^{(\beta s_0 - \gamma)t} \end{aligned}$$

$$\begin{aligned}
x(0) &= s_0 + \frac{\beta S_0 \epsilon}{\beta s_0 - \gamma}(1-1) = s_0 \\
x'(t) &= \frac{\beta S_0 \epsilon}{\beta s_0 - \gamma}\left(-(\beta s_0 - \gamma)\right) e^{(\beta s_0 - \gamma)t} \\
&= -\beta s_0 \epsilon e^{(\beta s_0 - \gamma)t} = -\beta s_0\, y(t) \\
y(0) &= \epsilon e^{(\beta s_0 - \gamma)0} = \epsilon \times 1 = \epsilon \\
y'(t) &= \epsilon(\beta s_0 - \gamma) e^{(\beta s_0 - \gamma)t} \quad = \quad (\beta s_0 - \gamma) y(t)
\end{aligned}$$

b. If $\beta s_0 - \gamma < 0$,

$$\lim_{t\to\infty} y(t) \to 0$$

The number of infected, $y(t)$ goes to zero. There is no epidemic.

Exercise 18.5.10,

$$\begin{aligned}
s'(t) &= \alpha r(t) - \beta \times s(t) \times i(t) \\
i'(t) &= \beta \times s(t) \times i(t) - \gamma i(t) \\
r'(t) &= \gamma i(t) - \alpha r(t)
\end{aligned}$$

a.

b. The Jacobian at $(s_0, 0, 0)$ is

$$\begin{bmatrix} 0 & -\beta\, s_0 & \alpha \\ 0 & \beta\, s_0 - \gamma & 0 \\ 0 & \gamma & -\alpha \end{bmatrix}$$

c.

d.

$$\begin{aligned}
x'(t) &= -\beta s_0\, y(t) + \alpha z(t) & x(0) &= s_0 \\
y'(t) &= (\beta s_0 - \gamma)\, y(t) & y(0) &= \epsilon \\
z'(t) &= \gamma y(t) - \alpha z(t) & z(0) &= 0
\end{aligned}$$

For convenience, let $K = \beta\, s_0 - \gamma$. Then

$$y'(t) = Ky(t), \qquad y(t) = Ce^{Kt}, \qquad y(0) = \epsilon, \qquad y(t) = \epsilon e^{Kt}.$$

$$z'(t) = \gamma y(t) - \alpha z(t) = \gamma\, \epsilon\, e^{Kt} - \alpha z(t)$$

$$\begin{aligned}
z'(t) + \alpha\, z(t) &= \gamma\epsilon e^{Kt} \\
z_p(t) &= Ae^{Kt} \qquad AKe^{Kt} + \alpha e^{Kt} = \gamma\epsilon e^{Kt} \\
AK + \alpha &= \gamma\epsilon, \qquad A = \frac{\gamma\epsilon - \alpha}{K} \\
z_p &= \frac{\gamma\epsilon - \alpha}{K} e^{Kt} \qquad \text{Assume} \quad \gamma\epsilon \neq \alpha. \\
z_h'(t) + \alpha z_h(t) &= 0, \qquad z_h(t) = C\, e^{-\alpha\, t} \\
z(t) &= \frac{\epsilon - \alpha}{K} e^{Kt} + Ce^{-\alpha\, t}, \qquad z(0) = 0 \\
z(t) &= \frac{\epsilon - \alpha}{K}\left(e^{Kt} - e^{-\alpha\, t}\right)
\end{aligned}$$

$$\begin{aligned}
x'(t) &= \gamma y(t) - \alpha z(t) \\
&= \gamma\epsilon e^{Kt} - \alpha\,\frac{\epsilon - \alpha}{K}\left(e^{Kt} - e^{-\alpha\, t}\right) \\
x(t) &= \frac{\gamma\epsilon}{K} e^{Kt} - \alpha\,\frac{\epsilon - \alpha}{K}\left(\frac{e^{Kt}}{K} + \frac{e^{-\alpha t}}{\alpha}\right) \\
&= \frac{\gamma\epsilon}{\beta\, s_0 - \gamma} e^{(\beta\, s_0 - \gamma)t} - \alpha\,\frac{\epsilon - \alpha}{\beta\, s_0 - \gamma}\left(\frac{e^{(\beta\, s_0 - \gamma)t}}{\beta\, s_0 - \gamma} + \frac{e^{-\alpha t}}{\alpha}\right)
\end{aligned}$$

Exercise 18.5.11.

a. As time progresses, the curve ascends toward the left, meaning that the number of infected is increasing and the number of susceptible is decreasing. The curve reaches a maximum at about 0.84 susceptible, 0.16 infected and decreases to the left from there. The number of infected decreases, as does the number of susceptible.

b. At E, the number of infected is zero, so no more susceptible get infected; it appears that about 0.68 susceptible escapes the virus.

c. The number, i of infected depends on the number of susceptible, and we write

$$i = i(s). \quad \text{By the chain rule,} \quad \frac{d\,i}{d\,t} = \frac{d\,i(s)}{d\,s}\,\frac{d\,s}{d\,t} \qquad \frac{d\,i(s)}{d\,s} = \frac{\frac{d\,i}{d\,t}}{\frac{d\,s}{d\,t}} = \frac{d\,i}{d\,s}$$

$$\begin{aligned}
\frac{d\,i}{d\,s} &= \frac{\dfrac{d\,i}{d\,t}}{\dfrac{d\,s}{d\,t}} = \frac{\beta \times s(t) \times i(t) - \gamma i(t)}{-\beta \times s(t) \times i(t)} \\
&= \frac{\beta \times s(t) \times i(t)}{-\beta \times s(t) \times i(t)} \quad + \quad \frac{-\gamma i(t)}{-\beta \times s(t) \times i(t)}
\end{aligned}$$

$$\begin{aligned}
\frac{d\,i}{d\,s} &= -1 + \frac{\gamma}{\beta}\frac{1}{s} \\
i &= -s + \frac{\gamma}{\beta}\ln s \,+\, C \\
\epsilon &= -s_0 + \frac{\gamma}{\beta}\ln s_0 \,+\, C \\
C &= \epsilon + s_0 - \frac{\gamma}{\beta}\ln s_0 \\
i &= -s + \frac{\gamma}{\beta}\ln s \,+\, \epsilon + s_0 - \frac{\gamma}{\beta}\ln s_0 \\
&= \epsilon + s_0 - s + \frac{\gamma}{\beta}\ln\frac{s}{s_0}
\end{aligned}$$

d. As $t \to \infty$, s decreases, potentially to zero. However, $\ln(s/s_0) \to -\infty$ as $s \to 0$. There is a positive value $\tilde{s}$ for which $\epsilon + s_0 - \tilde{s} + \frac{\gamma}{\beta}\ln\frac{\tilde{s}}{s_0}$ is negative, and a positive value $\hat{s}$ for which $\epsilon + s_0 - \hat{s} + \frac{\gamma}{\beta}\ln\frac{\hat{s}}{s_0} \quad = \quad i$ is zero (see the point, E, in Figure 18.5.11.) At least $\hat{s}$ susceptibles do not get infected.